AN $\mathfrak{sl}_2$-CATEGORIFICATION OF TENSOR PRODUCTS OF SIMPLE REPRESENTATIONS OF $\mathfrak{sl}_2$

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Abstract. In [12], Zheng gave a geometric categorification of tensor products of simple $U_q(\mathfrak{sl}_2)$-modules. We extend his work to a 2-categorical setting, in line with the higher representation theory programme of Rouquier.

1. Introduction

For any parabolic subgroup $P \subset GL_n(\mathbb{C})$, let $\mathcal{D}_P(\mathrm{Gr}(i))$ denote the bounded derived category of P-smooth constructible $\overline{\mathbb{Q}_l}$-sheaves on $\mathrm{Gr}(i)$. Let V denote a tensor product of simple representations of $\mathfrak{sl}_2$. We show that there is an integer $n \geq 0$ and a parabolic $P \subset GL_n(\mathbb{C})$ such that $\bigoplus_{i=0}^n \mathcal{D}_P(\mathrm{Gr}(i))$, equipped with certain natural endofunctors E and F, is a triangulated categorification of V. This is based on work of Zheng ([12]).

Realising E and E^2 as Fourier-Mukai transforms, we explain how to define 2-morphisms $X \in \mathrm{End}^\bullet(E)$ and $T \in \mathrm{End}^\bullet(E^2)$ using Chern classes of canonical vector bundles. This extends our construction to an $\mathfrak{sl}_2$-categorification, in the sense of Chuang and Rouquier ([6]).

Keeping track of the Tate twist, we pass to a categorification of tensor products of simple representations of $U_q(\mathfrak{sl}_2)$. Via Koszul duality, as in [12], we obtain an abelian categorification of these representations.

Naturally, our results should generalise to the case of highest weight integrable representations of arbitrary quantum groups. Using a notion of micro-local perverse sheaves on quiver varieties, Zheng has shown ([11]) how to generalise the weak categorification.

Let $\mathcal{D}^b_{P\times\mathbb{C}^*}(T^*(\mathrm{Gr}(i))\text{-coh})$ denote the bounded derived category of $(P \times \mathbb{C}^*)$-equivariant coherent sheaves on the cotangent bundle. Via Saito's mixed Hodge modules, $\bigoplus_{i=0}^n \mathcal{D}_P(\mathrm{Gr}(i))$ can be replaced by $\bigoplus_{i=0}^n \mathcal{D}^b_{P\times\mathbb{C}^*}(T^*(\mathrm{Gr}(i))\text{-coh})$ in the constructions above. In this picture, generalising to an arbitrary quantum group should be carried out by replacing $T^*(\mathrm{Gr}(i))$ by other quiver varieties.

The definition of X and T, and the proof that they satisfy the Hecke relations, was explained to me by Raphaël Rouquier. I thank him very much for his generosity, and for the many things I have learnt from him.

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2. Triangulated $\mathfrak{sl}_2$-categorification of tensor products

2.1. **Abelian $\mathfrak{sl}_2$-categorification.** Let $H_n(q)$, for $q \neq 0, 1$, denote the affine Hecke algebra of type $\widetilde{A}_{n-1}$ over a field k. Let $H_n(0)$ (resp. $H_n(1)$) denote the nil (resp. degenerate) affine Hecke algebra of type $\widetilde{A}_{n-1}$. In particular, $H_n(1)$ is not the specialisation of the affine Hecke algebra to $q = 1$.

We shall mainly need $H_n(0)$, which has generators $T_1, \ldots, T_{n-1}, X_1, \ldots, X_n$, subject to the following relations:

$$\begin{aligned} T_i^2 &= 0 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \\ T_iTj &= T_jT_i \;\text{ if } |i-j| > 1 \\ X_iX_j &= X_jX_i \\ T_iX_j &= X_jT_i \;\text{ if } |i-j| > 1 \\ T_iX_i &= X_{i+1}T_i - 1 \\ T_iX_{i+1} &= X_iT_i + 1. \end{aligned}$$

In [6], Chuang and Rouquier introduced the notion of an abelian $\mathfrak{sl}_2$-categorification.

Definition 2.1. A weak abelian $\mathfrak{sl}_2$-categorification is the data of

- a k-linear abelian category $\mathcal{A}$, with finite dimensional complexified Grothendieck group $K = \mathbb{C} \otimes K_0(\mathcal{A})$
- an adjoint pair of exact endofunctors (E, F)

such that

- E and F induce an action of $\mathfrak{sl}_2$ on K
- F is isomorphic to a left adjoint of E.

Example 2.2. Let $Gr(i)$ denote the Grassmannian variety of i-dimensional subspaces of $\mathbb{C}^n$, and let $\mathrm{Gr}(i, i+1)$ denote the partial flag variety $\{V, W \subset \mathbb{C}^n \mid V \subset W,\ \dim(V) = i,\ \dim(W) = i+1\}$. We have the following diagram, where p and q are the canonical projections.

$$Gr(i) \xleftarrow{\;p\;} Gr(i, i+1) \xrightarrow{\;q\;} Gr(i+1)$$

The (singular) cohomology algebra $H^\bullet(\mathrm{Gr}(i))$ has a unique simple module up to isomorphism, induced by the projection

$$H^\bullet(\mathrm{Gr}(i)) \twoheadrightarrow H^0(\mathrm{Gr}(i)) \xrightarrow{\simeq} \mathbb{C}\,.$$

The morphism q induces an inclusion of algebras $H^\bullet(\mathrm{Gr}(i+1)) \hookrightarrow H^\bullet(\mathrm{Gr}(i, i+1))$. The morphism p induces an inclusion of algebras $H^\bullet(\mathrm{Gr}(i)) \hookrightarrow H^\bullet(\mathrm{Gr}(i, i+1))$.

Regarding $H^\bullet(\mathrm{Gr}(i,i+1))$ as an $H^\bullet(\mathrm{Gr}(i+1))$-$H^\bullet(\mathrm{Gr}(i))$-bimodule, define

$$E_i\colon H^\bullet(\mathrm{Gr}(i))\text{-mod} \to H^\bullet(\mathrm{Gr}(i+1))\text{-mod}$$

by $H^\bullet(\mathrm{Gr}(i,i+1)) \otimes_{H^\bullet(\mathrm{Gr}(i))} -$. Regarding $H^\bullet(\mathrm{Gr}(i,i+1))$ as an $H^\bullet(\mathrm{Gr}(i))$-$H^\bullet(\mathrm{Gr}(i+1))$-bimodule, define

$$F_i\colon H^\bullet(\mathrm{Gr}(i+1))\text{-mod} \to H^\bullet(\mathrm{Gr}(i))\text{-mod}$$

by $H^\bullet(\mathrm{Gr}(i,i+1)) \otimes_{H^\bullet(\mathrm{Gr}(i+1))} -$. Then $E = \bigoplus_{i=0}^n E_i$ and $F = \bigoplus_{i=0}^n F_i$ define endofunctors of $\mathcal{A}(n) = \bigoplus_{i=0}^n H^\bullet(\mathrm{Gr}(i))$-mod.

It is classical (see 3.4 in [7], for example) that, as an $H^\bullet(\mathrm{Gr}(i))$-module,

$$H^\bullet(\mathrm{Gr}(i,i+1)) \cong \bigoplus_{j=0}^{n-i-1} H^\bullet(\mathrm{Gr}(i)).$$

As an $H^\bullet(\mathrm{Gr}(i))$-module,

$$H^\bullet(\mathrm{Gr}(i-1,i)) \cong \bigoplus_{j=0}^{i-1} H^\bullet(\mathrm{Gr}(i)).$$

It follows that

$$EF(H^\bullet(\mathrm{Gr}(i))) = i(n-i+1)H^\bullet(\mathrm{Gr}(i))$$

and that

$$FE(H^\bullet(\mathrm{Gr}(i)) = (n-i)(i+1)H^\bullet(\mathrm{Gr}(i)).$$

Let e and f denote the endomorphisms of $K_0(\mathcal{A}(n))$ induced by E and F. Then

$$(ef-fe)([H^\bullet(\mathrm{Gr}(i))]) = (2i-n)\,[H^\bullet(\mathrm{Gr}(i))]\,.$$

Since $\mathbb{C}\otimes K_0(\mathcal{A}(n)) = \bigoplus_{i=0}^n \mathbb{C}\,[H^\bullet(\mathrm{Gr}(i))]$, we have shown that $ef-fe$ acts on $K_0(\mathcal{A}(n)_\lambda)$ by λ, where $\mathcal{A}(n)_\lambda = H^\bullet(\mathrm{Gr}(\frac{\lambda+n}{2}))$-mod.

The endofunctors (E,F) are an adjoint pair, with F isomorphic to a left adjoint of E. We will see this later. Alternatively, one can prove it algebraically, as in Proposition 3.5 of [7].

Thus we have a weak $\mathfrak{sl}_2$-categorification of the simple representation of $\mathfrak{sl}_2$ of dimension n. In this example, we merged the arguments of 5.3 in [6] and 6.2 in [7]. In the latter paper, the weak $\mathfrak{sl}_2$-categorification is modified to a weak categorification of the simple representation of $U_q(\mathfrak{sl}_2)$ of dimension n, using graded versions of the functors and categories above. We will pass to $U_q(\mathfrak{sl}_2)$ slightly differently later.

Definition 2.3. An abelian $\mathfrak{sl}_2$-categorification is the data of

- a weak $\mathfrak{sl}_2$-categorification $\mathcal{A}$
- natural transformations $X \in \mathrm{End}(E)$, $T \in \mathrm{End}^2(E)$

such that

- $X-a$ is locally nilpotent for some $a \in k$
- $T_i \to 1_{E^{n-i-1}}T1_{E^{i-1}}$ and $X_i \to 1_{E^{n-i}}X1_{E^i}$ define a morphism $H_n(q) \to \mathrm{End}(E^n)$ for all n and a fixed q.

Remark 2.4. The last property is key, allowing abelian $\mathfrak{sl}_2$-categorifications to be controlled.

Example 2.5. Via Koszul duality and a special case of our main result, we will see (3.2) that the weak categorification 2.2 can be extended to an $\mathfrak{sl}_2$-categorification.

Subquotients of the affine Hecke algebra can be used to give an algebraic abelian $\mathfrak{sl}_2$-categorification of the simple representation of $\mathfrak{sl}_2$ of dimension n (see 5.3 in [6]). These 'minimal' categorifications play a central role in the theory of abelian higher representations of $\mathfrak{sl}_2$ (see 5.24 in [6]) .

Remark 2.6. The paper [6] of Chuang and Rouquier illustrates that abelian $\mathfrak{sl}_2$-categorifications yield derived equivalences of importance in representation theory. There are other motivations for higher representation theory beyond classical representation theory, as we remark briefly in 2.12.

2.2. **Triangulated $\mathfrak{sl}_2$-categorification.** We now define triangulated $\mathfrak{sl}_2$-categorifications, after Rouquier.

Definition 2.7. Let V denote a finite dimensional representation of $\mathfrak{sl}_2$. A *weak triangulated $\mathfrak{sl}_2$-categorification* of V is the data of

- a triangulated category $\mathcal{A}$
- an adjoint pair (E, F) of triangulated endofunctors of $\mathcal{A}$

such that

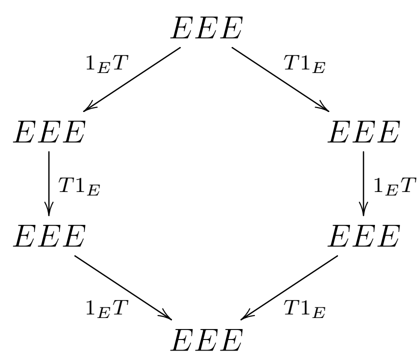
- F is isomorphic to a left adjoint of E
- $V = \mathbb{C} \otimes K_0(\mathcal{A})$.

Definition 2.8. A *triangulated $\mathfrak{sl}_2$-categorification* of V is the data of

- a weak triangulated $\mathfrak{sl}_2$-categorification $\mathcal{A}$
- natural transformations $X \in \mathrm{End}^\bullet(E)$ and $T \in \mathrm{End}^\bullet(E^2)$

such that

- the following diagram in $\mathrm{End}^\bullet(E^3)$ commutes



- $T^2 = 0$
- $T(X1_E) - (1_E X)T = 1 = (X1_E)T - T(1_E X)$
- X is nilpotent.

Remark 2.9. This is the case $q = 0$, so that we obtain a morphism $H_n(0) \to \mathrm{End}^\bullet(E^n)$. We shall not need the other two cases.

Remark 2.10. One should also ensure that $\mathcal{T}$ admits a weight decomposition compatible with E and F. We explain what holds for abelian categorifications.

Suppose that $\mathcal{A}$ equipped with endofunctors E and F is an abelian $\mathfrak{sl}_2$-categorification of a representation V of $\mathfrak{sl}_2$. If V_λ is a weight space of V, let $\mathcal{A}_\lambda$ denote the full subcategory of $\mathcal{A}$ of objects whose class belongs to V_λ in $\mathbb{C} \otimes K_0(\mathcal{A})$. It is proved in 5.5 of [6] that $\mathcal{A} = \bigoplus_\lambda \mathcal{A}_\lambda$, so that the class of an indecomposable object of $\mathcal{A}$ is a weight vector.

Furthermore, E and F are compatible with the weight decomposition of $\mathcal{A}$. Indeed, it is proved in 5.27 of [6] that if $\lambda \geq 0$, then

$$EF\mathrm{Id}_{\mathcal{A}_{-\lambda}} \bigoplus \mathrm{Id}_{\mathcal{A}_\lambda}^{\oplus\lambda} \cong FE\mathrm{Id}_{\mathcal{A}_{-\lambda}}$$

and

$$EF\mathrm{Id}_{\mathcal{A}_\lambda} \cong FE\mathrm{Id}_{\mathcal{A}_\lambda} \bigoplus \mathrm{Id}_{\mathcal{A}_\lambda}^{\oplus\lambda}.$$

In the triangulated case, one (probably) cannot deduce these facts from the axioms, so a stronger condition is needed. We avoid the question of what such a condition should be.

Example 2.11. Let $\mathcal{A}$ be an abelian $\mathfrak{sl}_2$-categorification of V, with endofunctors E and F and natural transformations X and T. The functors E and F pass to endofunctors $E^\bullet$ and $F^\bullet$ on the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$. Similarly, X and T pass to natural transformations $X^\bullet \in \mathrm{End}^\bullet(E^\bullet)$ and $T^\bullet \in \mathrm{End}^\bullet((E^2)^\bullet)$, giving $\mathcal{D}(\mathcal{A})$ the structure of a triangulated $\mathfrak{sl}_2$-categorification of V.

Remark 2.12. Following a suggestion of Crane and Frenkel, Rouquier has conjectured that, after passing from triangulated categories to dg-categories, higher representations (of which $\mathfrak{sl}_2$-categorifications are a special case) should give rise to a 4-dimensional TQFT. The decategorification of the TQFT should recover the 3-dimensional TQFT of Reshetikhin-Turaev.

Rouquier has also suggested that higher representation theory should allow moduli space constructions to be bypassed. This would give an algebraic approach to Donaldson-Thomas and Gromov-Witten invariants.

2.3. **Weak categorification.** Let $G = GL_n(\mathbb{C})$, and fix a Borel subgroup $B \subset G$. Fix a prime number l, and let $\overline{\mathbb{Q}_l}$ denote the algebraic closure of the field of l-adic numbers. Given a complex algebraic variety X (with its étale topology) equipped with an action of B, let $\mathcal{D}(X)$ denote the bounded derived category of B-smooth constructible $\overline{\mathbb{Q}_l}$-sheaves on X. Thus $\mathcal{D}(X)$ is the full subcategory of the bounded derived category of constructible $\overline{\mathbb{Q}_l}$-sheaves (see 2.2.18 in [3]) consisting of complexes whose cohomology sheaves are locally constant on B-orbits.

Let $X \xrightarrow{f} Y$ be a morphism of B-schemes of finite type over $\mathbb{C}$. The usual induced functors between the bounded derived categories of constructible $\overline{\mathbb{Q}_l}$-sheaves on X and Y restrict to functors $f_*, f_! \colon \mathcal{D}(X) \to \mathcal{D}(Y)$, and $f^*, f^! \colon \mathcal{D}(Y) \to \mathcal{D}(X)$.

Fix a positive integer n. Let $\mathrm{Gr}(i)$ and $\mathrm{Gr}(i, i+1)$ be as in §2.2. We have the following diagram, where p and q are the canonical projections.

$$Gr(i) \xleftarrow{\;p\;} Gr(i,i+1) \xrightarrow{\;q\;} Gr(i+1)$$

We shall use the fact that p and q are proper, so that $p_* = p_!$ and $q_* = q_!$, without further mention. Let $E_i = q_! p^* \colon \mathcal{D}(\mathrm{Gr}(i)) \to \mathcal{D}(\mathrm{Gr}(i+1))$ and $F_i = p_! q^* \colon \mathcal{D}(\mathrm{Gr}(i+1)) \to \mathcal{D}(\mathrm{Gr}(i))$. Let $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{D}(\mathrm{Gr}(i))$, and define $E = \bigoplus_{i=0}^n E_i \colon \mathcal{T} \to \mathcal{T}$, $F = \bigoplus_{i=0}^n F_i \colon \mathcal{T} \to \mathcal{T}$.

The proof of the following proposition is borrowed from 3.3.4 in [12]. We denote the constant $\overline{\mathbb{Q}_l}$-sheaf on X, regarded as an object of $\mathcal{D}(X)$ concentrated in degree zero, by $\overline{\mathbb{Q}_l}$ or $(\overline{\mathbb{Q}_l})_X$.

Proposition 2.13. *There is an isomorphism of functors*

$$F_i E_i \oplus \bigoplus_{n-i \le j < i} \mathrm{Id}[-2j](-j) \cong E_i F_i \oplus \bigoplus_{i \le j < n-i} \mathrm{Id}[-2j](-j),$$

where $[-]$ *denotes the shift functor of* $\mathcal{T}$*, and* $(-)$ *denotes the Tate twist.*

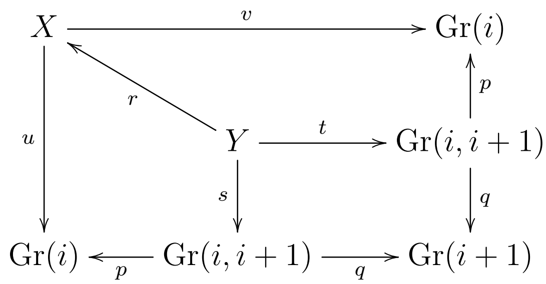
Proof. Let

$$X = \{V_1, V_2 \in \mathrm{Gr}(i) \mid \dim(V_1 + V_2) \le i+1\}$$

and

$$Y = \{V_1, V_2, V_3 \mid V_1, V_2 \in \mathrm{Gr}(i),\ V_3 \in \mathrm{Gr}(i+1),\ V_1 \subset V_3,\ V_2 \subset V_3\}.$$

We have the following commutative diagram.



Here p and q are as above, and

$$\begin{aligned} s(V_1, V_2, V_3) &= (V_1, V_3) \\ t(V_1, V_2, V_3) &= (V_2, V_3) \\ r(V_1, V_2, V_3) &= (V_1, V_2) \\ u(V_1, V_2) &= V_1 \\ v(V_1, V_2) &= V_2. \end{aligned}$$

Note that Y is the fibred product $\mathrm{Gr}(i, i+1) \times_{\mathrm{Gr(i+1)}} \mathrm{Gr}(i, i+1)$. Hence, by proper base change, (see XII, 5.1 in [1]) $q^* q_! \simeq t_! s^*$. Thus $F_i E_i = p_! q^* q_! p^* \simeq p_! t_! s^* p^* = v_! r_! r^* u^*$.

By sheafified Poincaré duality (see XVIII 3.2.5 in [1], and II 7.5 in [9]), $r_! r^*(-) \simeq r_! \overline{\mathbb{Q}_l} \otimes -$. Hence $F_i E_i \simeq v_!(r_! \overline{\mathbb{Q}_l} \otimes u^*(-))$.

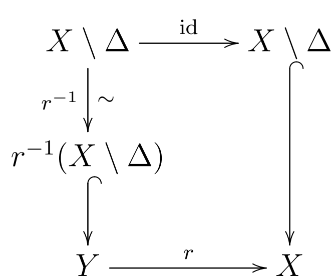
Let $i \colon \Delta \hookrightarrow X$ denote the inclusion of the diagonal Δ in X. Note that

$$\mathrm{Id}_{\mathrm{Gr}(i)} \simeq (vi)_!(ui)^* \simeq v_! i_! i^* u^*.$$

We deduce from sheafified Poincaré duality that

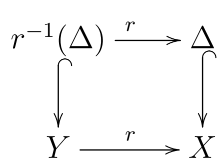
$$\mathrm{Id}_{\mathrm{Gr}(i)}(-) \simeq v_!(i_! \overline{\mathbb{Q}_l} \otimes u^*(-)).$$

Since r is an isomorphism above $X \setminus \Delta$, we have the following commutative diagram.



By proper base change, the restriction $(r_! \overline{\mathbb{Q}_l})_{|X \setminus \Delta}$ is isomorphic to $(\overline{\mathbb{Q}_l})_{X \setminus \Delta}$.

Over Δ, r is a $\mathbb{P}^{n-i-1}$-bundle. Applying proper base change to the diagram



we see that the restriction $(r_! \overline{\mathbb{Q}_l})_{|\Delta}$ is isomorphic to $r_!((\overline{\mathbb{Q}_l})_{r^{-1}(\Delta)})$. Hence (see Lemma 5.4.12 of [3]),

$$(r_! \overline{\mathbb{Q}_l})_{|\Delta} \simeq \bigoplus_{j=0}^{n-i-1} (\overline{\mathbb{Q}_l})_{\Delta}[-2j](-j).$$

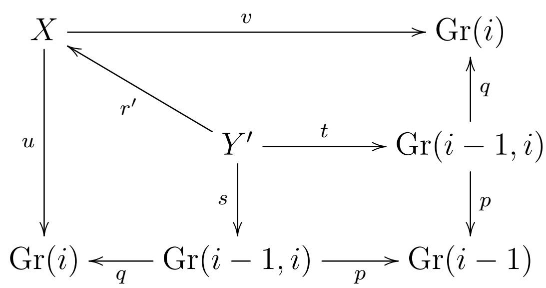
Let

$$Y' = \{V_1, V_2, V_3 \mid V_1 \in \mathrm{Gr}(i-1),\ V_2, V_3 \in \mathrm{Gr}(i),\ V_1 \subset V_2,\ V_1 \subset V_3\}.$$

Note also that

$$X = \{V_1, V_2 \in \mathrm{Gr}(i) \mid \dim(V \cap V') \geq i-1\}.$$

We have the following canonical commutative diagram.



Here u and v are as in the commutative diagram at the start of the proof, p and q are the canonical projections, and

$$s(V_1, V_2, V_3) = (V_1, V_2)$$
$$t(V_1, V_2, V_3) = (V_1, V_3)$$
$$r'(V_1, V_2, V_3) = (V_2, V_3).$$

As above, we find that

$$E_iF_i(-) \simeq v_!(r'_!\overline{\mathbb{Q}_l} \otimes u^*(-))$$
$$(r'_!\overline{\mathbb{Q}_l})_{|X\setminus\Delta} \simeq (\overline{\mathbb{Q}_l})_{X\setminus\Delta}$$
$$(r'_!\overline{\mathbb{Q}_l})_{|\Delta} \simeq \bigoplus_{j=0}^{i-1} (\overline{\mathbb{Q}_l})_{\Delta}[-2j](-j).$$

We have shown that

$$(r_!\overline{\mathbb{Q}_l})_{|\Delta} \oplus \bigoplus_{n-i\leq j<i} (\overline{\mathbb{Q}_l})_{|\Delta}[-2j](-j) \simeq (r'_!\overline{\mathbb{Q}_l})_{|\Delta} \oplus \bigoplus_{i\leq j<n-i} (\overline{\mathbb{Q}_l})_{|\Delta}[-2j](-j).$$

This isomorphism, the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber (6.2.5 in [3]), and the fact that

$$(r_!\overline{\mathbb{Q}_l})_{|X\setminus\Delta} \simeq (r'_!\overline{\mathbb{Q}_l})_{|X\setminus\Delta} \simeq (\overline{\mathbb{Q}_l})_{X\setminus\Delta},$$

imply that

$$r_!\overline{\mathbb{Q}_l} \oplus \bigoplus_{n-i\leq j<i} i_!\overline{\mathbb{Q}_l}[-2j](-j) \simeq r'_!\overline{\mathbb{Q}_l} \oplus \bigoplus_{i\leq j<n-i} i_!\overline{\mathbb{Q}_l}[-2j](-j).$$

The result follows by comparing this isomorphism with the realisations of E_iF_i, F_iE_i, and $\mathrm{Id}_{\mathrm{Gr}(i)}$ above. □

Corollary 2.14. *The functors E and F induce an action of $\mathfrak{sl}_2$ on $\mathbb{C} \otimes K_0(\mathcal{A})$.*

The functors E and F are adjoint to one another in the following sense.

Proposition 2.15. *Up to a shift and a twist, (E, F) and (F, E) are adjoint pairs of functors.*

Proof. Note that $p\colon \mathrm{Gr}(i, i+1) \to \mathrm{Gr}(i)$ is a $\mathbb{P}^{n-i-1}$-fibre bundle, and $q\colon \mathrm{Gr}(i, i+1) \to \mathrm{Gr}(i+1)$ is a $\mathbb{P}^i$-fibre bundle. Hence $p^! \simeq p^*[2(n-i-1)](n-i-1)$ and $q^! \simeq q^*[2i](i)$ (see, for example, II.8.1 in [9]). The result follows from the adjointness of (p^*, p_*), $(p_!, p^!)$, (q^*, q_*), and $(q_!, q^!)$. □

Using the weight space decomposition of $\mathbb{C} \otimes K_0(\mathcal{T})$, we now determine the action of $\mathfrak{sl}_2$ on $\mathbb{C} \otimes K_0(\mathcal{T})$ induced by E and F. A different approach was taken in [12].

Let $\mathcal{P}(\mathrm{Gr}(i))$ denote the category of B-smooth perverse sheaves on $\mathrm{Gr}(i)$, and let $\mathcal{D}^b(\mathcal{P}(\mathrm{Gr}(i)))$ denote the bounded derived category of $\mathcal{P}(\mathrm{Gr}(i))$ with its standard t-structure. There exists (see [2]) a canonical t-exact triangulated functor $\mathcal{D}^b(\mathcal{P}(\mathrm{Gr}(i))) \to \mathcal{D}(\mathrm{Gr}(i))$, which is the identity on $\mathcal{P}(\mathrm{Gr}(i))$. The existence follows from the existence of a filtered counterpart to $\mathcal{D}(\mathrm{Gr}(i))$, via the formalism of filtered triangulated categories.

Proposition 2.16. *The canonical functor $\mathcal{D}^b(\mathcal{P}(\mathrm{Gr(i)})) \to \mathcal{D}(\mathrm{Gr(i)})$ is an equivalence of categories.*

Proof. This is 1.3 in [2]. □

Proposition 2.17. *The category $\mathcal{D}^b(\mathcal{P}(\mathrm{Gr}(i)))$ is generated as a triangulated category by the projective objects in $\mathcal{P}(\mathrm{Gr}(i))$.*

Proof. Indeed, $\mathcal{P}(\mathrm{Gr}(i))$ has enough projectives (3.3.1 in [4]), and has finite global dimension (3.2.2 in [4]). □

Corollary 2.18. *As a $\mathbb{C}$-vector space, $\mathbb{C} \otimes K_0(\mathcal{T})$ has dimension 2^{n+1}.*

Proof. Let $\mathcal{T}' = \bigoplus_{i=0}^n \mathcal{D}^b(\mathcal{P}(\mathrm{Gr}(i)))$. By 2.16, $\mathbb{C} \otimes K_0(\mathcal{T}) \simeq \mathbb{C} \otimes K_0(\mathcal{T}')$. It follows from 2.17 that a basis of $\mathbb{C} \otimes K_0(\mathcal{T}')$ is given by the classes of indecomposable projective perverse sheaves in $\mathcal{T}'$.

Indecomposable projective perverse sheaves in $\mathcal{T}'$ are in bijection with simple perverse sheaves in $\mathcal{T}'$, which are in bijection with orbits of B on $\bigoplus_{i=0}^n \mathrm{Gr}(i)$. It is classical that there are $\binom{n}{i}$ orbits of B on $\mathrm{Gr}(i)$, and the result follows. □

Proposition 2.19. *Let L denote the standard representation of $\mathfrak{sl}_2$. As a representation of $\mathfrak{sl}_2$, $\mathbb{C} \otimes K_0(\mathcal{T}) \simeq L^{\otimes n}$.*

Proof. By 2.18, $\mathbb{C} \otimes K_0(\mathcal{T})$ has the correct dimension. Let $h = ef - fe$, where e and f are the endomorphisms of $\mathbb{C} \otimes K_0(\mathcal{T})$ induced by E and F. By 2.13, and the fact that $\mathbb{C} \otimes K_0(\mathrm{Gr}(i))$ has dimension $\binom{n}{i}$, the eigenvalues of h on $\mathbb{C} \otimes K_0(\mathcal{T})$ are correct. □

We have proved the following result.

Corollary 2.20. *The endofunctors E and F give $\mathcal{T}$ the structure of a weak $\mathfrak{sl}_2$-categorification of $L^{\otimes n}$.*

Given a parabolic subgroup P of $GL_n(\mathbb{C})$ for some n, let $\mathcal{D}_P(\mathrm{Gr}(i))$ denote the category of P-smooth constructible $\overline{\mathbb{Q}_l}$-sheaves. Let V be a tensor product of arbitrary simple representations of $\mathfrak{sl}_2$.

Proposition 2.21. *There is an integer $n \geq 0$, a parabolic subgroup $P \subset GL_n(\mathbb{C})$, and a pair of endofunctors (E_P, F_P) of $\mathcal{T}_P = \bigoplus_{i=0}^n \mathcal{D}_P(\mathrm{Gr}(i))$ giving $\mathcal{T}_P$ the structure of a weak $\mathfrak{sl}_2$-categorification of V.*

Proof. Exactly as in the case $V = L^{\otimes n}$, $P = B \subset GL_n(\mathbb{C})$ above. The results 2.18 and 2.19 must be modified, but we omit this. Given V, the interested reader will have no difficulty finding the corresponding parabolic and checking the details. □

2.4. **2-morphisms X and T.** We now explain how to extend the weak $\mathfrak{sl}_2$-categorification $\mathcal{T}$ of $L^{\otimes n}$ to a Chuang-Rouquier $\mathfrak{sl}_2$-categorification. In order to define $X \in \mathrm{End}^\bullet(E)$, we realise E_i as a Fourier-Mukai transform for every i.

The following diagram commutes, where p, q, p' and q' are the canonical projections, and j is the canonical map $(V_1 \subset V_2) \to (V_1, V_2)$.

$$\begin{array}{ccccc} & \swarrow^{p} & Gr(i, i+1) & \searrow^{q} & \\ & & \downarrow{\scriptstyle j} & & \\ Gr(i) & \xleftarrow[p']{} & Gr(i) \times Gr(i+1) & \xrightarrow[q']{} & Gr(i+1) \end{array}$$

Proposition 2.22. *There is an isomorphism of functors* $E_i \simeq q'_*(j_*\overline{\mathbb{Q}_l} \otimes p'^*(-))$.

Proof. Straightforward. $\square$

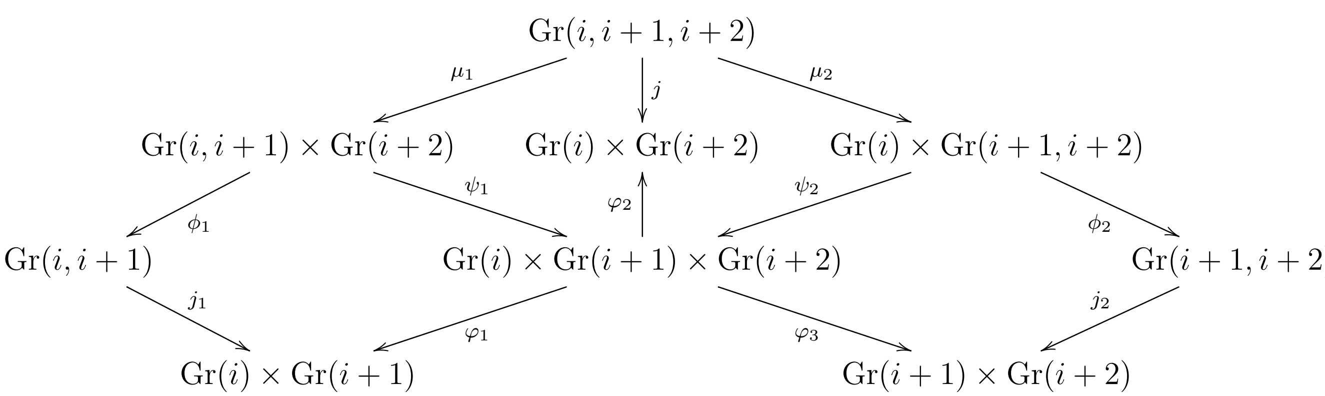
Let $\mathscr{L}$ denote the tautological line bundle on $\mathrm{Gr}(i, i+1)$. The fibre above the point $V_i \subset V_{i+1}$ is V_{i+1}/V_i. The first Chern class $c_1(\mathscr{L}) \in H^2(\mathrm{Gr}(i, i+1), \overline{\mathbb{Q}_l}(1))$ of $\mathscr{L}$ can be viewed as a morphism, belonging to $\mathrm{Hom}_{\mathcal{D}(\mathrm{Gr}(i,i+1))}(\overline{\mathbb{Q}_l}, \overline{\mathbb{Q}_l}[2](1))$. By functoriality, $c_1(\mathscr{L})$ determines a morphism in $\mathrm{Hom}_{\mathcal{D}(\mathrm{Gr}(i)\times\mathrm{Gr}(i+1))}(j_*\overline{\mathbb{Q}_l}, j_*\overline{\mathbb{Q}_l}[2](1))$ and hence, by the proposition, determines an endomorphism of E_i. Assembling these endomorphisms, we obtain an endomorphism X of E.

In order to define $T \in \mathrm{End}^\bullet(E^2)$, we realise $E_{i+1}E_i$ as a Fourier-Mukai transform for every i. We have the following commutative diagram, where p, q, p' and q' are the canonical projections, and j is the canonical map $(V_1 \subset V_2 \subset V_3) \to (V_1, V_3)$.

$$\begin{array}{ccccc} & \swarrow^{p} & Gr(i, i+1, i+2) & \searrow^{q} & \\ & & \downarrow{\scriptstyle j} & & \\ Gr(i) & \xleftarrow[p']{} & Gr(i) \times Gr(i+2) & \xrightarrow[q']{} & Gr(i+2) \end{array}$$

Proposition 2.23. *There is an isomorphism of functors* $E_{i+1}E_i \simeq q'_*(j_*\overline{\mathbb{Q}_l} \otimes p'^*(-))$.

Proof. We have the following commutative diagram, where the μ_i, ψ_i and j_i are the canonical maps, and j is the same as in the diagram above.



It follows from 2.22 (see 12.2.2 in [10]) that there is an isomorphism of functors $E_{i+1}E_i \simeq q'_*(K \otimes p'^*(-))$, where $K = \varphi_{2*}({\varphi_1}^* j_{1*}\overline{\mathbb{Q}_l} \otimes {\varphi_3}^* j_{2*}\overline{\mathbb{Q}_l})$. By proper base change with respect to the diamonds on the lower left and lower right of the diagram, $K \simeq \varphi_{2*}(\psi_{1*}\overline{\mathbb{Q}_l} \otimes \psi_{2*}\overline{\mathbb{Q}_l})$.

Thus $K \simeq \varphi_{2*}\psi_{1*}({\psi_1}^*\psi_{2*}\overline{\mathbb{Q}_l})$. By proper base change with respect to the upper diamond (ignoring the morphisms inside), $K \simeq \varphi_{2*}\psi_{1*}(\mu_{1*}\overline{\mathbb{Q}_l})$. The result follows from the commutativity of the upper diamond. $\square$

Remark 2.24. A different characterisation of $E_{i+1}E_i$ is given in 3.3.3 of [12]. We will see it later.

The map j factors through the canonical map $\pi\colon \mathrm{Gr}(i,i+1,i+2) \to \mathrm{Gr}(i,i+2)$ given by $(V_1 \subset V_2 \subset V_3) \to (V_1 \subset V_3)$. Let $R^k\pi_*$ denote the k^{th} higher direct image of π, and regard $R^2\pi_*(\overline{\mathbb{Q}_l})$ as a complex concentrated in degree zero. Since $R^k\pi_*$ vanishes for $k > 2$, there is a canonical morphism $\pi_*(\overline{\mathbb{Q}_l}[2]) \to R^2\pi_*(\overline{\mathbb{Q}_l})$ in $\mathcal{D}(\mathrm{Gr}(i,i+2))$.

Let $\eta\colon R^2\pi_*(\overline{\mathbb{Q}_l}(1)) \to \overline{\mathbb{Q}_l}$ denote the trace morphism, which is an isomorphism of $\overline{\mathbb{Q}_l}$-sheaves (see XVIII 2.9 in [1]). By composition, we obtain a canonical morphism $t'\colon \pi_!(\overline{\mathbb{Q}_l}[2](1)) \to R^2\pi_!(\overline{\mathbb{Q}_l}) \to \overline{\mathbb{Q}_l}$.

Moreover, t' extends to a natural transformation $\pi_!\pi^!(K) \to K$ for any $K \in \mathcal{D}(\mathrm{Gr}(i,i+2))$, via the following commutative diagram (cf. II.8 in [9]).

$$\begin{array}{ccc} \pi_!\pi^! K & \dashrightarrow & K \\ {\scriptstyle\sim}\downarrow & & \uparrow{\scriptstyle t'\otimes \mathrm{id}_K} \\ \pi_!(\overline{\mathbb{Q}_l}[2](1) \otimes \pi^*(K)) & \xrightarrow[\sim]{} & \pi_!(\overline{\mathbb{Q}_l}[2](1)) \otimes K \end{array}$$

Composing with the adjunction morphism $K \to \pi_*\pi^*(K)$, we get a natural transformation $T'\colon \pi_!\pi^!(K) \to \pi_*\pi^*(K)$. Let t denote the morphism obtained by taking $K = \overline{\mathbb{Q}_l}[-2](-1)$. By 2.23, t induces an endomorphism of $E_{i+1}E_i$ for every i. Assembling these endomorphisms, we obtain an endomorphism T of E^2.

Remark 2.25. The definitions of X and T were outlined to the author by Rouquier.

Let $\mathscr{E}$ denote the canonical rank two vector bundle on $\mathrm{Gr}(i, i+2)$ whose fibre above $(V_i \subset V_{i+2})$ is V_{i+2}/V_i. The $\mathbb{P}^1$-bundle π is the projectivisation of $\mathscr{E}$, and thus gives rise to a tautological line bundle $\mathcal{O}_\pi(-1)$ on $\mathrm{Gr}(i, i+1, i+2)$. Indeed, $\mathcal{O}_\pi(-1)$ is a subbundle of the pull-back bundle $\pi^*\mathscr{E}$, whose fibre above $(V_i \subset V_{i+1} \subset V_{i+2})$ is V_{i+1}/V_i. The quotient bundle $\pi^*\mathscr{E}/\mathcal{O}_\pi(-1)$ is the line bundle on $\mathrm{Gr}(i,i+1,i+2)$ corresponding to the twisting sheaf $\mathcal{O}_\pi(1)$. The fibre of $\pi^*\mathscr{E}/\mathcal{O}_\pi(-1)$ above $(V_i \subset V_{i+1} \subset V_{i+2})$ is V_{i+2}/V_{i+1}.

By 2.23, the first Chern classes $c_1(\mathcal{O}_\pi(-1))$ and $c_1(\pi^*\mathscr{E}/\mathcal{O}_\pi(-1))$ induce endomorphisms of E^2, which we denote by x and y respectively.

Proposition 2.26. *In* $\mathrm{End}^\bullet(E^2)$*, we have* $1_E X = x$ *and* $X 1_E = y$*.*

Proof. The second relation can be seen by inspecting the proof of 2.23. Indeed, $X1_E$ is determined by the morphism $\varphi_{2*}({\varphi_1}^* j_{1*}\overline{\mathbb{Q}_l} \otimes {\varphi_3}^* j_{2*} c_1(\mathscr{L}))$, where $\mathscr{L}$ is the tautological line bundle on $\mathrm{Gr}(i+1, i+2)$. By proper base change, this morphism identifies with $\varphi_{2*}(\psi_{1*}\overline{\mathbb{Q}_l} \otimes \psi_{2*} c_1(\phi_2^*\mathscr{L}))$, and hence with $\varphi_{2*}\psi_{1*}({\psi_1}^*\psi_{2*}c_1(\phi_2^*\mathscr{L}))$. By proper base change once more, this morphism identifies with $j_* c_1(\mu_2^*\phi_2^*\mathscr{L})$. The pull-back bundle $(\phi_2\mu_2)^*\mathscr{L}$ is exactly $\pi^*\mathscr{E}/\mathcal{O}_\pi(-1)$, as required.

The endofunctor $1_E X$ is determined by the morphism $\varphi_{2*}({\varphi_1}^* j_{1*} c_1(\mathscr{L}) \otimes {\varphi_3}^* j_{2*}\overline{\mathbb{Q}_l})$, where $\mathscr{L}$ is the tautological line bundle on $\mathrm{Gr}(i, i+1)$. As above, this morphism identifies with $\varphi_{2*}(\psi_{1*}c_1(\phi_1^*\mathscr{L}) \otimes \psi_{2*}\overline{\mathbb{Q}_l})$, and hence with $\varphi_{2*}\psi_{2*}({\psi_2}^*\psi_{1*}c_1(\phi_1^*\mathscr{L}))$. By proper base change, this morphism identifies with $j_*c_1(\mu_1^*\phi_1^*\mathscr{L})$. The pull-back bundle $(\phi_1\mu_1)^*\mathscr{L}$ is exactly $\mathcal{O}_\pi(-1)$, as required. □

We now show that T, x and y satisfy the defining relations 2.1 of the affine nilHecke algebra $H_2(0)$. The proof was outlined to the author by Rouquier.

Proposition 2.27. *In* $\mathrm{End}^\bullet(E^2)$*, we have*

$$T^2 = 0, \;\; yT - Tx = 1, \;\; T(x+y) = (x+y)T.$$

Proof. By the naturality of T', the composition

$$\pi_*\pi^*\overline{\mathbb{Q}_l} \xrightarrow{\sim} \pi_*\overline{\mathbb{Q}_l} \xrightarrow{t} \pi_*\overline{\mathbb{Q}_l}[-2](-1) \xrightarrow{\sim} \pi_*\pi^*\overline{\mathbb{Q}_l}[-2](-1)$$

fits into the following commutative diagram, for any $\alpha \in \mathrm{End}^2_{\mathcal{D}(\mathrm{Gr}(i,i+2))}(\overline{\mathbb{Q}_l})$.

$$\begin{array}{ccccccc}
\pi_*\pi^*\overline{\mathbb{Q}_l} & \xrightarrow{\sim} & \pi_*\overline{\mathbb{Q}_l} & \xrightarrow{t} & \pi_*\overline{\mathbb{Q}_l}[-2](-1) & \xrightarrow{\sim} & \pi_*\pi^*\overline{\mathbb{Q}_l}[-2](-1) \\
\downarrow{\scriptstyle \pi_*\pi^*\alpha} & & & & & & \downarrow{\scriptstyle \pi_*\pi^*\alpha[-2]} \\
\pi_*\pi^*\overline{\mathbb{Q}_l}[2] & \xrightarrow{\sim} & \pi_*\overline{\mathbb{Q}_l}[2] & \xrightarrow{t[2]} & \pi_*\overline{\mathbb{Q}_l}(-1) & \xrightarrow{\sim} & \pi_*\pi^*\overline{\mathbb{Q}_l}(-1)
\end{array}$$

We deduce that T commutes with $x+y$, since

$$c_1(\mathcal{O}_\pi(-1)) + c_1(\pi^*\mathscr{E}/\mathcal{O}_\pi(-1)) = \pi^*c_1(\mathscr{E}).$$

Furthermore, $t[-2](-1)\circ t$ factors through a morphism $\overline{\mathbb{Q}_l}[-2](-1) \to \overline{\mathbb{Q}_l}[-4](-2)$. This is the zero morphism, since shifting $\overline{\mathbb{Q}_l}$ by the dimension of $\mathrm{Gr}(i,i+2)$ is a simple perverse sheaf on $\mathrm{Gr}(i,i+2)$. Thus $T^2 = 0$.

Let $\alpha = \pi_*c_1(\mathcal{O}_\pi(-1))$ and $\beta = \pi_*c_1(\mathcal{O}_\pi(1))$. We claim that the following composition is the identity in $\mathrm{End}^\bullet_{\mathcal{D}(\mathrm{Gr}(i,i+2))}(\overline{\mathbb{Q}_l})$.

$$\overline{\mathbb{Q}_l} \xrightarrow{\mathrm{adj}} \pi_*\overline{\mathbb{Q}_l} \xrightarrow{\beta} \pi_*\overline{\mathbb{Q}_l}[2](1) \xrightarrow{t} \overline{\mathbb{Q}_l}$$

Given a point $z \in \mathrm{Gr}(i,i+2)$, let $\pi'\colon \pi^{-1}(z) \simeq \mathbb{P}^1 \to \{z\}$ denote the fibre map. Taking the fibre of the above composition at z, and applying proper base change, we obtain a morphism of the following form.

$$\overline{\mathbb{Q}_l} \xrightarrow{\mathrm{adj}} \pi'_*\overline{\mathbb{Q}_l} \xrightarrow{c_1(\mathcal{O}_{\mathbb{P}^1}(1))} \pi'_*\overline{\mathbb{Q}_l}[2](1) \longrightarrow \overline{\mathbb{Q}_l}$$

Via the natural isomorphism between π' and the global sections functor $\Gamma(\mathbb{P}^1,-)$, the above morphism identifies with the following morphism, where τ denotes the trace morphism on cohomology.

$$\overline{\mathbb{Q}_l} \xrightarrow{c_1(\mathcal{O}_{\mathbb{P}^1}(1))} H^2(\mathbb{P}^1,\overline{\mathbb{Q}_l}) \xrightarrow{\tau} \overline{\mathbb{Q}_l}$$

This is the identity, since the trace of the class of $c_1(\mathcal{O}_{\mathbb{P}^1})$ in $H^2(\mathbb{P}^1,\overline{\mathbb{Q}_l})$ is 1 (see Cycle, 2.1.5 in [5]).

The following composition is also the identity morphism, since $-c_1(\mathcal{O}_\pi(-1)) = c_1(\mathcal{O}_\pi(1))$.

$$\overline{\mathbb{Q}_l} \xrightarrow{\mathrm{adj}} \pi_*\overline{\mathbb{Q}_l} \xrightarrow{-\alpha} \pi_*\overline{\mathbb{Q}_l}[2](1) \xrightarrow{t} \overline{\mathbb{Q}_l}$$

The composition $\overline{\mathbb{Q}_l} \xrightarrow{\mathrm{adj}} \pi_*\overline{\mathbb{Q}_l} \xrightarrow{t} \overline{\mathbb{Q}_l}[-2](-1)$ is zero, since it factors through a morphism $\overline{\mathbb{Q}_l} \to \overline{\mathbb{Q}_l}[-2](-1)$.

It follows that the following composition is equal to the adjunction morphism $\overline{\mathbb{Q}_l} \to \pi_*\overline{\mathbb{Q}_l}$.

$$\overline{\mathbb{Q}_l} \xrightarrow{\mathrm{adj}} \pi_*\overline{\mathbb{Q}_l} \xrightarrow{\beta t - t[2](1)\alpha} \pi_*\overline{\mathbb{Q}_l}$$

The following diagram commutes, since $c_2(\pi^*\mathscr{E}) = c_1(\mathcal{O}_\pi(-1))c_1(\mathcal{O}_\pi(1))$.

$$\begin{array}{ccccccc}
 & \overline{\mathbb{Q}_l}[4](2) & & & & & \\
{\scriptstyle c_2(\mathscr{E})}\nearrow & & \searrow{\scriptstyle \mathrm{adj}} & & & & \\
\overline{\mathbb{Q}_l} & \xrightarrow{\mathrm{adj}} & \pi_*\overline{\mathbb{Q}_l} & \xrightarrow{\alpha\beta} & \pi_*\overline{\mathbb{Q}_l}[4](2) & \xrightarrow{t[2](1)} & \overline{\mathbb{Q}_l}[2](1)
\end{array}$$

In particular, the row in the above diagram factors through a morphism $\overline{\mathbb{Q}_l}[4](2) \to \overline{\mathbb{Q}_l}[2](1)$. It is therefore zero, and the composition

$$\overline{\mathbb{Q}_l} \xrightarrow{\text{adj}} \pi_*\overline{\mathbb{Q}_l} \xrightarrow{\beta} \pi_*\overline{\mathbb{Q}_l}[2](1) \xrightarrow{\beta t - t[2](1)\alpha} \pi_*\overline{\mathbb{Q}_l}[2](1)$$

is equal to the composition

$$\overline{\mathbb{Q}_l} \xrightarrow{\text{adj}} \pi_*\overline{\mathbb{Q}_l} \xrightarrow{\beta} \pi_*\overline{\mathbb{Q}_l}[2](1) \ .$$

This proves that $yT - Tx = 1$. □

Corollary 2.28. *In* $\mathrm{End}^\bullet(E^2)$*, we have* $Ty - xT = 1$ *.*

Proposition 2.29. *In* $\mathrm{End}^\bullet(E^3)$*, the following diagram commutes.*

$$\begin{array}{ccccc} & & EEE & & \\ & \swarrow^{1_E T} & & \searrow^{T 1_E} & \\ EEE & & & & EEE \\ \downarrow T1_E & & & & \downarrow 1_E T \\ EEE & & & & EEE \\ & \searrow_{1_E T} & & \swarrow_{T1_E} & \\ & & EEE & & \end{array}$$

Proof. Follows from the compatibility of the trace morphism with base change and composition. We omit the details. □

Combining 2.20, 2.27, 2.28, and 2.29, we have the following result.

Proposition 2.30. *The endofunctors E and F, and the endomorphisms X and T, give $\mathcal{T}$ the structure of an $\mathfrak{sl}_2$-categorification of $L^{\otimes n}$.*

Let V be a tensor product of arbitrary simple representations of $\mathfrak{sl}_2$.

Proposition 2.31. *There is an integer $n \geq 0$, a parabolic subgroup $P \subset GL_n(\mathbb{C})$, a pair of endofunctors (E_P, F_P) of $\mathcal{T}_P = \bigoplus_{i=0}^n \mathcal{D}_P(\mathrm{Gr}(i))$, and a pair of endomorphisms $X_P \in \mathrm{End}^\bullet(E_P)$, $T_P \in \mathrm{End}^\bullet(E_P^2)$ giving $\mathcal{T}_P$ the structure of an $\mathfrak{sl}_2$-categorification of V.*

Proof. The integer $n \geq 0$, the parabolic subgroup $P \subset GL_n(\mathbb{C})$, and the endofunctors (E_P, F_P) are given by 2.21. The 2-morphisms X_P and T_P are obtained exactly as in the case $V = L^{\otimes n}$ and $P = B \subset GL_n(\mathbb{C})$ above. □

2.5. **Action of $\mathbb{Z}[q, q^{-1}]$.** We explain how to pass to an $\mathfrak{sl}_2$-categorification of the quantum group $U_q(\mathfrak{sl}_2)$, where $q \in \mathbb{C}$ is neither zero nor a root of unity. A slightly different approach via shifting E and F is taken in [7] and [12].

Fix a parabolic subgroup $P \subset GL_n(\mathbb{C})$, and let $\mathcal{T}_P(q) = \bigoplus_{i=0}^n \mathcal{D}_P(\mathrm{Gr}(i))$. Choosing an isomorphism $\tau \colon \overline{\mathbb{Q}_l} \to \mathbb{C}$, fix an element $q^{1/2} \in \overline{\mathbb{Q}_l}$. This allows us to define a half-integral Tate twist $(-)(\frac{n}{2})$ on $\mathcal{D}_P(\mathrm{Gr}(i))$. For even n, this is the usual Tate twist.

We have the following diagram, where p and r are the canonical projections.

$$\begin{array}{ccc} & Gr(i, i+1) & \\ {}^{p}\swarrow & & \searrow^{r} \\ Gr(i) & & Gr(i+1) \end{array}$$

Let $E_i = r_! p^*(\frac{n-i-1}{2})\colon \mathcal{D}_P(\mathrm{Gr}(i)) \to \mathcal{D}_P(\mathrm{Gr}(i+1))$, $F_i = p_! r^*(\frac{i}{2})\colon \mathcal{D}_P(\mathrm{Gr}(i+1)) \to \mathcal{D}_P(\mathrm{Gr}(i))$, and $G_i = (\frac{2i-n}{2})\colon \mathcal{D}_P(\mathrm{Gr}(i)) \to \mathcal{D}_P(\mathrm{Gr}(i))$. Define $E, F, G\colon \mathcal{T}_P(q) \to \mathcal{T}_P(q)$ by $E = \bigoplus_{i=0}^n E_i$, $F = \bigoplus_{i=0}^n F_i$, and $G = \bigoplus_{i=0}^n G_i$.

There is an action of $\mathbb{Z}[q, q^{-1}]$ on $K_0(\mathcal{T}_P(q))$ given by $q \cdot [K] = \left[K(-\frac{1}{2})\right]$. We prove that E, F, and G induce an action of $U_q(\mathfrak{sl}_2)$ on $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(\mathcal{T}_P(q))$.

Proposition 2.32. *We have* $[GG^{-1}] = [G^{-1}G] = 1,\ [GEG^{-1}] = q^{-2}[E],\ [GFG^{-1}] = q^2[F]$, *and* $[EF] - [FE] = \frac{[G]-[G^{-1}]}{q-q^{-1}}$.

Proof. The last relation follows from 2.13, which implies the following isomorphism of functors.

$$F_i E_i \oplus \bigoplus_{n-i \le j < i} \mathrm{Id}[-2j](-j)(\tfrac{n-1}{2}) \simeq E_i F_i \oplus \bigoplus_{i \le j < n-i} \mathrm{Id}[-2j](-j)(\tfrac{n-1}{2}).$$

The other relations are easily checked. ☐

Let V denote a tensor product of simple representations of $U_q(\mathfrak{sl}_2)$.

Proposition 2.33. *There is an integer $n \ge 0$ and a parabolic subgroup $P \subset GL_n(\mathbb{C})$, together with endofunctors E, F and G of $\mathcal{T}_P(q)$, such that the induced action of $U_q(\mathfrak{sl}_2)$ on $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(\mathcal{T}_P(q))$ is isomorphic to the action of $U_q(\mathfrak{sl}_2)$ on V.*

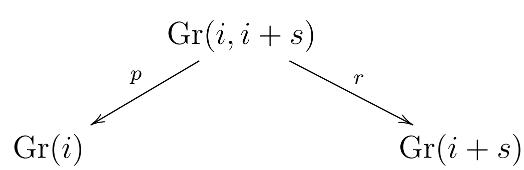
Proof. After 2.32, this is a question of combinatorics (cf. 2.16 - 2.21). The Clebsch-Gordon decomposition of V into a direct sum of simple representations (see 1.4.4 in [8]) gives a means to calculate the dimensions of the weight spaces of V. ☐

Remark 2.34. The proposition can be proved geometrically. This is the approach taken in [12].

Exactly as in §2.4, there are 2-morphisms $X \in \mathrm{End}(E)$ and $T \in \mathrm{End}(E^2)$ satisfying the relations 2.27 and 2.29, giving rise to a morphism $H_n(0) \to \mathrm{End}(E^n)$ for every n. Furthermore, (G, G^{-1}), (G^{-1}, G), (E, GF), and $(F, G^{-1}E)$ are adjoint pairs of functors, up to a shift (see 2.15).

Remark 2.35. We have, in essence, constructed a $U_q(\mathfrak{sl}_2)$-categorification of V, in the sense of the higher representation theory programme of Chuang and Rouquier. However, we refrain from using this terminology. Properly justifying it would take us too far afield.

Fix a parabolic subgroup $P \subset GL_n(\mathbb{C})$. We show that the divided powers (see 1.2 in [8]) of $[E]$ and $[F]$ are induced by endofunctors of $\mathcal{T}_P(q)$. We have the following diagram, where p and r are the canonical projections.



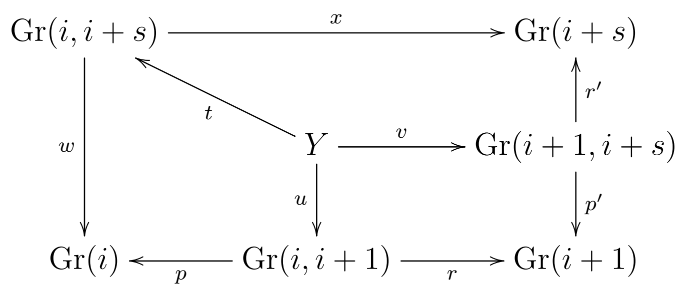
Let $E_i^{(s)} = r_! p^*(\frac{s(n-i-s)}{2})\colon \mathrm{Gr}(i) \to \mathrm{Gr}(i+s)$, and $F_i^{(s)} = p_! r^*(\frac{is}{2})\colon \mathrm{Gr}(i+s) \to \mathrm{Gr}(i)$. Define $E^{(s)}, F^{(s)}\colon \mathcal{T}_P(q) \to \mathcal{T}_P(q)$ by $E^{(s)} = \bigoplus_{i=0}^n E_i^{(s)}$, $F^{(s)} = \bigoplus_{i=0}^n F_i^{(s)}$. The proof of the following proposition is borrowed from 3.3.3 in [12].

Proposition 2.36. *We have isomorphisms of functors* $E^{(s-1)}E \simeq \bigoplus_{j=0}^{s-1} E^{(s)}(\frac{s-1-2j}{2})$ *and* $F^{(s-1)}F \simeq \bigoplus_{j=0}^{s-1} F^{(s)}(\frac{s-1-2j}{2})$.

Proof. Let

$$Y = \{V_1 \subset V_2 \subset V_3 \mid V_1 \in \mathrm{Gr}(i),\ V_2 \in \mathrm{Gr}(i+1),\ V_3 \in \mathrm{Gr}(i+s),\ V_1 \subset V_3,\ V_2 \subset V_3\}.$$

We have the following commutative diagram. The maps are the canonical projections.



We have that $E_{i+1}^{(s-1)}E_i = r'_! p'^* r_! p^*(\frac{s(n+1-i-s)-1}{2})$. By proper base change, $r'_! p'^* r_! p^* \simeq r'_! v_! u^* p^*$, and hence $E_{i+1}^{(s-1)}E_i \simeq x_! t_! t^* w^*(\frac{s(n+1-i-s)-1}{2})$. Since t is a $\mathbb{P}^{s-1}$-bundle, $t_!\overline{\mathbb{Q}_l} \simeq \bigoplus_{j=0}^{s-1} \overline{\mathbb{Q}_l}[-2j](-j)$. Thus $t_! t^* \simeq t_!\overline{\mathbb{Q}_l} \otimes - \simeq \bigoplus_{j=0}^{s-1}[-2j](-j)$, and

$$E_{i+1}^{(s-1)}E_i \simeq \bigoplus_{j=0}^{s-1} x_! w^*[-2j](-j)(\tfrac{s(n+1-i-s)-1}{2}) \simeq \bigoplus_{j=0}^{s-1} E_i^{(s)}(\tfrac{s-1-2j}{2}).$$

The second isomorphism is proved similarly. □

Corollary 2.37. *We have* $[E^{(s)}] = \frac{[E^s]}{[s]_q!}$ *and* $[F^{(s)}] = \frac{[F^s]}{[s]_q!}$.

3. Koszul duality

3.1. **Koszul duality.** Given a tensor product V of simple representations of $U_q(\mathfrak{sl}_2)$, let $\mathcal{T} = \mathcal{T}_P(q)$ denote the corresponding triangulated categorification of 2.33. In the spirit of Soergel, we outline how to pass to an abelian categorification via Koszul duality, as in 3.6 of [12].

Let L_I denote the direct sum of the simple perverse sheaves in $\mathcal{T}$. Let $\mathcal{L}_I$ denote the full subcategory of $\mathcal{T}$ consisting of the semisimple perverse sheaves in $\mathcal{T}$ and their shifts and Tate twists. Let $A = \mathrm{End}^\bullet_{\mathcal{T}}(L_I)$, regarded as an algebra via composition. Then $\mathrm{Ext}^\bullet_{\mathcal{T}}(L_I, -)$ defines a fully faithful functor from $\mathcal{L}_I$ to the category $\mathcal{A}$ of finitely generated graded left modules over A.

By the decomposition theorem of [3], the endofunctors E, F of $\mathcal{T}$ preserve $\mathcal{L}_I$, as does G. Let $x, z \in A$, and $y \in \mathrm{Ext}^\bullet_{\mathcal{T}}(L_I, E(L_I))$. The action $x \cdot y \cdot z = xyE(z)$ gives $\mathrm{Ext}^\bullet_{\mathcal{T}}(L_I, E(L_I))$ the structure of a graded A-bimodule, and $\mathrm{Ext}^\bullet_{\mathcal{T}}(L_I, E(L_I)) \otimes_A -$ defines an exact endofunctor E_a of $\mathcal{A}$. In the same way, F and G give rise to exact endofunctors F_a and G_a of $\mathcal{A}$.

After 2.32, E_a, F_a, and G_a induce an action of $U_q(\mathfrak{sl}_2)$ on $K_0(\mathcal{A})$. The endomorphisms $X \in \mathrm{End}^\bullet(E)$ and $T \in \mathrm{End}^\bullet(E^2)$ induce endomorphisms $X_a \in \mathrm{End}(E_a)$ and $T_a \in \mathrm{End}((E_a)^2)$ satisfying the relations 2.27 and 2.29. This gives $\mathcal{A}$ the structure of an abelian categorification. As in 2.5, $\mathcal{A}$ essentially has the structure of a $U_q(\mathfrak{sl}_2)$-categorification, in the sense of the higher representation theory programme of Chuang and Rouquier.

The indecomposable projective objects of $\mathcal{A}$ are the modules $\mathrm{Ext}^\bullet_{\mathcal{T}}(L_I, K)$, where K is a simple perverse sheaf in $\mathcal{T}$. Thus there is an action of $\mathbb{Z}[q, q^{-1}]$ on $K_0(\mathcal{A})$, defined by $q \cdot [\mathrm{Ext}^\bullet_{\mathcal{T}}(L_I, K)] = [\mathrm{Ext}^\bullet_{\mathcal{T}}(L_I, K(-\frac{1}{2})]$. Let $K_0(\mathcal{L}_I)$ denote the Grothendieck group of $\mathcal{L}_I$ as an additive category. There is also an action of $\mathbb{Z}[q, q^{-1}]$ on $K_0(\mathcal{L}_I)$, defined by $q \cdot [K] = [K(-\frac{1}{2})]$.

The functor $\mathrm{Ext}^\bullet_{\mathcal{T}}(L_I, -)$ induces an isomorphism $\mathbb{Q}_q \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(\mathcal{L}_I) \simeq \mathbb{Q}_q \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(\mathcal{A})$ of $U_q(\mathfrak{sl}_2)$-modules. The $U_q(\mathfrak{sl}_2)$-modules $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(\mathcal{T})$ and $\mathbb{Q}_q \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(\mathcal{L}_I)$ are also isomorphic, so the decategorification of $\mathcal{A}$ is V.

Remark 3.1. There is a canonical basis of V consisting of isomorphism classes of the indecomposable projectives in $\mathcal{A}$. In 3.5.9 of [11], Zheng identifies this basis with Lusztig's canonical basis.

3.2. **An abelian $\mathfrak{sl}_2$-categorification of simple representations of $\mathfrak{sl}_2$.** Let V denote the simple representation of $\mathfrak{sl}_2$ of dimension $n+1$. In this case, Proposition 2.31 takes the following form.

Proposition 3.2. *There are endofunctors E and F and endomorphisms $X \in \mathrm{End}^\bullet(E)$, $T \in \mathrm{End}^\bullet(E^2)$ giving $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{D}_{GL_n(\mathbb{C})}(\mathrm{Gr}(i))$ the structure of an $\mathfrak{sl}_2$-categorification of V.*

Via Koszul duality (as in 3.1, ignoring Tate twists), $\mathcal{T}$ gives rise to an abelian $\mathfrak{sl}_2$-categorification $\mathcal{A}$ of V. Indeed, $\mathcal{A}$ is the category of finitely generated graded left modules over the algebra $A = \bigoplus_{i=0}^n \mathrm{End}^\bullet_{\mathcal{D}(\mathrm{Gr}(i))}(\overline{\mathbb{Q}_l}) \simeq \bigoplus_{i=0}^n H^\bullet(\mathrm{Gr}(i))$.

Let E_a and F_a denote the structural endofunctors of $\mathcal{A}$. Regarding $H^\bullet(\mathrm{Gr}(i,i+1))$ as an $H^\bullet(\mathrm{Gr}(i))$-$H^\bullet(\mathrm{Gr}(i+1))$-bimodule, we have an isomorphism of functors

$$E_a \simeq \bigoplus_{i=0}^n H^\bullet(\mathrm{Gr}(i,i+1)) \otimes_A -.$$

Regarding $H^\bullet(\mathrm{Gr}(i,i+1))$ as an $H^\bullet(\mathrm{Gr}(i+1))$-$H^\bullet(\mathrm{Gr}(i))$-bimodule, we have an isomorphism of functors $F_a \simeq \bigoplus_{i=0}^n H^\bullet(\mathrm{Gr}(i,i+1)) \otimes_A -$.

This agrees with the weak $\mathfrak{sl}_2$-categorification of V given in 2.2. The structural endomorphisms $X_a \in \mathrm{End}^\bullet(E_a)$ and $T_a \in \mathrm{End}^\bullet((E_a)^2)$ enhance 2.2 to an $\mathfrak{sl}_2$-categorification of V.

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