Combinatorial homotopy theory

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I. Introduction

Lecture notes for an advanced course currently taking place at NTNU, Trondheim.

I.1. Acknowledgements

I thank Finn Faye Knudsen for his questions concerning cubical sets with connections, which led to the inclusion of §III.3.5 into these notes.

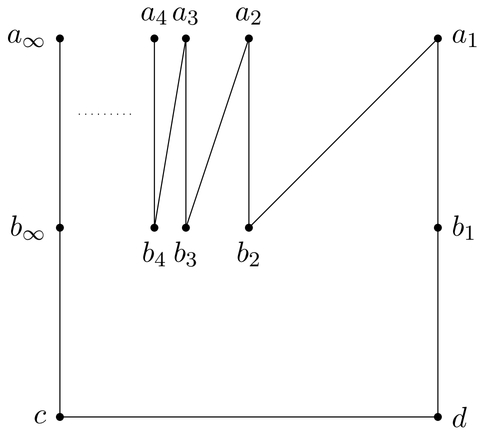
I thank Marius Thaule for his proof reading of the notes from the first lecture — the notes have not yet been revised accordingly!

II. A zoo of shapes

II.1. Introduction

II.1.1. Warsaw circle

Topological spaces can be very wild — at least from the point of view of ordinary homotopy theory! We might for instance encounter the Warsaw circle.



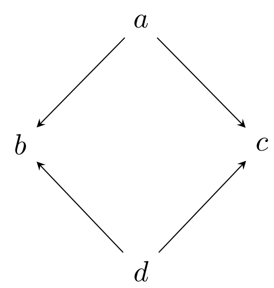
Regarded as a subspace of the plane, in this picture we can take a_∞ to be $(0,1)$, b_∞ to be $(0, \frac{1}{2})$, c to be $(0,0)$, d to be $(1,0)$, a_n to be $(\frac{1}{n}, 1)$, and b_n to be $(\frac{1}{n}, \frac{1}{2})$. The canonical map from the Warsaw circle to the point gives rise to an isomorphism on all homotopy groups, but the Warsaw circle is not contractible[1].

II.1.2. A moral

We may draw a moral from this example — maps from the topological interval into the Warsaw circle are not appropriate for detecting the global nature of the Warsaw circle. Intuitively, a path in the Warsaw circle cannot 'jump' from the vertical line from a_n to b_n to the vertical line from a_∞ to b_∞, for any n, and thus the fundamental group cannot detect any loops. With a more appropriate notion of fundamental group — for example, one cooked out of open coverings — a loop will be detected[2].

II.1.3. A finite circle

Let us explore another example. Let S^1_{fin} denote the poset defined by the picture below[3]



From any poset we can cook up a space[4] with the same underlying set, and with U declared to be an open set if for any $x \in U$ and $x \leq y$, we have $y \in U$. Thus the topology on S^1_{fin} consists of exactly the following open sets: $\emptyset$, $\{b\}$, $\{c\}$, $\{b, c\}$, $\{a, b, c\}$, $\{b, c, d\}$, and $\{a, b, c, d\}$. There is a map

$$S^1 \longrightarrow S^1_{fin}$$

which gives rise to an isomorphism on all homotopy groups, where S^1 is the usual topological circle[5]. On the other hand, every map

$$S^1_{fin} \longrightarrow S^1$$

is constant, so that S^1_{fin} cannot be homotopy equivalent to S^1.

II.1.4. Our moral revisited

Though this example is of a very different nature to that of the Warsaw circle, we may draw from it the same moral as in II.1.1 — maps from the topological interval into S^1_{fin} are not appropriate for detecting the global nature of S^1_{fin}. Intuitively, to build a good homotopy theory of finite spaces, our interval should be a finite space! The Sierpiński interval, namely the poset defined by the following diagram, is a more suitable gadget to look at[6].

$$0 \longrightarrow 1$$

II.1.5. Spaces for which Whitehead's theorem holds?

We thus see that if we hope for our classical theory of homotopy groups of a space X to behave well — in particular, if we wish for Whitehead's theorem to hold, namely that homotopy equivalence can be detected by homotopy groups — we must work with a space X for which maps from I^n to X are an appropriate setting to be working in. How might we arrive at the notion of such a space?

II.1.6. Lego

As a first guess, we might begin with the topological n-cubes themselves, and try constructing spaces out of these in a nice way. Perhaps the single most important foundational insight of algebraic topology is that most familiar spaces can be constructed by glueing together copies of the topological n-cubes (or other choices of building blocks) along their faces. For example, the torus can be obtained by glueing together a copy of I^2 to two copies of I, as indicated by the colours or arrowheads in the figure below.

I $\qquad$ I $\qquad$ I^2

It turns out that Whitehead's theorem indeed holds for spaces built up in this way. Moreover these spaces behave well in many respects, and are very general — every space can be shown to be weakly homotopy equivalent to one obtained by glueing together copies of the topological n-cubes along their faces. The key step in the proof is an approximation theorem — variously described as cellular, cubical, simplicial, and so on, depending on the context — which goes back at least to the 1920s.

II.1.7. Towards a conceptual lego!

We have now arrived at our motivating idea: good spaces for homotopy theory can be constructed by glueing together a small collection of building blocks. In this lecture we will explain that, thinking about this idea from a conceptual point of view, we can move from topology to a purely abstract setting.

II.2. Presheaves

II.2.1. Introduction

Let us discuss a little category theory. We will show that the category of presheaves on $\mathcal{A}$ has a universal property — it can be viewed as asserting that $\mathsf{Set}^{\mathcal{A}^{op}}$ is freely built from $\mathcal{A}$ by formally adding in colimits. More precisely, we will establish this universal property by proving that any presheaf is a colimit of representable presheaves. One can go far in category theory with only a few tools — vital, but few. This is one of them!

A colimit is a categorical notion of glueing. If we view the objects of $\mathcal{A}$ as building blocks, and the arrows of $\mathcal{A}$ as a prescription for how we may glue these building blocks to one another, we can — by virtue of the universal property of $\mathsf{Set}^{\mathcal{A}^{op}}$ — exactly view a presheaf on $\mathcal{A}$ as a gadget glued out of these building blocks following this prescription.

It is this intuition that leads to the idea that presheaves may model homotopy types. If we can capture the building blocks of topology — simplexes, or cubes, or more exotic gadgets — in an abstract way as the objects of a category, and capture the recipe for glueing these shapes to one another as the arrows of this category, we might hope that the corresponding presheaves are a good abstract setting for homotopy theory. In the rest of this course we will work towards showing that this hope can be fulfilled!

II.2.2. Recollections

II.2.2.1. Let Set denote the category of sets.

II.2.2.2. Given a category $\mathcal{A}$, we denote its opposite category by $\mathcal{A}^{op}$.

II.2.2.3. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be categories. We denote by $\mathcal{A}_1^{\mathcal{A}_0}$ the category of functors from $\mathcal{A}_0$ to $\mathcal{A}_1$.

II.2.2.4. For the remainder of §II.2, let $\mathcal{A}$ be a category.

II.2.2.5. A *presheaf* on $\mathcal{A}$ is a functor

$$\mathcal{A}^{op} \longrightarrow \mathsf{Set}.$$

II.2.2.6. The *category of presheaves on* $\mathcal{A}$ is the functor category $\mathsf{Set}^{\mathcal{A}^{op}}$. Recall that an arrow of $\mathsf{Set}^{\mathcal{A}^{op}}$ is a natural transformation.

II.2.2.7. There is a functor

$$\mathcal{A} \hookrightarrow \mathsf{Set}^{\mathcal{A}^{op}}$$

defined by $a \mapsto \mathrm{Hom}_{\mathcal{A}}(-, a)$ and $f \mapsto \mathrm{Hom}_{\mathcal{A}}(-, f)$ for an object a of $\mathcal{A}$ and an arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$. By the Yoneda lemma, this functor is fully faithful. We say that a presheaf on $\mathcal{A}$ is *representable* if it belongs to the essential image of this functor, namely if it is isomorphic in $\mathsf{Set}^{\mathcal{A}^{op}}$ to a presheaf $\mathrm{Hom}_{\mathcal{A}}(-, a)$ for some object a of $\mathcal{A}$.

II.2.2.8. As a consequence of the fact that Set has all limits and colimits, so does $\mathsf{Set}^{\mathcal{A}^{op}}$. These limits and colimits are computed *levelwise*[7]. Recall that this has the following meaning — we will treat the case of colimits, the case of limits is entirely analogous. Suppose that we have a category $\mathcal{D}$ and a functor

$$\mathcal{D} \xrightarrow{F} \mathsf{Set}^{\mathcal{A}^{op}}.$$

Given an object d of $\mathcal{D}$, let F_d denote the presheaf $F(d)$. Given an arrow

$$d_0 \xrightarrow{f} d_1$$

of $\mathcal{D}$, let

$$F_{d_0} \xrightarrow{f_f} F_{d_1}$$

denote the natural transformation

$$F(d_0) \xrightarrow{F(f)} F(d_1).$$

For any object a of $\mathcal{A}$, let

$$\mathcal{D} \xrightarrow{F_a} \mathsf{Set}$$

denote the functor defined by $d \mapsto F_d(a)$ and $f \mapsto F_f(a)$ for an object d of $\mathcal{D}$ and an arrow f of $\mathcal{D}$. Then the colimit of F is the presheaf on $\mathcal{A}$ defined by $a \mapsto \operatorname{colim} F_a$, where $\operatorname{colim} F_a$ denotes the colimit of F_a.

As an alternative point of view on the functor F_a, let 1_{Cat} denote the category with exactly one object and one (identity) arrow, and let

$$1_{\mathsf{Cat}} \xrightarrow{1_a} \mathcal{A}^{op}$$

denote the functor sending the unique object of 1_{Cat} to a. Then the following diagram in the category of categories commutes.

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{F} & \mathsf{Set}^{\mathcal{A}^{op}} \\ & \searrow^{F_a} & \downarrow{\scriptstyle \mathsf{Set}^{1_a}} \\ & & \mathsf{Set} \end{array}$$

II.2.3. Presheaves as glueings

II.2.3.1. The following theorem expresses a universal property of the category $\mathsf{Set}^{\mathcal{A}^{op}}$. As we mentioned earlier, the passage from $\mathcal{A}$ to $\mathsf{Set}^{\mathcal{A}^{op}}$ freely adds in all colimits to $\mathcal{A}$ [8].

II.2.3.2. Theorem Let $\mathcal{C}$ be a category with all (small) colimits, and suppose that we have a functor

$$\mathcal{A} \xrightarrow{F} \mathcal{C}.$$

Then there is a unique functor

$$\mathsf{Set}^{\mathcal{A}^{op}} \xrightarrow{F'} \mathcal{C}$$

such that the following diagram in the category of categories commutes.

$$\begin{array}{ccc} \mathcal{A} & \hookrightarrow & \mathsf{Set}^{\mathcal{A}^{op}} \\ & \searrow^{F} & \downarrow F' \\ & & \mathcal{C} \end{array}$$

II.2.3.3. The main step in the proof of Theorem II.2.3.2 is the following theorem.

II.2.3.4. Theorem Let X be a presheaf on $\mathcal{A}$. Then there is a category $\mathcal{D}$ and a functor

$$\mathcal{D} \xrightarrow{F} \mathsf{Set}^{\mathcal{A}^{op}}$$

such that X can be equipped with the structure of a co-cone of F, and such that this co-cone defines a colimit of F.

II.2.3.5. Proof Let

$$\mathcal{A} \overset{j}{\hookrightarrow} \mathsf{Set}^{\mathcal{A}^{op}}$$

denote the functor of II.2.2.7. Given an object a of $\mathcal{A}$, let Y_a denote the presheaf $\mathrm{Hom}_{\mathcal{A}}(-, a)$. Let $\mathcal{D}$ denote the category [9] defined as follows.

(i) The objects of $\mathcal{D}$ are pairs of an object a of $\mathcal{A}$ and an arrow

$$Y_a \longrightarrow X$$

in $\mathsf{Set}^{\mathcal{A}^{op}}$.

(ii) An arrow of $\mathcal{D}$ from (a_0, f_0) to (a_1, f_1), where

$$Y_{a_0} \xrightarrow{f_0} X$$

and

$$Y_{a_1} \xrightarrow{f_1} X$$

are arrows of $\mathsf{Set}^{\mathcal{A}^{op}}$, is an arrow

$$a_0 \xrightarrow{g} a_1$$

of $\mathcal{A}$ such that the following diagram in $\mathsf{Set}^{\mathcal{A}^{op}}$ commutes.

$$\begin{array}{ccc} Y_{a_0} & \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(-,g)} & Y_{a_1} \\ & \searrow^{f_0} & \downarrow f_1 \\ & & X \end{array}$$

(iii) Composition of arrows of $\mathcal{D}$ and identity arrows of $\mathcal{D}$ are as in $\mathcal{A}$.

The following defines a functor [10]

$$D \xrightarrow{G} \mathcal{A}.$$

(i) To an (a, f) of $\mathcal{D}$ is associated the object a of $\mathcal{A}$.

(ii) To an arrow

$$a_0 \xrightarrow{g} a_1$$

of $\mathcal{D}$ is associated g itself, viewed as an arrow of $\mathcal{A}$.

Let F denote the following composite functor.

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{G} & \mathcal{A} \\ & \searrow^{F} & \downarrow j \\ & & \mathsf{Set}^{\mathcal{A}^{op}} \end{array}$$

We claim that X defines a colimit of F. To be precise, the arrows

$$Y_a \xrightarrow{g} X$$

of $\mathsf{Set}^{\mathcal{A}^{op}}$ for an object a of $\mathcal{A}$ assemble into a co-cone for the functor F, by definition of the arrows of $\mathcal{D}$. We claim that X equipped with this co-cone defines a colimit of F.

To prove the claim, suppose that we have an object X' of $\mathsf{Set}^{\mathcal{A}^{op}}$ and arrows

$$Y_a \xrightarrow{\Gamma_g} X'$$

of $\mathsf{Set}^{\mathcal{A}^{op}}$ for every object a of $\mathcal{A}$ and every arrow

$$Y_a \xrightarrow{g} X$$

of $\mathsf{Set}^{\mathcal{A}^{op}}$, which together define a co-cone of F. We construct a morphism of presheaves

$$X \xrightarrow{\Gamma_{can}} X'$$

as follows. Let a be an object of $\mathcal{A}$, and let x belong to $X(a)$. By the Yoneda lemma, x corresponds uniquely to a morphism of presheaves

$$Y_a \xrightarrow{g_x} X.$$

We define $\Gamma_{can}(x)$ to be the element of $X'(a)$ corresponding to Γ_{g_x} under the Yoneda lemma. That Γ_{can} defines a morphism of presheaves can be seen as follows. Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, let x belong to $X(a_0)$, and let x' belong to $X(a_1)$. The Yoneda lemma gives us that the following diagram in $\mathsf{Set}^{\mathcal{A}^{op}}$ commutes.

$$\begin{array}{ccc} Y_{a_0} & \xrightarrow{\mathrm{Hom}(-,f)} & Y_{a_1} \\ & \searrow^{g_x} & \downarrow{\scriptstyle g_{x'}} \\ & & X \end{array}$$

By definition of a co-cone, we thus have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{array}{ccc} Y_{a_0} & \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(-,f)} & Y_{a_1} \\ & \searrow^{\Gamma_{g_x}} & \downarrow{\scriptstyle \Gamma_{g_{x'}}} \\ & & X' \end{array}$$

By the Yoneda lemma once more, we deduce that $f\big(\Gamma_{can}(x)\big) = \Gamma_{can}(x')$, as required. Finally, let us verify that for every object

$$Y_a \xrightarrow{g} X$$

of $\mathcal{D}$, the following diagram in $\mathsf{Set}^{\mathcal{A}^{op}}$ commutes.

$$
\begin{array}{ccc}
Y_a & \xrightarrow{\;g\;} & X \\
& \underset{\Gamma_g}{\searrow} & \Big\downarrow{\scriptstyle \Gamma_{can}} \\
& & X'
\end{array}
$$

Let x denote the element of $X(a)$ corresponding to g under the Yoneda lemma, and let x' denote the element of $X'(a)$ corresponding to Γ_g under the Yoneda lemma. The Yoneda lemma gives us that the commutativity of the above triangle is equivalent to the assertion that $\Gamma_{can}(x) = x'$. This holds by definition of Γ_{can}.

II.2.3.6. We usually express Theorem II.2.3.4 as: *a presheaf on $\mathcal{A}$ is a colimit of representable presheaves.*

II.3. Semi-cubical sets

II.3.1. Why the topological n-cubes?

Let us consider the following question: from an abstract point of view, what are the properties of the topological n-cubes I^0, I^1, I^2, $\ldots$ that we draw upon in classical homotopy theory? Firstly, we have that $I^n \simeq I^{n-1} \times I$, for any $n \geq 1$. This leads us to a simpler question: what structure does the topological interval I have that we draw upon in classical homotopy theory?

II.3.2. Interval in a category

Most fundamentally, we can observe that we rely upon the fact that I can be thought of as having two endpoints, 0 and 1. Reformulating this, we rely on the existence of a pair of maps

$$
I^0 \underset{i_1}{\overset{i_0}{\rightrightarrows}} I,
$$

where i_0 sends the unique point of I^0 to 0, and i_1 sends the unique point of I^0 to 1.

Thus, as a point of depature, we might define an *interval* in a category $\mathcal{A}$ to be the data $\left(I^0, I^1, i_0, i_1\right)$ of a pair of objects I^0 and I^1 of $\mathcal{A}$ together with a pair of arrows

$$
I^0 \underset{i_1}{\overset{i_0}{\rightrightarrows}} I^1
$$

of $\mathcal{A}$.

II.3.3. Homotopy with respect to an interval

Suppose that $\mathcal{A}$ is equipped with the structure $(\otimes, 1, \lambda, \rho, \alpha)$ of a monoidal category, where 1 is a unit object, λ is a natural isomorphism

$$- \otimes 1 \longrightarrow -,$$

ρ is a natural isomorphism

$$1 \otimes - \longrightarrow -,$$

and α is a natural isomorphism

$$- \otimes (- \otimes -) \longrightarrow (- \otimes -) \otimes -.$$

For example, if $\mathcal{A}$ has finite products, we may regard $\mathcal{A}$ as equipped with its cartesian monoidal structure. Given an interval $\widehat{I} = (1, I^1, i_0, i_1)$ in $\mathcal{A}$, we can define a notion of homotopy between arrows of $\mathcal{A}$ with respect to $\widehat{I}$. Indeed, suppose that

$$a_0 \underset{f_1}{\overset{f_0}{\rightrightarrows}} a_1$$

are arrows of $\mathcal{A}$. Then a *homotopy* from f_0 to f_1 with respect to $\widehat{I}$ to be an arrow

$$a_0 \otimes I \xrightarrow{h} a_1$$

of $\mathcal{A}$ such that the following diagrams in $\mathcal{A}$ commute.

$$\begin{array}{ccc} a_0 & \xrightarrow{a_0 \otimes i_0} & a_0 \otimes I \\ & \searrow^{f_0} & \downarrow h \\ & & a_1 \end{array} \qquad \begin{array}{ccc} a_0 & \xrightarrow{a_0 \otimes i_1} & a_0 \otimes I \\ & \searrow^{f_1} & \downarrow h \\ & & a_1 \end{array}$$

We (harmlessly!) implicitly identify a_0 with $a_0 \otimes 1$ via λ here. In the case of the category of spaces equipped with its cartesian monoidal structure, we recover the classical notion of homotopy.

II.3.4. Towards the category of semi-cubes

In this way, we find that from a conceptual point of view the structure (I^0, I, i_0, i_1) with which the topological interval I is equipped, together with the property that $I^n \simeq I^{n-1} \times I^1$ for all $n \geq 1$, already furnishes us with the fundamental ingredients of classical homotopy theory: maps

$$I^n \longrightarrow X$$

up to homotopy, for any space X.

II.3.5. An outline of the category of semi-cubes

Thus we are led to the following idea, as our first guess at an appropriate category of building blocks for combinatorial homotopy theory: we would like to define the category of semi-cubes as the free strict monoidal category on an interval. Let us work towards constructing such a category.

We begin with the formal data of an interval, namely a directed graph with exactly two objects 1 and I, and two arrows

$$1 \underset{i_1}{\overset{i_0}{\rightrightarrows}} I,$$

and build a strict monoidal category as follows, roughly speaking. We first add in an object $I \otimes I$ and four arrows

$$I \underset{I \otimes i_1}{\overset{I \otimes i_0}{\rightrightarrows}} I \otimes I,$$

and

$$I \underset{i_1 \otimes I}{\overset{i_0 \otimes I}{\rightrightarrows}} I \otimes I.$$

Next we add in an object $I \otimes I \otimes I$, and six arrows

$$I \otimes I \underset{I \otimes I \otimes i_1}{\overset{I \otimes I \otimes i_0}{\rightrightarrows}} I \otimes I \otimes I,$$

$$I \otimes I \underset{I \otimes i_1 \otimes I}{\overset{I \otimes i_0 \otimes I}{\rightrightarrows}} I \otimes I \otimes I,$$

and

$$I \otimes I \underset{i_1 \otimes I \otimes I}{\overset{i_0 \otimes I \otimes I}{\rightrightarrows}} I \otimes I \otimes I.$$

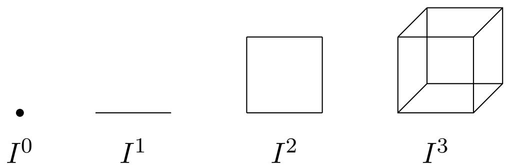
We continue in this manner, ending up with an object $\underbrace{I \otimes \ldots \otimes I}_{n}$ for every $n \geq 0$, which we will denote by I^n for short, and arrows

$$I^{n-1} \underset{I^{i-1} \otimes i_1 \otimes I^{n-i}}{\overset{I^{i-1} \otimes i_0 \otimes I^{n-i}}{\rightrightarrows}} I^n$$

for every $1 \leq i \leq n$. It remains only to take care of composition and identities.

II.3.6. A topological picture

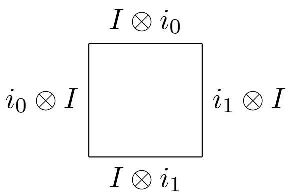
Before we do so, let us develop a little more feeling for our proposed category of cubes. Intuitively, we can think of I^n as corresponding to the topological n-cube.



We can think of the arrows

$$I^{n-1} \underset{I^{i-1} \otimes i_1 \otimes I^{n-i}}{\overset{I^{i-1} \otimes i_0 \otimes I^{n-i}}{\rightrightarrows}} I^n$$

for some $1 \leq i \leq n$ as a pair of faces of the topological n-cube, as illustrated below.



We make an arbitrary choice to make when depicting the faces in this way. In the picture above, we could equally regard $i_0 \otimes I$ and $i_1 \otimes I$ as the horizontal faces, and regard $I \otimes i_0$ and $I \otimes i_1$ as the vertical faces.

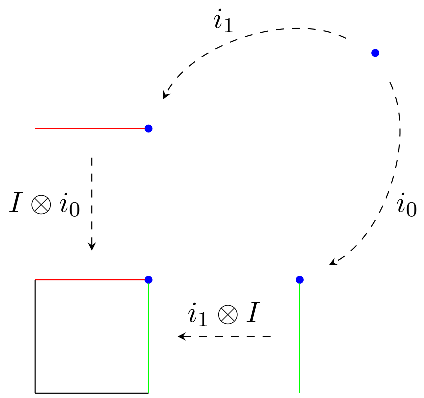
II.3.7. A need to impose relations

Our first idea to take care of compositions and identities in our proposed category of cubes might simply be to take the free category on the directed graph with the objects and arrows we arrived at by the end of II.3.5. We might then try to equip this category with a strict monoidal structure in the obvious way, defining $I^m \otimes I^n$ to be I^{m+n}, and similarly on arrows.

However, this will not quite work yet. In order to obtain a strict monoidal category, we must have that the following diagram commutes.

$$\begin{array}{ccc} I^0 & \xrightarrow{i_0} & I^1 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_1 \otimes I} \\ I^1 & \xrightarrow[I \otimes i_0]{} & I^2 \end{array}$$

Topologically, this corresponds to the fact that we have the following two ways of arriving at the top right point of I^2.



We must also have that three more diagrams commute, corresponding to the three other vertices of I^2. Moreover we must have that further diagrams, corresponding in topology to higher dimensional analogues of the above picture. Finally, we are led to the following definition.

II.3.8. The category of semi-cubes

II.3.8.1. Definition Let Υ denote the directed graph defined as follows.

(i) We have an object for every $n \geq 0$, which we denote by I^n.

(ii) For every $1 \leq i \leq n$ and $0 \leq \epsilon \leq 1$, we have an arrow

$$I^{n-1} \xrightarrow{f^n_{i,\epsilon}} I^n.$$

Let $\mathcal{F}(\Upsilon)$ denote the free category upon Υ. The *category of semi-cubes*, which we will denote by $\square_s$, is the quotient of $\mathcal{F}(\Upsilon)$ by the relation $\sim$ defined inductively as follows.

(i) For any $1 \leq i \leq n$ and $1 \leq j \leq n+1$, and any $0 \leq \delta, \epsilon \leq 1$, we have that

$$f^{n+1}_{j,\epsilon} \circ f^n_{i,\delta} \sim \begin{cases} f^{n+1}_{i,\delta} \circ f^n_{j-1,\epsilon} & \text{if } j > i, \\ f^{n+1}_{i+1,\delta} \circ f^n_{j,\epsilon} & \text{if } j \leq i. \end{cases}$$

(ii) Suppose that we have arrows

$$I^q \underset{g_0'}{\overset{g_0}{\rightrightarrows}} I^r$$

and

$$I^r \underset{g_1'}{\overset{g_1}{\rightrightarrows}} I^s$$

of $\mathcal{F}(\Upsilon)$ with $q, r, s \geq 0$, such that $g_0 \sim g_1$ and $g_0' \sim g_1'$. Then we have that $g_1 \circ g_0 \sim g_1' \circ g_0'$.

II.3.8.2. Here we think of the arrow

$$I^{n-1} \xrightarrow{f_{i,0}^n} I^n$$

as shorthand for our earlier arrow

$$I^{n-1} \xrightarrow{I^{i-1} \otimes i_0 \otimes I^{n-i}} I^n.$$

Similarly, we think of the arrow

$$I^{n-1} \xrightarrow{f_{i,1}^n} I^n$$

as shorthand for our earlier arrow

$$I^{n-1} \xrightarrow{I^{i-1} \otimes i_1 \otimes I^{n-i}} I^n.$$

II.3.9. A universal property of the category of semi-cubes

We now work towards rigorously equipping $\square_s$ with a strict monoidal structure, and proving that $\square_s$ with this monoidal structure is the free strict monoidal category upon an interval, in an appropriate sense.

II.3.9.1. To help us on our way, let us recall an observation or two about a product of categories. Given categories $\mathcal{A}_0$ and $\mathcal{A}_1$, let $\mathcal{A}_0 \times \mathcal{A}_1$ denote the category defined as follows.

(i) The objects are pairs (a, a') of an object a of $\mathcal{A}_0$ and an object a' of $\mathcal{A}_1$.

(ii) An arrow from (a_0, a'_0) to (a_1, a'_1) is a pair (f, f') of an arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}_0$ and an arrow

$$a'_0 \xrightarrow{f'} a'_1$$

of $\mathcal{A}_1$.

(iii) The composition $(g, g') \circ (f, f')$ of an arrow

$$(a_0, a'_0) \xrightarrow{(f, f')} (a_1, a'_1)$$

and an arrow

$$(a_1, a'_1) \xrightarrow{(g, g')} (a_2, a'_2)$$

is the arrow

$$(a_0, a'_0) \xrightarrow{(f' \circ f, g' \circ g)} (a_2, a'_2).$$

(iv) The identity arrow from (a, a') to itself is $\big(id(a), id(a')\big)$.

To define a functor F from $\mathcal{A}_0 \times \mathcal{A}_1$ to a category $\mathcal{A}_2$, it suffices to specify $F(a, a')$ for any object a of $\mathcal{A}_0$ and any object a' of $\mathcal{A}_1$, and to specify $F(f, id)$ and $F(id, f')$ for any arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}_0$ and any arrow

$$a'_0 \xrightarrow{f'} a'_1$$

of $\mathcal{A}_1$, in such a way that the following diagram in $\mathcal{A}_2$ commutes.

$$
\begin{array}{ccc}
F(a_0, a_0') & \xrightarrow{F\left(f, id(a_0')\right)} & F(a_1, a_0') \\
{\scriptstyle F\left(id(a_0), f'\right)}\downarrow & & \downarrow{\scriptstyle F\left(id(a_1), f'\right)} \\
F(a_0, a_1') & \xrightarrow[F\left(f, id(a_1')\right)]{} & F(a_0', a_1')
\end{array}
$$

Indeed we can then define $F(f, f')$ to be the arrow of $\mathcal{A}_2$ obtained by taking either route through this diagram, we can define $F(gf, id)$ to be $F(g, id) \circ F(f, id)$, and can define $F(id, g'f')$ to be $F(id, g') \circ F(id, f')$. This observation will be useful for us below.

Before we get to this, let us observe that we have a functor

$$
\mathcal{A}_0 \times \mathcal{A}_1 \xrightarrow{pr_0} \mathcal{A}_0
$$

given by $(a, a') \mapsto a$ and $(f, g) \mapsto f$. Likewise we have a functor

$$
\mathcal{A}_0 \times \mathcal{A}_1 \xrightarrow{pr_1} \mathcal{A}_1
$$

given by $(a, a') \mapsto a'$ and $(f, f') \mapsto f'$. Suppose that we have a commutative diagram as follows in the category of categories.

$$
\begin{array}{ccc}
\mathcal{A}_0 \times \mathcal{A}_1 & \xrightarrow{pr_0} & \mathcal{A}_0 \\
{\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle F_0} \\
\mathcal{A}_1 & \xrightarrow[F_1]{} & \mathcal{A}_2
\end{array}
$$

Then for any object (a, a') of $\mathcal{A}_0 \times \mathcal{A}_1$ we have that $F_0(a) = F_1(a')$. Let us denote this object of $\mathcal{A}_2$ by $r_{F_0,F_1}(a, a')$. Moreover for any arrow f of $\mathcal{A}_0$ and any arrow f' of $\mathcal{A}_1$, we have that $F_0(f) = F_1(f')$. Let us denote this arrow of $\mathcal{A}_2$ by $r_{F_0,F_1}(f, f')$. It is clear that this recipe defines a functor

$$
\mathcal{A}_0 \times \mathcal{A}_1 \xrightarrow{r_{F_0,F_1}} \mathcal{A}_2
$$

and that r_{F_0,F_1} is moreover the unique functor such that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{array}{ccc}
\mathcal{A}_0 \times \mathcal{A}_1 & \xrightarrow{pr_0} & \mathcal{A}_0 \\
{\scriptstyle pr_1}\downarrow & \searrow^{r_{F_0,F_1}} & \downarrow{\scriptstyle F_0} \\
\mathcal{A}_1 & \xrightarrow[F_1]{} & \mathcal{A}_2
\end{array}
$$

Thus $\mathcal{A}_0 \times \mathcal{A}_1$ together with the functors pr_0 and pr_1 defines a product of $\mathcal{A}_0$ and $\mathcal{A}_1$ in the category of categories.

II.3.9.2. Definition Let

$$\square_s \times \square_s \xrightarrow{- \otimes -} \square_s.$$

denote the functor defined inductively by the following recipe, by virtue of our observations in II.3.9.1.

(i) For $m, n \geq 0$, we define $I^m \otimes I^n$ to be I^{m+n}.

(ii) For $1 \leq i \leq m$ and $n \geq 0$, and $0 \leq \epsilon \leq 1$, we define $f^m_{i,\epsilon} \otimes I^n$ to be $f^{m+n}_{i,\epsilon}$.

(iii) For $1 \leq i \leq n$ and $m \geq 0$, and $0 \leq \epsilon \leq 1$, we define $I^m \otimes f^n_{i,\epsilon}$ to be $f^{m+n}_{m+i,\epsilon}$.

(iv) For $m, m', m'', n \geq 0$, and arrows

$$I^m \xrightarrow{g_0} I^{m'}$$

and

$$I^{m'} \xrightarrow{g_1} I^{m''}$$

of $\square_s$, we define $(g_1 \circ g_0) \otimes I^n$ to be $(g_1 \otimes I^n) \circ (g_0 \otimes I^n)$.

(v) For $m, n, n', n'' \geq 0$, and arrows

$$I^n \xrightarrow{g_0} I^{n'}$$

and

$$I^{n'} \xrightarrow{g_1} I^{n''}$$

of $\square_s$, we define $I^m \otimes (g_1 \circ g_0)$ to be $(I^m \otimes g_1) \circ (I^m \otimes g_0)$.

II.3.9.3. Note that by definition of the relation $\sim$ in the definition of $\square_s$, the following diagram in $\mathcal{A}$ commutes for any $1 \leq i \leq m$ and $1 \leq j \leq n$, and any $0 \leq \delta, \epsilon \leq 1$.

$$\begin{array}{ccc} I^{m+n-1} & \xrightarrow{f^{m+n}_{i,\delta}} & I^{m+n} \\ {\scriptstyle f^{m+n}_{m+j,\epsilon}} \downarrow & & \downarrow {\scriptstyle f^{m+n+1}_{m+j+1,\epsilon}} \\ I^{m+n} & \xrightarrow[f^{m+n+1}_{i,\delta}]{} & I^{m+n+1} \end{array}$$

In other words, we have that the following diagram in $\square_s$ commutes.

$$\begin{array}{ccc} I^{m+n-1} & \xrightarrow{f^m_{i,\epsilon} \otimes I^n} & I^{m+n} \\ {\scriptstyle I^m \otimes f^n_{j,\delta}} \big\downarrow & & \big\downarrow {\scriptstyle I^m \otimes f^n_{j,\delta}} \\ I^{m+n} & \xrightarrow[f^m_{i,\epsilon} \otimes I^n]{} & I^{m+n+1} \end{array}$$

The definition of $\sim$ also gives us that the following diagram in $\square_s$ commutes for any $1 \leq i \leq m$ and $1 \leq j \leq n$.

$$\begin{array}{ccc} I^{m+n-1} & \xrightarrow{f^{m+n}_{m+j,\epsilon}} & I^{m+n} \\ {\scriptstyle f^{m+n}_{i,\delta}} \big\downarrow & & \big\downarrow {\scriptstyle f^{m+n+1}_{i,\delta}} \\ I^{m+n} & \xrightarrow[f^{m+n+1}_{m+j+1,\epsilon}]{} & I^{m+n+1} \end{array}$$

In other words, we have that the following diagram in $\square_s$ commutes.

$$\begin{array}{ccc} I^{m+n-1} & \xrightarrow{I^m \otimes f^n_{j,\epsilon}} & I^{m+n} \\ {\scriptstyle f^m_{i,\epsilon} \otimes I^n} \big\downarrow & & \big\downarrow {\scriptstyle f^m_{i,\delta} \otimes I^n} \\ I^{m+n} & \xrightarrow[I^m \otimes f^n_{j,\delta}]{} & I^{m+n+1} \end{array}$$

Thus the recipe of II.3.9.2 does allow us to cook up a functor

$$\square_s \times \square_s \xrightarrow{- \otimes -} \square_s$$

as in II.3.9.1, and it is the relation $\sim$, whose definition was motivated by the intuitive considerations of II.3.7, which exactly ensures this.

II.3.9.4. Proposition The functor

$$\square_s \times \square_s \xrightarrow{- \otimes -} \square_s$$

equips the category $\square_s$ with the structure of a strict monoidal category with unit I^0.

II.3.9.5. Proof Immediate verification.

II.3.9.6. It is now a straightforward matter to put on firm ground our intuitive conception of $\square_s$ as the free strict monoidal category upon an interval. Let us get down to it straight away!

II.3.9.7. Let $(\square_s)^{\leq 1}$ denote the free category on the following directed graph.

$$0 \underset{\iota_1}{\overset{\iota_0}{\rightrightarrows}} 1$$

We think of $(\square_s)^{\leq 1}$ as the *free-standing interval.* If $\left(I^0, I^1, i_0, i_1\right)$ defines an interval in a category $\mathcal{A}$, then we obtain a functor

$$(\square_s)^{\leq 1} \longrightarrow \mathcal{A}$$

given by $0 \mapsto I^0$, $1 \mapsto I^1$, $\iota_0 \mapsto i_0$, and $\iota_1 \mapsto i_1$. In particular, we obtain in this way a functor

$$(\square_s)^{\leq 1} \hookrightarrow \square_s.$$

II.3.9.8. Proposition Let $\mathcal{A}$ be a category equipped with a strict monoidal structure $(\otimes, 1)$. For any functor

$$(\square_s)^{\leq 1} \xrightarrow{\mathsf{int}} \mathcal{A},$$

there is a unique strict monoidal functor

$$\square_s \xrightarrow{\mathsf{can}} \mathcal{A}$$

such that the following diagram in the category of categories commutes.

$$\begin{array}{ccc} (\square_s)^{\leq 1} & \hookrightarrow & \square_s \\ & \searrow^{\mathsf{int}} & \downarrow \mathsf{can} \\ & & \mathcal{A} \end{array}$$

II.3.9.9. Proof The unique possible recipe for a strictly monoidal functor

$$\square_s \xrightarrow{\mathsf{can}} \mathcal{A}$$

fitting into the above commutative diagram is the following.

(i) We define $\mathsf{can}(I^0)$ to be $\mathsf{int}(0)$, and for $n \geq 1$ we define $\mathsf{can}(I^n)$ to be

$$\underbrace{\mathsf{int}(1) \otimes \cdots \otimes \mathsf{int}(1)}_{n}.$$

(ii) For $1 \leq i \leq n$, we define $\mathsf{can}(f^n_{i,0})$ to be the arrow

$$\mathsf{can}(I^{n-1}) \xrightarrow{\mathsf{can}(I^{i-1}) \otimes i_0 \otimes \mathsf{can}(I^{n-i})} \mathsf{can}(I^n)$$

of $\mathcal{A}$.

(iii) For $1 \leq i \leq n$, we define $\mathsf{can}(f^n_{i,1})$ to be the arrow

$$\mathsf{can}(I^{n-1}) \xrightarrow{\mathsf{can}(I^{i-1}) \otimes i_1 \otimes \mathsf{can}(I^{n-i})} \mathsf{can}(I^n)$$

of $\mathcal{A}$.

(iv) For $q, r, s \geq 0$, and arrows

$$I^q \xrightarrow{g_0} I^r$$

and

$$I^r \xrightarrow{g_1} I^s$$

of $\square$, we define $\mathsf{can}(g_1 \circ g_0)$ to be $\mathsf{can}(g_1) \circ \mathsf{can}(g_0)$.

It remains to verify that this recipe indeed defines a functor from $\square_s$ to $\mathcal{A}$. Evidently it defines a functor from $\mathcal{F}(\Upsilon)$ to $\mathcal{A}$. In addition, for any $1 \leq i < j \leq n+1$, and $0 \leq \delta, \epsilon \leq 1$, it follows immediately from the fact that $(\otimes, 1)$ defines a strict monoidal structure upon $\mathcal{A}$ that the following diagram in $\mathcal{A}$ commutes.

$$\begin{array}{ccc} \mathsf{can}(I^{n-1}) & \xrightarrow{\mathsf{can}(f^n_{i,\delta})} & \mathsf{can}(I^n) \\ {\scriptstyle \mathsf{can}(f^n_{j-1,\epsilon})} \downarrow & & \downarrow {\scriptstyle \mathsf{can}(f^{n+1}_{j,\epsilon})} \\ \mathsf{can}(I^n) & \xrightarrow[\mathsf{can}(f^{n+1}_{i,\delta})]{} & \mathsf{can}(I^{n+1}) \end{array}$$

Moreover, for any $1 \leq j \leq i \leq n$, and $0 \leq \delta, \epsilon \leq 1$, it again follows immediately from the fact that $(\otimes, 1)$ defines a strict monoidal structure upon $\mathcal{A}$ that the following diagram in $\mathcal{A}$ commutes.

$$\begin{array}{ccc} \mathsf{can}(I^{n-1}) & \xrightarrow{\mathsf{can}(f^n_{i,\delta})} & \mathsf{can}(I^n) \\ \Big\downarrow \mathsf{can}(f^n_{j,\epsilon}) & & \Big\downarrow \mathsf{can}(f^{n+1}_{j,\epsilon}) \\ \mathsf{can}(I^n) & \xrightarrow[\mathsf{can}(f^{n+1}_{i,\delta})]{} & \mathsf{can}(I^{n+1}) \end{array}$$

This completes our proof.

II.3.10. Semi-cubical sets

II.3.10.1. Let us summarise our story so far. We have seen that presheaves precisely express the idea of glueing from a category of building blocks. By considering abstract properties of the topological n-cubes which we rely upon in classical homotopy theory, we have arrived at a first guess for a candidate category of building blocks for an abstract theory of homotopy — our category $\square_s$ of semi-cubes.

II.3.10.2. Definition A *semi-cubical set* is a presheaf on $\square_s$.

II.3.10.3. Can we construct a homotopy theory of semi-cubical sets? If so, how closely does it resemble the classical homotopy theory of topological spaces?

II.4. Cubical sets

II.4.1. Further structure of the topological interval?

To develop a feeling for these questions, we may pose another — from an abstract point of view, what further structure of the topological interval do we rely upon in developing the deeper homotopy theory of spaces? So far, we have seen that we can understand the notion of homotopy itself in a purely conceptual way. But homotopy theory is much more than the definition of homotopy!

II.4.2. Constant homotopies

Let us consider the question of constructing a homotopy from a map

$$X \xrightarrow{f} Y$$

between spaces to itself, in other words a constant homotopy. We cannot even do this in the presence of the interval $\left(I^0, I^1, i_0, i_1\right)$ alone! Indeed, let

$$I \xrightarrow{p} I^0$$

denote the canonical map. Then we have that the following diagrams in the category of spaces commute.

$$\begin{array}{ccc} I^0 & \xrightarrow{i_0} & I^1 \\ & {\scriptstyle id}\searrow & \big\downarrow {\scriptstyle p} \\ & & I^0 \end{array} \qquad \begin{array}{ccc} I^0 & \xrightarrow{i_1} & I^1 \\ & {\scriptstyle id}\searrow & \big\downarrow {\scriptstyle p} \\ & & I^0 \end{array}$$

Thus we see that the composite map

$$\begin{array}{ccc} X \times I & \xrightarrow{X \times p} & X \\ & \searrow & \big\downarrow {\scriptstyle f} \\ & & Y \end{array}$$

defines a homotopy from f to itself.

II.4.3. Contraction structure upon an interval

Let $\mathcal{A}$ be a category equipped with a monoidal structure $(\otimes, 1, \lambda, \rho, \alpha)$ as in II.3.3. Let $\widehat{I} = \big(1, I, i_0, i_1\big)$ be an interval in $\mathcal{A}$. Motivated by II.4.2, we define a *contraction structure* upon $\widehat{I}$ is an arrow

$$I \xrightarrow{p} 1$$

of $\mathcal{A}$ such that the following diagrams in $\mathcal{A}$ commute.

$$\begin{array}{ccc} I^0 & \xrightarrow{i_0} & I^1 \\ & {\scriptstyle id}\searrow & \big\downarrow {\scriptstyle p} \\ & & I^0 \end{array} \qquad \begin{array}{ccc} I^0 & \xrightarrow{i_1} & I^1 \\ & {\scriptstyle id}\searrow & \big\downarrow {\scriptstyle p} \\ & & I^0 \end{array}$$

As in II.4.2, a contraction structure upon $\widehat{I}$ allows us to construct constant homotopies between arrows of $\mathcal{A}$, with homotopy with respect to $\widehat{I}$ being the notion we discussed in II.3.3.

II.4.4. Towards the category of cubes

We can now carry out a construction of a category of cubes entirely in analogy with our construction of the category $\square_s$ of semi-cubes. Rather than beginning with an interval alone, we begin with an interval equipped with a contraction structure, and pass to the free strict monoidal category upon this initial data. This will introduce a need for further relations.

We will shortly jump straight into the details. Before we do, we might note that the ability to reverse and to compose homotopies is just as fundamental as the construction of constant homotopies in the classical homotopy theory of spaces. Can we capture this by means of extra structure upon an interval? Should we not build this structure into our category of cubes, just as we are proposing to build in a contraction structure?

The answer to the first of these questions is — yes! The interested reader may consult [11] or the book [3] of Kamps and Porter[11]. The answer to the second question could be a lecture or more in itself! We may certainly build an *involution* structure into our category of cubes, which allows us to reverse homotopies. I hope to have time to give the details in a future update to these notes. However, we cannot naively build a *subdivision* structure into our category of cubes which would allow us to compose homotopies — or more precisely we could, but it would be destroyed upon passing to presheaves. However, we can build in a subdivision structure in a more sophisticated way — this leads towards higher category theory. A paper [10] on this is in preparation.

It turns out, though, that building a contraction structure into our category of cubes is sufficient to construct a homotopy theory of cubical sets which we will eventually see will be equivalent, in an appropriate sense, to that of spaces. That this is possible revolves around the notion of a Kan complex, which we will introduce in the next lecture.

In fact, it is not even necessary to build in a contraction structure! In the setting of the simplicial analogues of semi-cubical sets, Rourke and Sanderson showed in [8] that we can still build a homotopy theory which is equivalent to that of spaces. However, building in a contraction structure is very useful — we will make use of it very frequently — giving rise to a richer theory, in which we have parallels of more aspects of classical homotopy theory.

II.4.5. The category of cubes

II.4.5.1. Definition Let Υ' denote the directed graph defined as follows.

(i) We have an object for every $n \geq 0$, which we denote by I^n.

(ii) For every $1 \leq i \leq n$ and $0 \leq \epsilon \leq 1$, we have an arrow

$$I^{n-1} \xrightarrow{f^n_{i,\epsilon}} I^n.$$

(iii) For every $1 \leq i \leq n$ we have an arrow

$$I^n \xrightarrow{d^n_i} I^{n-1}.$$

Let $\mathcal{F}(\Upsilon')$ denote the free category upon Υ'. The *category of cubes*, which we will denote by $\square$, is the quotient of $\mathcal{F}(\Upsilon')$ by the relation $\sim$ defined inductively as follows.

(i) For any $1 \leq i \leq n$ and $1 \leq j \leq n+1$, and any $0 \leq \delta, \epsilon \leq 1$, we have that

$$f^{n+1}_{j,\epsilon} \circ f^n_{i,\delta} \sim \begin{cases} f^{n+1}_{i,\delta} \circ f^n_{j-1,\epsilon} & \text{if } j > i, \\ f^{n+1}_{i+1,\delta} \circ f^n_{j,\epsilon} & \text{if } j \leq i. \end{cases}$$

(ii) For any $n \geq 2$, $1 \leq i \leq n$, and $1 \leq j \leq n-1$, we have that

$$d^{n-1}_j \circ d^n_i \sim \begin{cases} d^{n-1}_{i-1} \circ d^n_j & \text{if } j < i, \\ d^{n-1}_i \circ d^n_{j+1} & \text{if } j \geq i. \end{cases}$$

(iii) For any $n \geq 2$, $1 \leq i, j \leq n$, and $0 \leq \epsilon \leq 1$, we have that

$$d^n_j \circ f^n_{i,\epsilon} \sim \begin{cases} id & \text{if } j = i, \\ f^{n-1}_{i,\epsilon} \circ d^{n-1}_{j-1} & \text{if } j > i, \\ f^{n-1}_{i-1,\epsilon} \circ d^{n-1}_j & \text{if } j < i. \end{cases}$$

(iv) Suppose that we have arrows

$$I^q \underset{g_0'}{\overset{g_0}{\rightrightarrows}} I^r$$

and

$$I^r \underset{g_1'}{\overset{g_1}{\rightrightarrows}} I^s$$

of $\mathcal{F}(\Upsilon')$ with $q, r, s \geq 0$, such that $g_0 \sim g_1$ and $g_0' \sim g_1'$. Then we have that $g_1 \circ g_0 \sim g_1' \circ g_0'$.

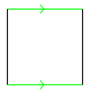
II.4.5.2. Here we think of d^n_i as the arrow

$$I^n \xrightarrow{I^{i-1} \otimes p \otimes I^{n-i}} I^{n-1}$$

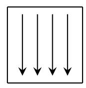
in our outline of the free strict monoidal category upon an interval with contraction, analogous to that of the free strict monoidal category upon an interval in II.3.5. This motivates the relations of (ii) as higher dimensional analogues of the defining axioms of an interval with contraction in II.4.3. The relations of (iii) arise analogously to those of (i) — see II.3.7.

II.4.6. A topological picture

As in II.3.6, we think of I^n as the topological n-cube, and as $f^n_{i,\epsilon}$ for $0 \leq \epsilon \leq 1$ as face inclusions. We then think of d^n_i as the projection of I^n onto its face $f^n_{i,1}$ — the choice of $f^n_{i,0}$ or $f^n_{i,1}$ doesn't matter, but it is important that we make the same choice for all i and n. For example, suppose that $f^2_{1,0}$ and $f^2_{1,1}$ are the inclusions onto the faces of I^2 depicted in colour or by an arrow below.



Then d^2_1 can be taken to be the following projection map.



The reader may wish to justify to themself that the relations in the definition of $\square$ really do hold in this topological picture, as in II.3.7.

II.4.7. A universal property of the category of cubes

II.4.7.1. Definition Let

$$\square \times \square \xrightarrow{- \otimes -} \square.$$

denote the functor defined inductively by the following recipe, by virtue of our observations in II.3.9.1.

(i) For $m, n \geq 0$, we define $I^m \otimes I^n$ to be I^{m+n}.

(ii) For $1 \leq i \leq m$ and $n \geq 0$, and $0 \leq \epsilon \leq 1$, we define $f^m_{i,\epsilon} \otimes I^n$ to be $f^{m+n}_{i,\epsilon}$.

(iii) For $1 \leq i \leq n$ and $m \geq 0$, and $0 \leq \epsilon \leq 1$, we define $I^m \otimes f^n_{i,\epsilon}$ to be $f^{m+n}_{m+i,\epsilon}$.

(iv) For $1 \leq i \leq m$ and $n \geq 0$, we define $d^m_i \otimes I^n$ to be d^{m+n}_i.

(v) For $1 \leq i \leq n$ and $m \geq 0$, we define $I^m \otimes d^n_i$ to be d^{m+n}_{m+i}.

(vi) For $m, m', m'', n \geq 0$, and arrows

$$I^m \xrightarrow{g_0} I^{m'}$$

and

$$I^{m'} \xrightarrow{g_1} I^{m''}$$

of $\square$, we define $(g_1 \circ g_0) \otimes I^n$ to be $(g_1 \otimes I^n) \circ (g_0 \otimes I^n)$.

(vii) For $m, n, n', n'' \geq 0$, and arrows

$$I^n \xrightarrow{g_0} I^{n'}$$

and

$$I^{n'} \xrightarrow{g_1} I^{n''}$$

of $\square_s$, we define $I^m \otimes (g_1 \circ g_0)$ to be $(I^m \otimes g_1) \circ (I^m \otimes g_0)$.

II.4.7.2. Note that by definition of the relation $\sim$ in the definition of $\square$, the following diagram in $\mathcal{A}$ commutes for any $1 \leq i \leq m$ and $1 \leq j \leq n$.

$$\begin{array}{ccc} I^{m+n} & \xrightarrow{d_i^{m+n}} & I^{m+n-1} \\ {\scriptstyle d_{m+j}^{m+n}} \downarrow & & \downarrow {\scriptstyle d_{m-1+j}^{m+n-1}} \\ I^{m+n-1} & \xrightarrow[d_i^{m+n-1}]{} & I^{m+n-2} \end{array}$$

In other words, we have that the following diagram in $\square$ commutes.

$$\begin{array}{ccc} I^{m+n} & \xrightarrow{d_i^m \otimes I^n} & I^{m+n-1} \\ {\scriptstyle I^m \otimes d_j^n} \downarrow & & \downarrow {\scriptstyle I^{m-1} \otimes d_j^n} \\ I^{m+n-1} & \xrightarrow[d_i^m \otimes I^{n-1}]{} & I^{m+n-2} \end{array}$$

The definition of $\sim$ also gives us that the following diagram in $\square$ commutes, for any $1 \leq i \leq m$ and $1 \leq j \leq n$.

$$\begin{array}{ccc} I^{m+n} & \xrightarrow{d_{m+j}^{m+n}} & I^{m+n-1} \\ {\scriptstyle d_i^{m+n}} \downarrow & & \downarrow {\scriptstyle d_i^{m+n-1}} \\ I^{m+n-1} & \xrightarrow[d_{m+j-1}^{m+n-1}]{} & I^{m+n-2} \end{array}$$

In other words, we have that the following diagram in $\square$ commutes.

$$\begin{CD}
I^{m+n} @>{I^m \otimes d^n_j}>> I^{m+n-1} \\
@V{d^m_i \otimes I^n}VV @VV{d^m_i \otimes I^{n-1}}V \\
I^{m+n-1} @>>{I^{m-1} \otimes d^n_j}> I^{m+n-2}
\end{CD}$$

In addition, the definition of $\sim$ gives us that the following diagram in $\square$ commutes, for any $1 \leq i \leq m$ and $1 \leq j \leq n$, and any $0 \leq \epsilon \leq 1$.

$$\begin{CD}
I^{m+n-1} @>{f^{m+n}_{i,\epsilon}}>> I^{m+n} \\
@V{d^{m+n-1}_{m+j-1}}VV @VV{d^{m+n}_{m+j}}V \\
I^{m+n-2} @>>{f^{m+n-1}_{i}}> I^{m+n-1}
\end{CD}$$

In other words, we have that the following diagram in $\square$ commutes.

$$\begin{CD}
I^{m+n-1} @>{f^m_{i,\epsilon} \otimes I^n}>> I^{m+n} \\
@V{I^{m-1} \otimes d^n_j}VV @VV{I^m \otimes d^n_j}V \\
I^{m+n-2} @>>{f^m_{i,\epsilon} \otimes I^{n-1}}> I^{m+n-1}
\end{CD}$$

Moreover, the definition of $\sim$ gives us that the following diagram in $\square$ commutes, for any $1 \leq i \leq m$ and $1 \leq j \leq n$, and any $0 \leq \delta \leq 1$.

$$\begin{CD}
I^{m+n} @>{d^{m+n}_i}>> I^{m+n-1} \\
@V{f^{m+n}_{m+j,\delta}}VV @VV{f^{m+n-1}_{m+j-1,\delta}}V \\
I^{m+n+1} @>>{d^{m+n+1}_i}> I^{m+n}
\end{CD}$$

In other words, we have that the following diagram in $\square$ commutes.

$$\begin{CD}
I^{m+n} @>{d^m_i \otimes I^n}>> I^{m+n-1} \\
@V{I^m \otimes f^n_{j,\delta}}VV @VV{I^{m-1} \otimes f^n_{j,\delta}}V \\
I^{m+n+1} @>>{d^m_i \otimes I^{n+1}}> I^{m+n}
\end{CD}$$

Together with our observations of II.3.9.3, we thus have that the recipe of II.4.7.1 does allow us to cook up a functor

$$\square \times \square \xrightarrow{- \otimes -} \square$$

by virtue of II.3.9.1. As in II.3.9.3, it is the relation $\sim$ which exactly ensures this.

II.4.7.3. Proposition The functor

$$\square \times \square \xrightarrow{- \otimes -} \square$$

equips the category $\square$ with the structure of a strict monoidal category with unit I^0.

II.4.7.4. Proof Immediate verification.

II.4.7.5. Let $\square^{\leq 1}$ denote the free category on the directed graph of two objects 0 and 1, a pair of arrows

$$0 \underset{\iota_1}{\overset{\iota_0}{\rightrightarrows}} 1,$$

and an arrow

$$1 \xrightarrow{\pi} 0.$$

We think of $\square^{\leq 1}$ as the *free-standing interval equipped with a contraction structure*, noting that the commutative diagrams in $\square^{\leq 1}$ other than those involving composition with an identity arrow are exactly the following.

$$\begin{array}{ccc} 0 & \xrightarrow{\iota_0} & 1 \\ & \underset{id}{\searrow} & \downarrow \pi \\ & & 0 \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{\iota_1} & 1 \\ & \underset{id}{\searrow} & \downarrow \pi \\ & & 0 \end{array}$$

If (I^0, I^1, i_0, i_1, p) defines an interval equipped with a contraction structure p in a category $\mathcal{A}$, then we obtain a functor

$$\square^{\leq 1} \longrightarrow \mathcal{A}$$

given by $0 \mapsto I^0$, $1 \mapsto I^1$, $\iota_0 \mapsto i_0$, $\iota_1 \mapsto i_1$, and $\pi \mapsto p$. In particular, we obtain in this way a functor

$$\square^{\leq 1} \hookrightarrow \square.$$

II.4.7.6. Proposition Let $\mathcal{A}$ be a category equipped with a strict monoidal structure $(\otimes, 1)$. For any functor

$$\square^{\leq 1} \xrightarrow{\mathsf{int}} \mathcal{A},$$

there is a unique strict monoidal functor

$$\square \xrightarrow{\mathsf{can}} \mathcal{A}$$

such that the following diagram in the category of categories commutes.

$$\begin{array}{ccc} \square^{\leq 1} & \hookrightarrow & \square \\ & \searrow^{\mathsf{int}} & \downarrow \mathsf{can} \\ & & \mathcal{A} \end{array}$$

II.4.7.7. Proof The unique possible recipe for a strictly monoidal functor

$$\square \xrightarrow{\mathsf{can}} \mathcal{A}$$

fitting into the above commutative diagram is the following.

(i) We define $\mathsf{can}(I^0)$ to be $\mathsf{int}(0)$, and for $n \geq 1$ we define $\mathsf{can}(I^n)$ to be

$$\underbrace{\mathsf{int}(1) \otimes \cdots \otimes \mathsf{int}(1)}_{n}.$$

(ii) For $1 \leq i \leq n$, we define $\mathsf{can}(f^n_{i,0})$ to be the arrow

$$\mathsf{can}(I^{n-1}) \xrightarrow{\mathsf{can}(I^{i-1}) \otimes i_0 \otimes \mathsf{can}(I^{n-i})} \mathsf{can}(I^n)$$

of $\mathcal{A}$.

(iii) For $1 \leq i \leq n$, we define $\mathsf{can}(f^n_{i,1})$ to be the arrow

$$\mathsf{can}(I^{n-1}) \xrightarrow{\mathsf{can}(I^{i-1}) \otimes i_1 \otimes \mathsf{can}(I^{n-i})} \mathsf{can}(I^n)$$

of $\mathcal{A}$.

(iv) For $1 \leq i \leq n$, we define $\mathsf{can}(d_i^n)$ to be the arrow

$$\mathsf{can}(I^{n-1}) \xrightarrow{\mathsf{can}(I^{i-1}) \otimes d_i^n \otimes \mathsf{can}(I^{n-i})} \mathsf{can}(I^n)$$

of $\mathcal{A}$.

(v) For $q, r, s \geq 0$, and arrows

$$I^q \xrightarrow{g_0} I^r$$

and

$$I^r \xrightarrow{g_1} I^s$$

of $\square_s$, we define $\mathsf{can}(g_1 \circ g_0)$ to be $\mathsf{can}(g_1) \circ \mathsf{can}(g_0)$.

It remains to verify that this recipe indeed defines a functor from $\square$ to $\mathcal{A}$. Evidently, it defines a functor from $\mathcal{F}(\Upsilon')$ to $\mathcal{A}$. We now observe that, as in the proof of Proposition II.3.9.8, it follows immediately from the fact that $(\otimes, 1)$ defines a strict monoidal structure upon $\mathcal{A}$ that the defining relations of $\square$ give rise under $\mathsf{can}(-)$ to commutative diagrams in $\mathcal{A}$.

II.4.8. Cubical sets

II.4.8.1. Definition A *cubical set* is a presheaf on $\square$.

II.4.8.2. We may now ask the same questions as in II.3.10.3. Can we construct a homotopy theory of cubical sets? If so, how closely does it resemble the classical homotopy theory of topological spaces? This will be the topic of the remainder of the course.

Notes

1 For example, the first Čech cohomology group of the Warsaw circle is $\mathbb{Z}$, the integers.

2 There is a notion of homotopy equivalence with respect to which the Warsaw circle is homotopy equivalent to the circle, as we might intuitively expect: this is an aspect of *shape theory*. Very closely related ideas are involved in the theory of étale homotopy types of topoi, the original examples being the petit étale topoi arising in algebraic geometry. A reference for shape theory, whose presentation of the Warsaw circle I have borrowed, is the book [5] of Mardešić and Segal. A reference for étale homotopy theory is the book [1] of Artin and Mazur.

3 The space S^1_{fin} is sometimes referred to as the *pseudocircle*.

4 This recipe actually gives rise to one half of an equivalence of categories between finite posets and finite spaces, and more generally between arbitrary posets and Alexandroff spaces.

5 Explicitly, thinking of S^1 as a subspace of $\mathbb{R}^2$ in the usual way, we can take the map

$$S^1 \longrightarrow S^1_{fin}$$

to be given, as suggested by the picture, by

$$(x,y) \mapsto \begin{cases} b & \text{if } x < 0, \\ c & \text{if } x > 0, \\ a & \text{if } (x,y) = (0,1), \\ d & \text{if } (x,y) = (0,-1). \end{cases}$$

6 The homotopy theory of finite spaces is very interesting — every space is weakly homotopy equivalent to a finite space, which is a priori truly remarkable! For more on the use of the Sierpiński interval in the homotopy theory of finite spaces, see for example §4 of the paper [7] of Raptis.

7 The terminology *pointwise* is sometimes used, rather than levelwise.

8 In the process of freely adding colimits to our category, we may destroy any that already existed. If we wish to prescribe that a colimit of a diagram in $\mathcal{A}$ remains a colimit of the corresponding diagram in $\mathsf{Set}^{\mathcal{A}^{op}}$ under the Yoneda embedding, we must instead take sheaves on $\mathcal{A}$ with respect to an appropriate Grothendieck topology. This is a significant reason for the use of the Nisnevich topology in Voevodsky's approach to motivic homotopy theory, as discussed for example in the notes [9] of Weibel to a series of lectures of Voevodsky in Seattle.

9 This category is an example of a *comma category*.

10 If we view $\mathcal{D}$ as a comma category, this functor can be constructed canonically.

11 One can go far with a small rainbow of structures — in [11] it is proven that, assuming a certain strictness hypothesis, one can cook up a model structure from an interval equipped with contraction, involution, subdivision, and connection structures. In addition to [11] and the book [3] of Kamps and Porter, the reader can find the definitions of these structures in §2 of the paper [2] of Grandis.

III. Kan complexes and homotopy groups

III.1. An attempt to define homotopy groups of a cubical set

III.1.1. Getting to know cubical sets

III.1.1.1. Scholium Putting together Definition II.4.5.1 and Definition II.4.8.1, we have that a cubical set is the data of a set X_n for every $n \geq 0$, a map

$$X_n \xrightarrow{f_n^{i,\epsilon}} X_{n-1}$$

for every $1 \leq i \leq n$ and every $0 \leq \epsilon \leq 1$, and a map

$$X_{n-1} \xrightarrow{d_n^i} X_n$$

for every $1 \leq i \leq n$, such that the following hold.

(i) For any $n \geq 2$, $1 \leq i \leq n$, $1 \leq j \leq n-1$, and $0 \leq \delta, \epsilon \leq 1$, we have that

$$f_{n-1}^{j,\epsilon} \circ f_n^{i,\delta} = \begin{cases} f_{n-1}^{i-1,\delta} \circ f_n^{j,\epsilon} & \text{if } i > j, \\ f_{n-1}^{i,\epsilon} \circ f_n^{j+1,\epsilon} & \text{if } i \leq j. \end{cases}$$

(ii) For any $1 \leq i \leq n$ and $1 \leq j \leq n+1$, we have that

$$d_{n+1}^j \circ d_n^i = \begin{cases} d_{n+1}^i \circ d_n^{j-1} & \text{if } i < j, \\ d_{n+1}^{i+1} \circ d_n^{j,\epsilon} & \text{if } i \geq j. \end{cases}$$

(iii) For any $n \geq 2$, $1 \leq i, j \leq n$, and $0 \leq \delta \leq$, we have that

$$f_n^{j,\delta} \circ d_n^i = \begin{cases} id & \text{if } i = j, \\ d_{i-1}^{n-1} \circ f_{j,\delta}^{n-1} & \text{if } i > j, \\ d_i^{n-1} \circ f_{j-1,\delta}^{n-1} & \text{if } i < j. \end{cases}$$

III.1.1.2. Definition Let X be a cubical set. We refer to an element x of X_n as an *n-cube* of X. We refer to

$$X_n \xrightarrow{f_n^{i,\epsilon}} X_{n-1}$$

for any $1 \leq i \leq n$ and $0 \leq \epsilon \leq 1$ as a *face map*. Given an n-cube x of X, we refer to the $(n-1)$-cube $f_n^{i,\epsilon}(x)$ of X as the $(i,\epsilon)^{th}$*-face* of x, or less precisely as a *face* of x. We refer to

$$X_{n-1} \xrightarrow{d_n^i} X_n$$

for any $1 \leq i \leq n$ as a *degeneracy map*. Given an $(n-1)$-cube x of X, we refer to $d_i^n(x)$ as a *degenerate* n-cube of X. If an n-cube of X is not obtainable as $d_n^i(x)$ for some $(n-1)$-cube x of X and some $1 \leq i \leq n$, we refer to it as *non-degenerate*.

III.1.1.3. To elaborate upon Scholium III.1.1.1, given a cubical set X, the set X_n is the image of the object I^n of $\square$ under X. The face map

$$X_n \xrightarrow{f_n^{i,\epsilon}} X_{n-1}$$

for some $1 \leq i \leq n$ and $0 \leq \epsilon \leq 1$ is the image of the arrow

$$I^{n-1} \xrightarrow{f_{i,\epsilon}^n} I^n$$

of $\square$ under X. The degeneracy map

$$X_n \xrightarrow{d_n} X_{n-1}$$

for some $1 \leq i \leq n$ is the image of the arrow

$$I^n \xrightarrow{d_i^n} I^{n-1}$$

of $\square$ under X. If we view $\square$ as the free strict monoidal category upon an interval equipped with a contraction structure — Proposition II.4.7.6 — then the face map $f_n^{i,0}$ is the image under X of the arrow

$$I^{n-1} \xrightarrow{I^{i-1} \otimes i_0 \otimes I^{n-i}} I^n$$

of $\square$, and the face map $f_n^{i,1}$ is the image under X of the arrow

$$I^{n-1} \xrightarrow{I^{i-1} \otimes i_1 \otimes I^{n-i}} I^n$$

of $\square$. The degeneracy map d_n^i is the image under X of the arrow

$$I^n \xrightarrow{I^{i-1} \otimes p \otimes I^{n-i}} I^{n-1}$$

of $\square$.

III.1.1.4. Definition We denote by $\square^n$ the presheaf on $\square$ represented by I^n. We refer to $\square^n$ as the *free-standing n-cube.*

III.1.1.5. Explicitly, $\square^0$ consists of a single 0-cube, corresponding to the identity arrow

$$I^0 \xrightarrow{id} I^0$$

in $\square$, and a single degenerate n-cube for every $n > 0$. For example, the degenerate 1-cube corresponds to the arrow

$$I^1 \xrightarrow{p} I^0$$

of $\square$. Although there are two arrows

$$I^2 \longrightarrow I^1$$

in $\square$, the following diagram in $\square$ commutes, so that $\square^0$ has a single degenerate 2-cube.

$$\begin{array}{ccc} I^2 & \xrightarrow{I^1 \otimes p} & I^1 \\ {\scriptstyle p \otimes I^1}\downarrow & & \downarrow{\scriptstyle p} \\ I^1 & \xrightarrow[p]{} & I^0 \end{array}$$

III.1.1.6. In a similar way, $\square^1$ can explicitly be described as follows.

(i) There are two 0-cubes, corresponding to the arrows

$$I^0 \underset{i_1}{\overset{i_0}{\rightrightarrows}} I^1$$

of $\square$.

(ii) There is a single non-degenerate 1-cube, corresponding to the identity arrow

$$I^1 \xrightarrow{id} I^1$$

of $\square$.

(iii) There are two degenerate 1-cubes, given by the following two composite arrows of $\square$.

$$\begin{array}{ccc} I^1 & \xrightarrow{p} & I^0 \\ & {\scriptstyle i_0 \circ p}\searrow & \downarrow{\scriptstyle i_0} \\ & & I^1 \end{array} \qquad \begin{array}{ccc} I^1 & \xrightarrow{p} & I^0 \\ & {\scriptstyle i_1 \circ p}\searrow & \downarrow{\scriptstyle i_1} \\ & & I^1 \end{array}$$

.

(iv) All n-cubes for $n > 1$ are degenerate. For example, there are exactly four degenerate 2-cubes. Two of these are given by the arrows

$$I^2 \underset{p \otimes I^1}{\overset{I^1 \otimes p}{\rightrightarrows}} I^1$$

of $\square$. The other two are obtained by lifting each of the two degenerate 1-cubes of (iii) to a degenerate 2-cube — we obtain a single lift rather than two in each case for exactly the same reason we discussed in III.1.1.5 when considering the 2-cubes of $\square^0$.

III.1.1.7. To give one more example, $\square^2$ can explicitly be described as follows.

(i) There are four 0-cubes, arrived at by taking either route through the following commutative diagrams in $\square$.

$$\begin{array}{ccc} I^0 & \xrightarrow{i_0} & I^1 \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle I^1 \otimes i_0} \\ I^1 & \xrightarrow[i_0 \otimes I^1]{} & I^2 \end{array} \qquad \begin{array}{ccc} I^0 & \xrightarrow{i_1} & I^1 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle I^1 \otimes i_1} \\ I^1 & \xrightarrow[i_1 \otimes I^1]{} & I^2 \end{array}$$

$$\begin{array}{ccc} I^0 & \xrightarrow{i_0} & I^1 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle I^1 \otimes i_1} \\ I^1 & \xrightarrow[i_0 \otimes I^1]{} & I^2 \end{array} \qquad \begin{array}{ccc} I^0 & \xrightarrow{i_1} & I^1 \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle I^1 \otimes i_0} \\ I^1 & \xrightarrow[i_1 \otimes I^1]{} & I^2 \end{array}$$

(ii) There are four non-degenerate 1-cubes, corresponding to the arrows

$$I^1 \underset{I^1 \otimes i_1}{\overset{I^1 \otimes i_0}{\rightrightarrows}} I^2$$

and

$$I^1 \underset{i_1 \otimes I^1}{\overset{i_0 \otimes I^1}{\rightrightarrows}} I^2$$

of $\square$.

(iii) There are four degenerate 1-cubes, corresponding to the four 0-cubes of $\square^2$ in the same way as the degenerate 1-cubes of $\square^1$ corresponded in III.1.1.6 to the 0-cubes of $\square^1$.

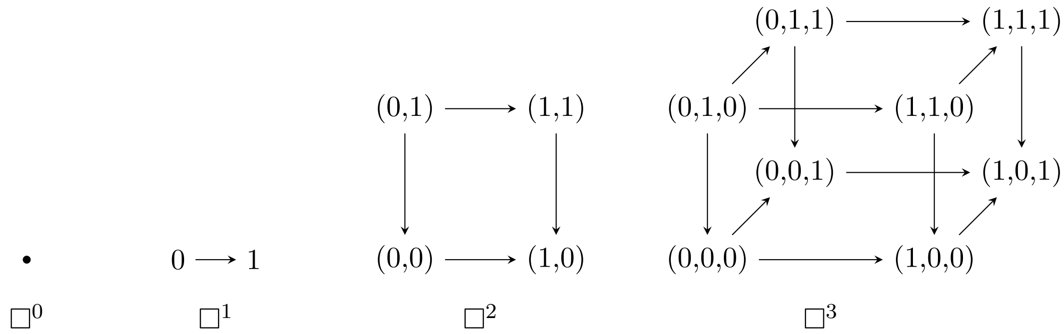
(iv) There is a single non-degenerate 2-cube of $\square^2$, corresponding to the identity arrow

$$I^2 \xrightarrow{id} I^2$$

of $\square$.

(v) All n-cubes for $n > 2$ are degenerate.

III.1.1.8. We think of the cubical sets $\square^n$ for $n \geq 0$ as combinatorial analogues of the topological n-cubes. Ignoring degenerate i-cubes, we may depict $\square^n$ as follows in low dimensions, thinking of the square in the picture of $\square^2$ as solid, and similarly thinking of the cube and the squares in the picture of $\square^3$ as solid.

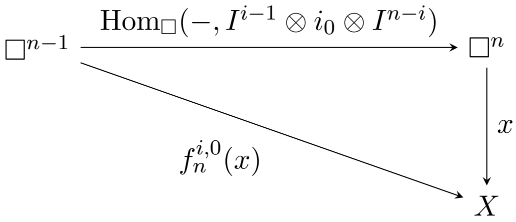


III.1.1.9. By a *morphism* from a cubical set X_0 to a cubical set X_1 we will mean a natural transformation from X_0 to X_1. By the *category of cubical sets* we will mean the category $\mathsf{Set}^{\square^{op}}$.

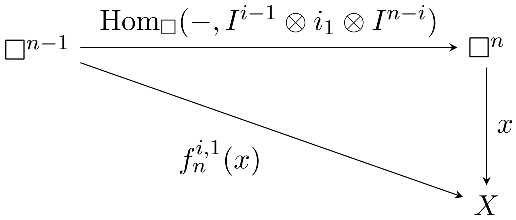
III.1.1.10. Let X be a cubical set, and let x be an n-cube of X. Now that we are acquainted with the free-standing n-cubes, we can understand x, its faces, and the degenerate $(n+1)$-cubes it gives rise to, more geometrically. Indeed, by the Yoneda lemma, x corresponds uniquely to a morphism

$$\square^n \longrightarrow X$$

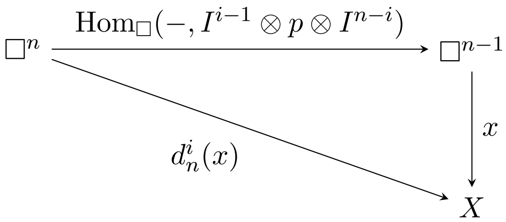
which we will also denote by x. Appealing to the Yoneda lemma once more, we have that the following diagram in $\mathsf{Set}^{\square^{op}}$ commutes for every $1 \leq i \leq n$.



Similarly we have that the following diagram in $\mathsf{Set}^{\square^{op}}$ commutes for every $1 \leq i \leq n$.



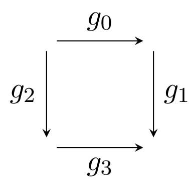
Finally we have that the following diagram in $\mathsf{Set}^{\square^{op}}$ commutes for every $1 \leq i \leq n$.



III.1.1.11. Let X be a cubical set. Taking into account III.1.1.10, we will frequently depict a 1-cube x of X as

$$x_0 \longrightarrow x_1,$$

if x_0 is $d_1^0(x)$, and x_1 is $d_1^1(x)$. Similarly, we will frequently depict a 2-cube x of X as follows.



Here g_0 is $f_2^{2,0}(x)$, g_1 is $f_2^{1,1}(x)$, g_2 is $f_2^{1,0}(x)$, and g_3 is $f_2^{2,1}(x)$. We might also depict the faces of g_1, g_2, g_3, and g_4 as vertices of this square.

III.1.2. A moral

We see that cubical sets can be thought of from two points of view — as algebraic gadgets consisting of specific data satisfying specific axioms, or as combinatorial encodings of geometric gadgets. As we proceed we will typically explore an idea geometrically in low dimensions, before working out the necessary combinatorics to express our idea in higher dimensions. In short — we shall think geometrically, and prove algebraically!

III.1.3. Towards homotopy groups of a cubical set

III.1.3.1. If we reflect upon the definition of the homotopy group $\pi_n(X, x)$ of a pointed topological space (X, x) from a cubical point of view, we see that there are three ingredients.

(i) Maps from the topological n-cube I^n to X.

(ii) A condition that the boundary of I^n be mapped to x under a map from I^n to X.

(iii) Homotopies between maps from I^n to X satisfying the requirement of (ii).

III.1.3.2. Let X be a cubical set. In this setting, we have arrived at the analogue of (i) — a morphsm from $\square^n$ to X¡ or in other words an n-cube of X, namely an element of X_n. In particular, let us define a *point* of X to be a 0-cube of X.

III.1.3.3. Let us now turn to an analogue of (ii) in our abstract setting. As we have already touched upon in III.1.1.5, the relations defining a cubical set — see III.1.1.1 — ensure that

$$d_n^{i_n} d_{n-1}^{i_{n-1}} \cdots d_1^1(x)$$

defines the same n-cube of X for any 0-cube x of X and any choice of $1 \leq i_j \leq j$ for $1 \leq j \leq n$. We will thus simply denote this n-cube by x.

III.1.3.4. Let X be a cubical set, and let $*$ be a point of X, and let x be a morphism

$$\square^n \longrightarrow X$$

of cubical sets. Appealing to our observation in III.1.3.3, we can express the idea that the boundary of x is mapped to $*$ as the requirement that $f_n^{i,\epsilon}(x) = *$ for every $1 \leq i \leq n$ and $0 \leq \epsilon \leq 1$. We will later — see Definition III.2.1.2 —- that we can rigorously define the boundary of $\square^n$ as a cubical set $\partial\square^n$, which will allow to re-formulate this definition slightly more geometrically, in III.2.1.4.

III.1.3.5. Let X be a cubical set, and let $*$ be a point of X. With our abstract understanding of (ii) in hand, let us define

$$Z_n(X,*) = \{x \in X_n \mid f_n^{i,\epsilon}(x) = * \text{ for all } 1 \leq i \leq n \text{ and } 0 \leq \epsilon \leq 1\}.$$

When $n = 0$ we interpret the conditions to be vacuous, so that $Z_0(X,*)$ is X_0. By analogy with (iii) above, we would like to define an equivalence relation upon $Z_n(X,*)$ asserting that two n-cubes

$$\square^n \underset{x_1}{\overset{x_0}{\rightrightarrows}} X$$

are homotopic, in an appropriate sense. We would then define $\pi_n(X,*)$ to be the quotient of $Z_n(X,*)$ by this equivalence relation, and hope to equip this quotient with a group structure, by further analogy with topology.

III.1.4. Combinatorially homotopic n-cubes in a cubical set

III.1.4.1. The interval in $\square$ equipped with its contraction structure gives rise to an interval equipped with a contraction structure in $\mathsf{Set}^{\square^{op}}$. Moreover, the monoidal structure upon $\square$ gvies rise to a monoidal structure upon $\mathsf{Set}^{\square^{op}}$. Thus we arrive — see II.3.3 — at a natural notion of homotopy in $\mathsf{Set}^{\square^{op}}$.

III.1.4.2. We will give the details later in the course. Let $(X,*)$ be a pointed cubical set, and let x_0 and x_1 be n-cubes of X belonging to $Z_n(X,*)$ — intuitively, n-cubes whose boundary is trivial. For now, let us take it on faith that a homotopy from the morphism

$$I^n \xrightarrow{x_0} X$$

of cubical sets to the morphism

$$I^n \xrightarrow{x_1} X$$

of cubical sets is exactly a morphism

$$I^{n+1} \xrightarrow{h} X$$

of cubical sets such that the following diagram in $\mathsf{Set}^{\square^{op}}$ commute

$$\begin{array}{ccc} I^n & \xrightarrow{I^n \otimes i_0} & I^{n+1} \\ & {\scriptstyle x_0}\searrow & \downarrow{\scriptstyle h} \\ & & X \end{array} \qquad \begin{array}{ccc} I^n & \xrightarrow{I^n \otimes i_1} & I^{n+1} \\ & {\scriptstyle x_1}\searrow & \downarrow{\scriptstyle h} \\ & & X \end{array}$$

and moreover the following diagrams in $\mathsf{Set}^{\square^{op}}$ commute for every $1 \leq i \leq n$.

$$\begin{array}{ccc} I^n & \xrightarrow{I^{i-1}\otimes i_0 \otimes I^{n-i}} & I^{n+1} \\ & \searrow^{p} & \downarrow h \\ & & X \end{array} \qquad \begin{array}{ccc} I^n & \xrightarrow{I^{i-1}\otimes i_1 \otimes I^{n-i}} & I^{n+1} \\ & \searrow^{*} & \downarrow h \\ & & X \end{array}$$

III.1.4.3. Thus we are led to the following definition[12] Let $(X, *)$ be a pointed cubical set, and let x_0 and x_1 be n-cubes of X belonging to $Z_n(X, *)$, for any $n \geq 0$. Then we define $x_0 \sim x_1$ if there is an $(n+1)$-cube h of X such that

$$f^{i,\epsilon}_{n+1}(h) = \begin{cases} x_0 & \text{if } i = n+1 \text{ and } \epsilon = 0, \\ x_1 & \text{if } i = n+1 \text{ and } \epsilon = 1, \\ * & \text{if } i \neq n. \end{cases}$$

In this case we write that x_0 is *homotopic* to x_1, and that h defines a *homotopy* from x_0 to x_1.

III.1.4.4. Let X be a cubical set, and let x_0 and x_1 be 0-cubes of X. Then $x_0 \sim x_1$ if there is a 1-cube h of X with the boundary depicted below.

$$x_0 \xrightarrow{h} x_1$$

III.1.4.5. Let X be a cubical set, let $*$ be a 0-cube of X, and let

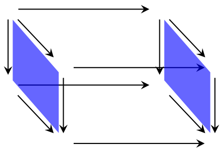
$$* \underset{x_1}{\overset{x_0}{\rightrightarrows}} *$$

be 1-cubes of X belonging to $Z_1(X, *)$. Then $x_0 \sim x_1$ if there is a 2-cube h of X with the boundary depicted below — the unlabelled 1-cubes are degenerate.

$$\begin{array}{ccc} * & \xrightarrow{x_0} & * \\ \downarrow & & \downarrow \\ * & \xrightarrow[x_1]{} & * \end{array}$$

Here we think of the upper horizontal face as $f^{2,0}_2(h)$, of the lower horizontal face as $f^{2,1}_2(h)$, of the left vertical face as $f^{1,0}_2(h)$, and of the right vertical face as $f^{1,1}_2(h)$. The reader may compare with II.3.6 — there is an arbitrary choice involved when drawing the picture.

III.1.4.6. Let X be a cubical set, let $*$ be a 0-cube of X, and let x_0 and x_1 be 2-cubes of X belonging to $Z_2(X, *)$. Then $x_0 \sim x_1$ if there is a 3-cube h of X with the following boundary, in which the left shaded 2-cube is x_0, the right shaded 2-cube is x_1, and the remaining four 2-cubes are degenerate, as are all the 1-cubes.



Here we think of the left, shaded face as $f_3^{3,0}(h)$, of the right, shaded face as $f_3^{3,1}(h)$, of the back face as $f_3^{1,0}(h)$, of the front face as $f_3^{1,1}(h)$, of the top face as $f_3^{2,0}(h)$, and of the bottom face as $f_3^{2,1}(h)$.

III.1.4.7. Let X be a cubical set, and let $*$ be a point of X. In order to fulfill our plan — see III.1.3.5 — for the definition of $\pi_n(X, *)$, we must have that $\sim$ defines an equivalence relation upon $Z_n(X, *)$. However, this is not the case for an arbitrary cubical set!

III.1.4.8. The degeneracy maps ensure that $\sim$ is reflexive. Indeed, if x is an n-cube of a cubical set X, then $d_{n+1}^{n+1}(x)$ defines a homotopy from x to itself. Recall that our topological motivation for introducing degeneracies was exactly in order to ensure this — see II.4.2.

III.1.4.9. However, $\sim$ is not symmetric. For a minimal counter-example, let X denote the cubical set uniquely defined by the following recipe[13].

(i) We have a single 0-cube $*$.

(ii) We have two non-degenerate 1-cubes

$$* \begin{array}{c} \xrightarrow{x_0} \\ \xrightarrow[x_1]{} \end{array} *,$$

.

(iii) We have a single non-degenerate 2-cube x with the boundary depicted below, in which the vertical 1-cubes are degenerate.

$$\begin{array}{ccc} * & \xrightarrow{x_0} & * \\ \downarrow & & \downarrow \\ * & \xrightarrow[x_1]{} & * \end{array}$$

(iv) Every n-cube for $n > 2$ is degenerate.

Then we have that $x_0 \sim x_1$ by virtue of the 2-cube x, but since there is no 2-cube with the following boundary

$$\begin{array}{ccc} * & \xrightarrow{x_1} & * \\ \downarrow & & \downarrow \\ * & \xrightarrow[x_0]{} & * \end{array}$$

we do not have that $x_1 \sim x_0$.

III.1.4.10. Morevoer $\sim$ is not transitive. For a minimal counter-example, let X denote the cubical set uniquely defined by the following recipe.

(i) We have a single 0-cube $*$.

(ii) We have exactly three non-degenerate 1-cubes x_0, x_1, and x_2, all with the following boundary.

$$* \longrightarrow *$$

(iii) We have exactly two non-degenerate 2-cubes: one, which we will denote by x, with the boundary depicted below

$$\begin{array}{ccc} * & \xrightarrow{x_0} & * \\ \downarrow & & \downarrow \\ * & \xrightarrow[x_1]{} & * \end{array}$$

and one, which we will denote by x', with the boundary depicted below.

$$\begin{array}{ccc} * & \xrightarrow{x_1} & * \\ \downarrow & & \downarrow \\ * & \xrightarrow[x_2]{} & * \end{array}$$

(iv) Every n-cube for $n > 2$ is degenerate.

Then $x_0 \sim x_1$ by virtue of the 2-cube x, and $x_1 \sim x_2$ by virtue of the 2-cube x', but since there is no 2-cube with the boundary

$$\begin{array}{ccc} * & \xrightarrow{x_0} & * \\ \downarrow & & \downarrow \\ * & \xrightarrow[x_2]{} & * \end{array}$$

we do not have that $x_0 \sim x_2$.

III.1.4.11. If we were to work not with cubical sets but with presheaves on the free strict monoidal category on an interval equipped with not only a contraction structure but also an *involution* structure allowing us to reverse homotopies — we already proposed this as a possibility in II.4.4 — our homotopy relation $\sim$ would be symmetric. Transitivity is much more involved — we cannot ensure that it holds by equipping $\square$ with more structure, as discussed in II.4.4.

III.1.4.12. A profound insight of Kan — which goes back to the paper [4] — is that although $\sim$ may not define an equivalence relation for arbitrary cubical sets, it will define an equivalence relation for cubical sets satisfying a certain extension property, now known as Kan complexes. It is to introducing these gadgets that we now turn.

III.2. Greeting Kan complexes

III.2.1. Horns and boundaries

III.2.1.1. We have become acquainted with the free-standing n-cubes $\square^n$. Let us now define a couple more families of cubical sets which will be of vital importance to us.

III.2.1.2. Definition For any $n \geq 1$, we denote by $\partial\square^n$ the cubical set obtained as follows, and refer to it as *boundary of the free-standing n-cube.* Let x denote the unique non-degenerate n-cube of $\square^n$.

(i) We remove x from $\square^n$.

(ii) For every $m \geq 1$, we remove the $(n+m)$-cube

$$d^{i_m}_{n+m} d^{i_{m-1}}_{n+m-1} \cdots d^{i_1}_{n+1}(x)$$

from $\square^n$, for every choice of $1 \leq i_j \leq n + j$ for $1 \leq j \leq m$.

III.2.1.3. If we ignore keeping track of degeneracies, the boundary of $\square^n$ exactly corresponds to our topological intuition. Thinking of $\square^1$ as

$$0 \longrightarrow 1$$

as in III.1.1.8, we have that $\partial\square^1$ is the coproduct of two copies of $\square^0$, corresponding to the 0-cubes 0 and 1 of $\square^1$. Thinking of $\square^2$ as a solid square

$$\begin{array}{ccc} (0,1) & \longrightarrow & (1,1) \\ \downarrow & & \downarrow \\ (0,0) & \longrightarrow & (1,0) \end{array}$$

as in III.1.1.8, $\partial\square^2$ can be pictured in the same way, except that we no longer think of there being a 2-cube filling in the square. We could obtain $\partial\square^n$ by glueing together four copies of $\square^1$.

III.2.1.4. Let X be a cubical set, and let $*$ be a 0-cube of X. We have that an n-cube x of X for any $n \geq 1$ belongs to $Z_n(X,*)$, namely has trivial boundary, if and only if the following diagram in $\mathsf{Set}^{\square^{op}}$ commutes.

$$\begin{array}{ccc} \partial\square^n & \hookrightarrow & \square^n \\ \downarrow & & \downarrow x \\ \square^0 & \xrightarrow[*]{} & X \end{array}$$

Here the morphism

$$\partial\square^n \hookrightarrow \square^n$$

is the evident inclusion, and the morphism

$$\partial\square^n \longrightarrow \square^0$$

is canonical, observing that $\square^0$ is a final object of $\mathsf{Set}^{\square^{op}}$.

III.2.1.5. Definition For any $1 \leq i \leq n$ and $0 \leq \epsilon \leq 1$, we denote by $\sqcap^n_{i,\epsilon}$ the cubical set obtained as follows, and refer to it as the $(i,\epsilon)^{\text{th}}$ *n-horn*. Let x denote the unique non-degenerate n-cube of $\square^n$.

(i) We remove the $(n-1)$-cube $f^{i,\epsilon}_n(x)$ from $\partial\square^n$.

(ii) For every $m \geq 0$, we remove the $(n+m)$-cube

$$d^{i_m}_{n+m} d^{i_{m-1}}_{n+m-1} \cdots d^{i_0}_n \big(f^{i,\epsilon}_n(x)\big)$$

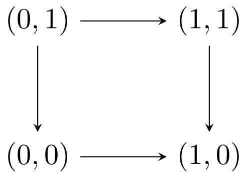
of $\square^n$ for any choice of $1 \leq i_j \leq n+j$ for $0 \leq j \leq m$.

III.2.1.6. The 1-horns are both, up to isomorphism, $\square^0$. If we think of $\square^1$ as

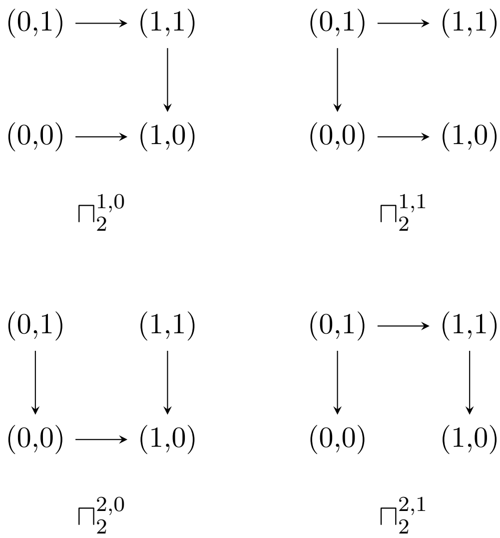
$$0 \longrightarrow 1$$

as usual, we can think of $\sqcap^{1,0}_1$ as corresponding to 1, and to $\sqcap^{1,1}_1$ as corresponding to 0.

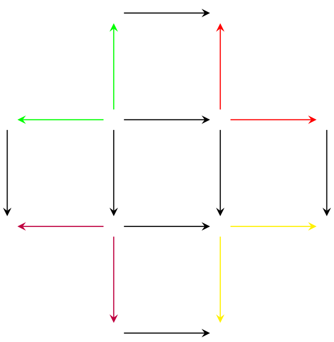
III.2.1.7. There are four 2-horns, one for each face of a square. Let us think of $\square^2$ as follows.



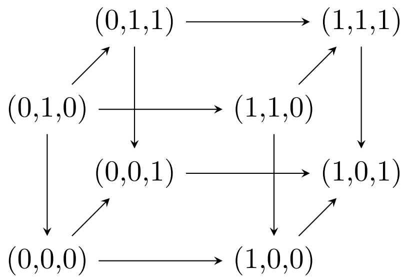
Then the four 2-horns are as depicted below.



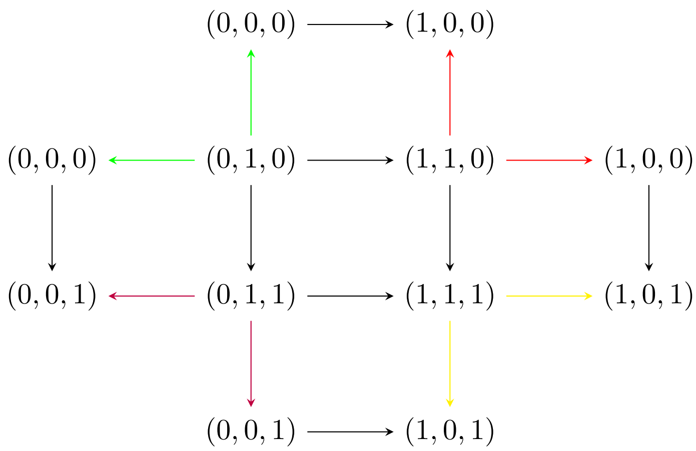
III.2.1.8. There are six 3-horns, one for each face of a cube. We will frequently need to construct 3-horns in which the five non-degenerate 2-cubes are labelled, and it will not be possible for us to indicate this clearly by attempting to depict the a 3-horn as a three dimensional gadget. Instead, we will depict 3-horns as 'nets' of five 2-cubes, as shown below — as recipes for assembling a cube-with-one-face-missing. The colours indicate 1-cubes which must be the same.



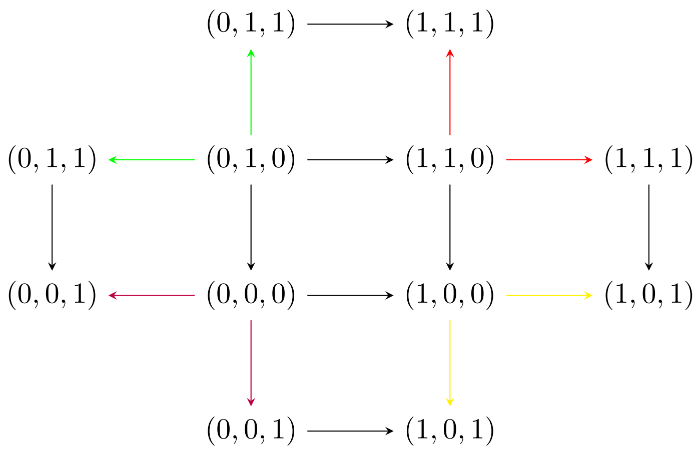
Let us think of the free-standing 3-cube in the following way.



Then we will depict the horn $\sqcap_3^{2,1}$ in the manner shown below — the bottom face of the cube is that which is missing when we assemble our 3-horn from this net.



To give another example, we will depict the horn $\sqcap_3^{3,1}$ as follows — the back face of the cube is that which is missing when we assemble our 3-horn from this net.



III.2.1.9. Definition Let X be a cubical set. A 3-*horn in* X is a morphism

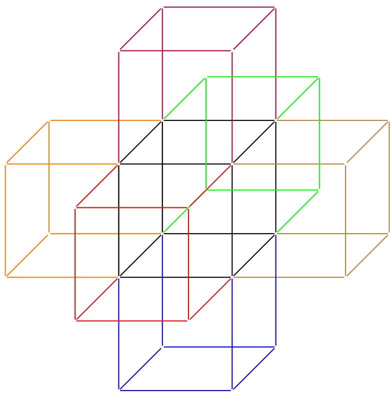
$$\sqcap_3^{i,\epsilon} \longrightarrow X$$

for some $1 \leq i \leq n$ and $0 \leq \epsilon \leq 1$.

III.2.1.10. Typically, when depicting a 3-horn in a cubical set X by a net, we will wish to label one or more 2-cubes. We will do this by placing a label at the centre of the relevant squares in the net. When we describe a 3-horn this way, the reader may find it helpful at first to try to draw for themself the 3-horn as a three dimensional gadget, namely a cube-with-one-face missing, and with some of the other five faces labelled.

III.2.1.11. The desperate reader, harkening back to the days of barneskole when mathematics was more hands-on, might find it helpful to cut out the horn along its boundary, and assemble it into an actual cube-with-one-face-missing!

III.2.1.12. We can the lift the idea of presenting 3-horns by two dimensional nets to allow us to visualise a 4-horn, which is a four dimensional gadget. We will not need this in the present lecture, but it will be helpful to us later on. For now, let us simply observe that a 'net' for a 4-horn is an arrangement of seven 3-cubes as below such that any two 2-cubes meeting orthogonally in the boundary of this gadget are the same. Each cube is coloured differently in the picture, in an attempt to make it easier to distinguish between them!



III.2.2. Kan complexes and the free-standing n-cubes

III.2.2.1. Definition Let X be a cubical set. Then X is a *Kan complex* if[14] for every $1 \leq i \leq n$ and $0 \leq \epsilon \leq 1$, and every morphism

$$\sqcap_{i,\epsilon}^{n} \xrightarrow{g} X$$

of cubical sets, there is a morphism

$$\square^n \xrightarrow{g'} X$$

of cubical sets such that the following diagram in $\mathsf{Set}^{\square^{op}}$ commutes.

$$\begin{array}{ccc} \sqcap^n_{i,\epsilon} & \hookrightarrow & \square^n \\ & \overset{g}{\searrow} & \downarrow g' \\ & & X \end{array}$$

III.2.2.2. Let X be a cubical set, and let

$$\sqcap^{i,\epsilon}_n \xrightarrow{g} X$$

be an n-horn in X. If there is an n-cube

$$\square^n \xrightarrow{g'} X$$

of X such that the diagram

$$\begin{array}{ccc} \sqcap^n_{i,\epsilon} & \hookrightarrow & \square^n \\ & \overset{g}{\searrow} & \downarrow g' \\ & & X \end{array}$$

in $\mathsf{Set}^{\square^{op}}$ commutes, we write that g *extends to* g'.

III.2.2.3. The free-standing 0-cube $\square^0$ is a Kan complex. Indeed, recalling the nxplicit description of $\square^0$ in III.1.1.5, there is a unique morphism

$$\sqcap^n_{i,\epsilon} \xrightarrow{g} \square^0$$

of cubical sets for any $1 \leq i \leq n$ and $0 \leq \epsilon \leq 1$, and a unique morphism

$$\square^n \xrightarrow{g'} \square^0$$

for any $n \geq 0$. We have that the following diagram in $\mathsf{Set}^{\square^{op}}$ commutes.

$$\begin{array}{ccc} \sqcap^n_{i,\epsilon} & \hookrightarrow & \square^n \\ & \overset{g}{\searrow} & \downarrow g' \\ & & \square^0 \end{array}$$

III.2.2.4. The free standing n-cube $\square^n$ is not a Kan complex for any $n \geq 1$. For instance, let us consider the possible 2-horns in $\square^1$. Let us view $\square^1$ as follows.

$$0 \xrightarrow{x} 1$$

Firstly, we have the following 2-horns, all of which can be extended to degenerate 2-cubes of $\square^1$. The unlabelled 1-cubes are degenerate.

$$\begin{array}{ccc} 0 & \xrightarrow{x} & 1 \\ \downarrow & & \downarrow \\ 0 & & 1 \end{array} \qquad \begin{array}{ccc} 0 & & 1 \\ \downarrow & & \downarrow \\ 0 & \xrightarrow[x]{} & 1 \end{array} \qquad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ {\scriptstyle x}\downarrow & & \\ 1 & \longrightarrow & 1 \end{array} \qquad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x} \\ 0 & \longrightarrow & 1 \end{array}$$

Secondly, we have the following 2-horns.

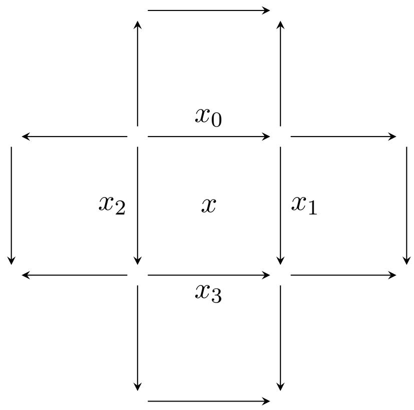
$$\begin{array}{ccc} 0 & \xrightarrow{x} & 1 \\ {\scriptstyle x}\downarrow & & \downarrow \\ 1 & & 1 \end{array} \qquad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow{\scriptstyle x} \\ 0 & \xrightarrow[x]{} & 1 \end{array}$$

Neither of these two 2-horns can be extended to a 2-cube of $\square^1$. Indeed, the 2-cubes of $\square^1$ are all degenerate, and degenerate 2-cubes cannot have a boundary as above.

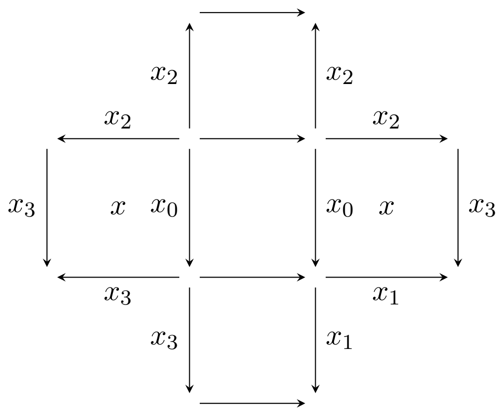
III.2.2.5. The same reasoning on 2-horns shows that $\square^n$ is not a Kan complex for $n \geq 1$. It is in fact only the 2-horns of $\square^3$ that cannot be extended! For instance, the possible 3-horns in $\square^2$ fall into two types. Let us view $\square^2$ as follows.

$$\begin{array}{ccc} & \xrightarrow{x_0} & \\ {\scriptstyle x_2}\downarrow & x & \downarrow{\scriptstyle x_1} \\ & \xrightarrow[x_3]{} & \end{array}$$

Firstly, we have those 3-horns in which only copy of the non-degenerate 2-cube x appears, described by the following net — the different horns correspond to the different choices of which face of the cube to take as x when assembling the net. All the unlabelled 2-cubes and 1-cubes are degenerate.



The 3-horns of this type can be extended to degenerate 3-cubes of $\square^2$. Secondly, we have the following 3-horns.



Again, all 3-horns of this type can be extended to 3-cubes of $\square^2$. It is not possible to cook up any 3-horn in which two copies of X meet at a common face — from the point of view of nets, the condition that two 1-cubes meeting at right angles in the boundary of the net must be the same cannot be satisified if two copies of x are adjacent. A similar story holds in higher dimensions.

III.2.2.6. We draw the attention of the reader acquainted with simplicial sets that whether or not the representable presheaves are Kan complexes exhibits a difference between the simplicial and cubical settings — the free-standing 1-simplex Δ^1*is* a Kan complex. [15].

III.2.2.7. Although $\square^n$ is not a Kan complex for any $n \geq 1$, we might wonder whether we could not simply add in 'extra' degeneracies to fill in the 2-horns which currently cannot be filled in. It turns out that this is a very good idea to pursue in cubical homotopy theory — let us discuss it, before renewing our efforts to define the homotopy groups of a cubical set, this time for Kan complexes.

III.3. Cubical sets with connections

III.3.1. Connections in topology

III.3.1.1. In the classical homotopy theory of spaces, many deeper theorems rely not only the structure of the topological interval $\left(I^0, I^1, i_0, i_1, p\right)$ equipped with its contraction structure — and also further structure allowing us to reverse and compose homotopies, for which we refer to [11] for more — but on the structure of the unit square I^2. An example is Dold's theorem on homotopy equivalences under an object, which allows us to conclude that a trivial cofibration is a section of a strong deformation retraction.

III.3.1.2. In proving these deeper results, we typically construct certain *double homotopies*, namely maps

$$X \times I^2 \longrightarrow Y$$

for some spaces X and Y. Rather beautifully, it turns out that we can build all the double homotopies we need from only two maps.

$$I^2 \underset{\Gamma_1}{\overset{\Gamma_0}{\rightrightarrows}} I$$

III.3.1.3. Before we describe, let us re-cast in topology the question of 'filling in' the 2-horns which in III.2.2.4 prevented $\square^1$ from being a Kan complex. Suppose that we have a path

$$I^1 \xrightarrow{f} X$$

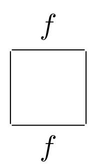
in a space X. Pre-composing with the map

$$I^2 \xrightarrow{I \times p} I$$

gives a map

$$I^2 \longrightarrow X$$

with the following boundary, in which the vertical sides are constant paths.



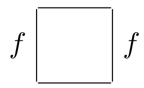
This corresponds to the degenerate 2-cube $d_2^2(x)$ for a 1-cube x of a cubical set X. Alternatively, pre-composing with the map

$$I^2 \xrightarrow{p \times I} I$$

gives a map

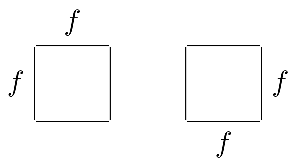
$$I^2 \longrightarrow X$$

with the following boundary, in which the horizontal sides are constant paths.



This corresponds to the degenerate 2-cube $d_2^1(x)$ for a 1-cube x of a cubical set X.

III.3.1.4. There are two other possible boundary configurations whose faces are either f or a constant path, namely the following.



III.3.1.5. Can we 'fill in' these boundary configurations in topology? To be more precise, can we construct a map

$$I^2 \longrightarrow I$$

whose composition with f will yield a double homotopy with boundary as in the leftmost of the above pictures? Can we construct a map

$$I^2 \longrightarrow I$$

whose composition with f will yield a double homotopy with boundary as in the rightmost of the above pictures?

III.3.1.6. The answer to both of these questions is — yes! For the picture on the left, we can take the map to be $(x, y) \mapsto x + y - xy$. This can be taken to be the map Γ_0 that we introduced in III.3.1.2. For the picture on the right, we can take the map to be $(x, y) \mapsto xy$. This can be taken to be the map Γ_1 that we introduced in III.3.1.2.

III.3.2. Connections in abstract homotopy theory

III.3.2.1. We refer to Γ_0 and Γ_1 as *connections*. As we have discussed, connections play a crucial role from an abstract point of view in homotopy theory — we refer to [11] for more, in which connections are involved throughout. Indeed, the whole character of classical homotopy theory — the reason for its richness — can be seen from an abstract point of view to rely indispensably on the structure not only of the unit intercal I^1 but the unit square I^2. Thus we can readily believe that it may be useful to expand upon our category of cubical shapes to include connections.

III.3.2.2. We can capture the fact that Γ_0 has the boundary we are looking for by the observation that the following diagrams in spaces commute.

$$\begin{array}{ccc} I^1 & \xrightarrow{I^1 \times i_0} & I^2 \\ & \overset{id}{\searrow} & \downarrow{\scriptstyle \Gamma_0} \\ & & I^1 \end{array} \qquad \begin{array}{ccc} I^1 & \xrightarrow{i_0 \times I^1} & I^2 \\ & \overset{id}{\searrow} & \downarrow{\scriptstyle \Gamma_0} \\ & & I^1 \end{array}$$

$$\begin{array}{ccc} I^1 & \xrightarrow{I^1 \times i_1} & I^2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Gamma_0} \\ I^0 & \xrightarrow[i_1]{} & I^1 \end{array} \qquad \begin{array}{ccc} I^1 & \xrightarrow{i_1 \times I^1} & I^2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Gamma_0} \\ I^0 & \xrightarrow[i_1]{} & I^1 \end{array}$$

III.3.2.3. This allows us to pass to the abstract setting — introduced in II.3.2 — of an interval $\widehat{I} = \left(1, I^1, i_0, i_1\right)$ in a category $\mathcal{A}$ equipped with a monoidal structure $(\otimes, 1, \lambda, \rho, \alpha)$. Indeed, we define an *upper connection structure* upon $\widehat{I}$ to be an arrow

$$I^1 \otimes I^1 \xrightarrow{\Gamma_0} I^1$$

such that the following diagrams in $\mathcal{A}$ commute.

$$\begin{array}{ccc} I^1 & \xrightarrow{I^1 \otimes i_0} & I^1 \otimes I^1 \\ & \overset{id}{\searrow} & \downarrow{\scriptstyle \Gamma_0} \\ & & I^1 \end{array} \qquad \begin{array}{ccc} I^1 & \xrightarrow{i_0 \otimes I^1} & I^1 \otimes I^1 \\ & \overset{id}{\searrow} & \downarrow{\scriptstyle \Gamma_0} \\ & & I^1 \end{array}$$

$$
\begin{array}{ccc}
I^1 & \xrightarrow{I^1 \otimes i_1} & I^1 \otimes I^1 \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Gamma_0} \\
I^0 & \xrightarrow[i_1]{} & I^1
\end{array}
\qquad\qquad
\begin{array}{ccc}
I^1 & \xrightarrow{i_1 \otimes I^1} & I^1 \otimes I^1 \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Gamma_0} \\
I^0 & \xrightarrow[i_1]{} & I^1
\end{array}
$$

III.3.2.4. Similarly, we can capture the fact that Γ_1 has the boundary we are looking for by the observation that the following diagrams in spaces commute.

$$
\begin{array}{ccc}
I^1 & \xrightarrow{I^1 \times i_1} & I^2 \\
 & {\scriptstyle id}\searrow & \downarrow{\scriptstyle \Gamma_1} \\
 & & I^1
\end{array}
\qquad\qquad
\begin{array}{ccc}
I^1 & \xrightarrow{i_1 \times I^1} & I^2 \\
 & {\scriptstyle id}\searrow & \downarrow{\scriptstyle \Gamma_1} \\
 & & I^1
\end{array}
$$

$$
\begin{array}{ccc}
I^1 & \xrightarrow{I^1 \times i_0} & I^2 \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Gamma_1} \\
I^0 & \xrightarrow[i_0]{} & I^1
\end{array}
\qquad\qquad
\begin{array}{ccc}
I^1 & \xrightarrow{i_0 \times I^1} & I^2 \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Gamma_1} \\
I^0 & \xrightarrow[i_0]{} & I^1
\end{array}
$$

III.3.2.5. This allows us to pass to the abstract setting of an interval $\widehat{I} = \big(1, I^1, i_0, i_1\big)$ in a category $\mathcal{A}$ equipped with a monoidal structure $(\otimes, 1, \lambda, \rho, \alpha)$. Indeed, we define an *upper connection structure* upon $\widehat{I}$ to be an arrow

$$I^1 \otimes I^1 \xrightarrow{\Gamma_1} I^1$$

such that the following diagrams in $\mathcal{A}$ commute.

$$
\begin{array}{ccc}
I^1 & \xrightarrow{I^1 \otimes i_1} & I^2 \\
 & {\scriptstyle id}\searrow & \downarrow{\scriptstyle \Gamma_1} \\
 & & I^1
\end{array}
\qquad\qquad
\begin{array}{ccc}
I^1 & \xrightarrow{i_1 \otimes I^1} & I^2 \\
 & {\scriptstyle id}\searrow & \downarrow{\scriptstyle \Gamma_1} \\
 & & I^1
\end{array}
$$

$$\begin{CD} I^1 @>{I^1 \otimes i_0}>> I^2 \\ @V{p}VV @VV{\Gamma_1}V \\ I^0 @>>{i_0}> I^1 \end{CD} \qquad \begin{CD} I^1 @>{i_0 \otimes I^1}>> I^2 \\ @V{p}VV @VV{\Gamma_1}V \\ I^0 @>>{i_0}> I^1 \end{CD}$$

III.3.2.6. In spaces, we have that pre-composing a constant path with Γ_0 or Γ_1 gives a constant map

$$I^2 \longrightarrow X.$$

Abstractly, we can capture this for Γ_0 by observing that the following diagram in the category of spaces commutes.

$$\begin{CD} I^2 @>{\Gamma_0}>> I \\ @V{I \times p}VV @VV{p}V \\ I @>>{p}> I \end{CD}$$

This is equivalent to the commutativity of the following diagram.

$$\begin{CD} I^2 @>{\Gamma_0}>> I \\ @V{p \times I}VV @VV{p}V \\ I @>>{p}> I \end{CD}$$

Similarly, we can capture this for Γ_1 by observing that the following diagram in the category of spaces commutes.

$$\begin{CD} I^2 @>{\Gamma_1}>> I \\ @V{I \times p}VV @VV{p}V \\ I @>>{p}> I \end{CD}$$

Again, this is equivalent to the commutativity of the following diagram.

$$\begin{array}{ccc} I^2 & \xrightarrow{\Gamma_1} & I \\ {\scriptstyle p \times I}\downarrow & & \downarrow{\scriptstyle p} \\ I & \xrightarrow[p]{} & I \end{array}$$

III.3.2.7. This allows us to pass to the abstract setting of an interval $\widehat{I} = \bigl(1, I, i_0, i_1, p, \Gamma_0\bigr)$ equipped with a contraction structure p and an upper connection structure Γ_0, in a category $\mathcal{A}$ equipped with a monoidal structure $(\otimes, 1, \lambda, \rho, \alpha)$. Indeed, we say that Γ_0 *is compatible with* p if the following diagram in $\mathcal{A}$ commutes.

$$\begin{array}{ccc} I \otimes I & \xrightarrow{\Gamma_0} & I \\ {\scriptstyle I \otimes p}\downarrow & & \downarrow{\scriptstyle p} \\ I & \xrightarrow[p]{} & I \end{array}$$

This is equivalent to requiring that the following diagram in $\mathcal{A}$ commutes.

$$\begin{array}{ccc} I \otimes A & \xrightarrow{\Gamma_0} & I \\ {\scriptstyle p \otimes I}\downarrow & & \downarrow{\scriptstyle p} \\ I & \xrightarrow[p]{} & I \end{array}$$

Suppose instead that $\widehat{I} = \bigl(1, I, i_0, i_1, p, \Gamma_0\bigr)$ is an interval in $\mathcal{A}$ equipped with a contraction structure p and a lower connection structure Γ_1. We say that Γ_1 *is compatible with* p if the following diagram in $\mathcal{A}$ commutes.

$$\begin{array}{ccc} I \otimes I & \xrightarrow{\Gamma_1} & I \\ {\scriptstyle I \otimes p}\downarrow & & \downarrow{\scriptstyle p} \\ I & \xrightarrow[p]{} & I \end{array}$$

This is equivalent to requiring that the following diagram in $\mathcal{A}$ commutes.

$$\begin{array}{ccc} I \otimes A & \xrightarrow{\Gamma_1} & I \\ {\scriptstyle p \otimes I}\downarrow & & \downarrow{\scriptstyle p} \\ I & \xrightarrow[p]{} & I \end{array}$$

III.3.3. Category of cubes with connections

III.3.3.1. We are now in a position to build a new category of shapes, exactly as we built the category of semi-cubes in II.3.5 and II.3.8, and as we built the category of cubes in II.4.4 and II.4.5.

III.3.3.2. Definition Let Υ'' denote the directed graph defined as follows.

(i) We have an object for every $n \geq 0$, which we denote by I^n.

(ii) For every $1 \leq i \leq n$ and every $0 \leq \epsilon \leq 1$, we have an arrow

$$I^{n-1} \xrightarrow{f^n_{i,\epsilon}} I^n.$$

(iii) For every $1 \leq i \leq n$, we have an arrow

$$I^n \xrightarrow{d^n_i} I^{n-1}.$$

(iv) For every $n \geq 2$, every $1 \leq i \leq n-1$, and every $0 \leq \epsilon \leq 1$, we have an arrow

$$I^n \xrightarrow{\Gamma^{n-1}_{i,\epsilon}} I^{n-1}.$$

Let $\mathcal{F}(\Upsilon'')$ denote the free category upon Υ''. The *category of cubes with connections*, which we will denote by $\square^c$, is the quotient of $\mathcal{F}(\Upsilon'')$ by the relation $\sim$ defined inductively as follows.

(i) For any $1 \leq i \leq n$ and $1 \leq j \leq n+1$, and any $0 \leq \delta, \epsilon \leq 1$, we have that

$$f^{n+1}_{j,\epsilon} \circ f^n_{i,\delta} \sim \begin{cases} f^{n+1}_{i,\delta} \circ f^n_{j-1,\epsilon} & \text{if } j > i, \\ f^{n+1}_{i+1,\delta} \circ f^n_{j,\epsilon} & \text{if } j \leq i. \end{cases}$$

(ii) For any $n \geq 2$, $1 \leq i \leq n$ and $1 \leq j \leq n-1$, we have that

$$d^{n-1}_j \circ d^n_i \sim \begin{cases} d^{n-1}_{i-1} \circ d^n_j & \text{if } j < i, \\ d^{n-1}_i \circ d^n_{j+1} & \text{if } j \geq i. \end{cases}$$

(iii) For any $1 \leq i, j \leq n$ and $0 \leq \epsilon \leq 1$, we have that

$$d^n_j \circ f^n_{i,\epsilon} \sim \begin{cases} id & \text{if } j = i, \\ f^{n-1}_{i,\epsilon} \circ d^{n-1}_{j-1} & \text{if } j > i, \\ f^{n-1}_{i-1,\epsilon} \circ d^{n-1}_j & \text{if } j < i. \end{cases}$$

(iv) For any $n \geq 2$, $1 \leq i \leq n-1$, $1 \leq j \leq n$, and $0 \leq \delta, \epsilon \leq 1$, we have that

$$\Gamma^{n-1}_{i,\epsilon} \circ f^{n}_{j,\delta} \sim \begin{cases} id & \text{if } j = i \text{ and } \delta = \epsilon, \text{ or } j = i+1 \text{ and } \delta = \epsilon, \\ f^{n-1}_{i,\delta} \circ d^{n}_{i} & \text{if } j = i \text{ and } \delta \neq \epsilon, \text{ or } j = i+1 \text{ and } \delta \neq \epsilon, \\ f^{n-2}_{j,\delta} \circ \Gamma^{n-2}_{i-1,\epsilon} & \text{if } n \geq 3 \text{ and } j < i, \\ f^{n-2}_{j-1,\delta} \circ \Gamma^{n-2}_{i,\epsilon} & \text{if } n \geq 3 \text{ and } j > i+1. \end{cases}$$

(v) For any $n \geq 2$, $1 \leq i \leq n-1$, $1 \leq j \leq n-1$, and $0 \leq \epsilon \leq 1$, we have that

$$d^{n-1}_{j} \circ \Gamma^{n-1}_{i,\epsilon} = \begin{cases} d^{n-1}_{i} \circ d^{n}_{i} & \text{if } i = j, \\ \Gamma^{n-2}_{i-1,\epsilon} \circ d^{n}_{j} & \text{if } n \geq 3 \text{ and } j < i, \\ \Gamma^{n-2}_{i,\epsilon} \circ d^{n}_{j+1} & \text{if } n \geq 3 \text{ and } j > i. \end{cases}$$

(vi) For any $n \geq 3$, $1 \leq i \leq n-1$, $1 \leq j \leq n-2$, and $0 \leq \delta, \epsilon \leq 1$, we impose that

$$\Gamma^{n-2}_{j,\delta} \circ \Gamma^{n-1}_{i,\epsilon} \sim \begin{cases} \Gamma^{n-2}_{i-1,\epsilon} \circ \Gamma^{n-1}_{j,\delta} & \text{if } j < i, \\ \Gamma^{n-2}_{i,\epsilon} \circ \Gamma^{n-1}_{j+1,\delta} & \text{if } j > i. \end{cases}$$

(vii) For any $n \geq 3$, $1 \leq i \leq n-2$, and $0 \leq \epsilon \leq 1$, we have that

$$\Gamma^{n-2}_{i,\epsilon} \circ \Gamma^{n-1}_{i,\epsilon} \sim \Gamma^{n-2}_{i,\epsilon} \circ \Gamma^{n-1}_{i+1,\epsilon}.$$

(viii) Suppose that we have arrows

$$I^{q} \underset{g_0'}{\overset{g_0}{\rightrightarrows}} I^{r}$$

and

$$I^{r} \underset{g_1'}{\overset{g_1}{\rightrightarrows}} I^{s}$$

of $\mathcal{F}(\Upsilon')$ with $q, r, s \geq 0$, such that $g_0 \sim g_1$ and $g_0' \sim g_1'$. Then we have that $g_1 \circ g_0 \sim g_1' \circ g_0'$.

III.3.3.3. Let $(\square^c)^{\leq 2}$ denote the full subcategory of $\square^c$ whose objects are I^0, I^1, and I^2. We think of $(\square^c)^{\leq 2}$ as the *free-standing interval equipped with a contraction structure and connections*[16].

III.3.3.4. To avoid confusion, let us adopt the following notation for the objects and arrows of $(\square^c)^{\leq 2}$.

(i) We denote I^0, I^1, and I^2 by 0, 1, and 2 respectively.

(ii) We denote i_0 and i_1 by ι_0 and ι_1 respectively.

(iii) We denote p by π.

(iv) We denote Γ_0 and Γ_1 by γ_0 and γ_1 respectively.

Let $\mathcal{A}$ be a category equipped with a monoidal structure $(\otimes, \lambda, \rho, \alpha)$. Suppose that $\big(I^0, I^1, i_0, i_1, p, \Gamma_0, \Gamma_1\big)$ defines an interval in $\mathcal{A}$ equipped with a contraction structure p, a lower connection structure Γ_0, and an upper connection structure Γ_1. Suppose moreover that Γ_0 is compatible with p, and that Γ_1 is compatible with p. Then we obtain a functor

$$(\square^c)^{\leq 2} \longrightarrow \mathcal{A}$$

given by $0 \mapsto I^0$. $1 \mapsto I^1$, $2 \mapsto I^2$, $\iota_0 \mapsto i_0$, $\iota_1 \mapsto i_1$, $\pi \mapsto p$, $\gamma_0 \mapsto \Gamma_0$, and $\gamma_1 \mapsto \Gamma_1$. Conversely, every functor

$$(\square^c)^{\leq 2} \longrightarrow \mathcal{A}$$

defines an interval in $\mathcal{A}$ equipped with a contraction structure, a lower connection structure, and an upper connection structure, such that both connection structures are compatible with the contraction structure.

III.3.3.5. Intuitively, our category of cubes with connections should be the free strict monoidal category upon $(\square^c)^{\leq 2}$. In this picture. we think of $\Gamma^n_{i,0}$ as the arrow

$$I^{n-1} \xrightarrow{I^{i-1} \otimes \Gamma_0 \otimes I^{n-i+1}} I^n.$$

We think of $\Gamma^n_{i,1}$ as the arrow

$$I^{n-1} \xrightarrow{I^{i-1} \otimes \Gamma_1 \otimes I^{n-i+1}} I^n.$$

III.3.3.6. The relations of (iv) in Definition III.3.3.2 arise by considering the defining axioms for an upper connection structure and a lower connection structure in III.3.2.3 and III.3.2.4 respectively, and formulating higher dimensional analogues. The relations of (v) in Definition III.3.3.2 arise by considering the defining axioms in III.3.2.7 for an upper connection structure compatible with a contraction structure, and for a lower connection structure compatible with a contraction structure. The relations of (vi) and (vii) arise by the same considerations as those of II.3.7.

III.3.4. A universal property of the category of cubes with connections

III.3.4.1. It remains to capture rigorously the idea that $\square^c$ should be the free strict monoidal category upon an interval equipped with a contraction structure and connection structures. We proceed exactly as in II.3.8 and II.4.5 — this time we will omit any justification, which is identical to that in II.3.8 and II.4.5.

III.3.4.2. Definition Let

$$\square^c \times \square^c \xrightarrow{- \otimes -} \square^c.$$

denote the functor defined inductively by the following recipe, by virtue of our observations in II.3.9.1.

(i) For $m, n \geq 0$, we define $I^m \otimes I^n$ to be I^{m+n}.

(ii) For $1 \leq i \leq m$ and $n \geq 0$, and $0 \leq \epsilon \leq 1$, we define $f^m_{i,\epsilon} \otimes I^n$ to be $f^{m+n}_{i,\epsilon}$.

(iii) For $1 \leq i \leq n$ and $m \geq 0$, and $0 \leq \epsilon \leq 1$, we define $I^m \otimes f^n_{i,\epsilon}$ to be $f^{m+n}_{m+i,\epsilon}$.

(iv) For $1 \leq i \leq m$ and $n \geq 0$, we define $d^m_i \otimes I^n$ to be d^{m+n}_i.

(v) For $1 \leq i \leq n$ and $m \geq 0$, we define $I^m \otimes d^n_i$ to be d^{m+n}_{m+i}.

(vi) For any $m \geq 2$, $1 \leq i \leq m$, $n \geq 0$, and $0 \leq \epsilon \leq 1$, we define $\Gamma^m_{i,\epsilon} \otimes I^n$ to be $\Gamma^{m+n}_{i,\epsilon}$.

(vii) For any $n \geq 2$, $1 \leq i \leq n$, $m \geq 0$, and $0 \leq \epsilon \leq 1$, we define $I^m \otimes \Gamma^n_{i,\epsilon}$ to be $\Gamma^{m+n}_{m+i,\epsilon}$.

(viii) For $m, m', m'', n \geq 0$, and arrows

$$I^m \xrightarrow{g_0} I^{m'}$$

and

$$I^{m'} \xrightarrow{g_1} I^{m''}$$

of $\square$, we define $(g_1 \circ g_0) \otimes I^n$ to be $(g_1 \otimes I^n) \circ (g_0 \otimes I^n)$.

(ix) For $m, n, n', n'' \geq 0$, and arrows

$$I^n \xrightarrow{g_0} I^{n'}$$

and

$$I^{n'} \xrightarrow{g_1} I^{n''}$$

of $\square_s$, we define $I^m \otimes (g_1 \circ g_0)$ to be $(I^m \otimes g_1) \circ (I^m \otimes g_0)$.

III.3.4.3. Proposition The functor

$$\square^c \times \square^c \xrightarrow{- \otimes -} \square^c$$

equips the category $\square^c$ with the structure of a strict monoidal category with unit I^0.

III.3.4.4. Proposition Let $\mathcal{A}$ be a category equipped with a strict monoidal structure $(\otimes, 1)$. For any functor

$$(\square^c)^{\leq 2} \xrightarrow{\mathsf{int}} \mathcal{A},$$

there is a unique strict monoidal functor

$$\square^c \xrightarrow{\mathsf{can}} \mathcal{A}$$

such that the following diagram in the category of categories commutes.

$$\begin{array}{ccc} (\square^c)^{\leq 2} & \hookrightarrow & \square^c \\ & \searrow^{\mathsf{int}} & \downarrow \mathsf{can} \\ & & \mathcal{A} \end{array}$$

III.3.5. A different perspective

III.3.5.1. We may think about the category of cubes with connections in a slightly different way — the reader not familiar with simplicial sets may wish to skip onto the next section. Recall that a simplicial set is a presheaf on a category Δ which can be defined as the full subcategory of the category of partially ordered sets whose objects are the partially ordered sets

$$0 \longrightarrow 1 \longrightarrow \cdots\cdots \longrightarrow n-1 \longrightarrow n$$

for $n \geq 0$.

III.3.5.2. We can define $\square$ in a similar way. For any $n \geq 0$, let $\mathcal{I}^n$ denote the set $\{0,1\}^n$ equipped with the partial ordering defined $(\delta_0, \ldots, \delta_n) \leq (\epsilon_0, \ldots, \epsilon_n)$ if $\delta_i \leq \epsilon_i$ for all $0 \leq i \leq n$. For any $1 \leq i \leq n$ and $0 \leq \epsilon \leq 1$, we denote by $f^n_{i,\epsilon}$ the functor

$$\mathcal{I}^{n-1} \longrightarrow \mathcal{I}^n$$

defined by $(\delta_0, \ldots, \delta_n) \mapsto (\delta_0, \ldots, \delta_{i-2}, \epsilon, \delta_{i-1}, \ldots, \delta_n)$. For any $1 \leq i \leq n$, we denote by d^n_i the functor

$$\mathcal{I}^n \longrightarrow \mathcal{I}^{n-1}$$

defined by $(\delta_0, \ldots, \delta_n) \mapsto (\delta_0, \ldots, \delta_{i-2}, \delta_i, \ldots, \delta_n)$. We can then view $\square$ as the subcategory of the category of posets whose objects are the posets $\mathcal{I}^n$ for $n \geq 0$, and whose arrows are generated by the functors $f^n_{i,\epsilon}$ and d^n_i for $1 \leq i \leq n$ and $0 \leq \epsilon \leq 1$.

III.3.5.3. However, $\square$ is not a full subcategory of the category of posets. The full subcategory of the category of posets whose objects are the $\mathcal{I}^n$ is exactly $\square^c$.

III.3.5.4. Thus by passing from the category of cubes to the category of cubes with connections intuitively should bring us closer to the simplicial world.

III.3.6. Cubical sets with connections

III.3.6.1. A *cubical set with connections* is a presheaf upon $\square^c$.

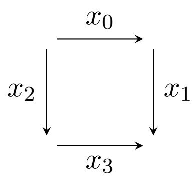
III.3.6.2. An explicit description of a cubical set with connections X can be given in direct analogy to the explicit description of a cubical set in III.1.1.1. The new data is a map

$$X_{n-1} \longrightarrow X_n$$

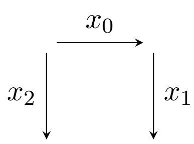
for every $n \geq 2$, $1 \leq i \leq n-1$, and $0 \leq \epsilon \leq 1$, which we will denote by $\Gamma^{i,\epsilon}_n$. These maps satisfy the dual relations to those of III.3.3.2.

III.3.6.3. We have put our idea of III.2.2.7 on solid ground. If we work with cubical sets with connections, the representable presheaf $\square^1$ is a Kan complex.

III.3.6.4. It remains the case that $\square^n$ is not a Kan complex for $n \geq 2$ if we work with cubical sets with connections. or example, let x denote the unique non-degenerate 2-cube of $\square^2$, and let us denote its boundary faces as follows.



The following defines a 2-horn in $\square^2$, but this 2-horn does not extend to a 2-cube of $categorycubes^2$.



III.3.6.5. This gives some evidence for our feeling of III.3.5.4 — for simplicial sets it is also the case that the representable n-simplex Δ^n is Kan complex for $n = 0$ and $n = 1$, and is not a Kan complex for $n \geq 2$.

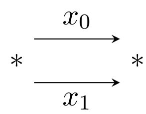
III.4. Homotopy groups of a Kan complex

III.4.1. Combinatorial homotopy defines an equivalence relation for a Kan complex

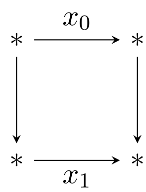
III.4.1.1. Let us now return to our attempt in III.1.3.5 to define the homotopy groups of a cubical set, this time assuming that our cubical set is a Kan complex. Our stumbling block earlier was that our homotopy relation $\sim$ did not define an equivalence relation — see III.1.4.11.

III.4.1.2. Proposition Let X be a Kan complex, and let $*$ be a 0-cube of X. Let x_0 and x_1 be n-cubes of X belonging to $Z_n(X, *)$, namely with trivial boundary. Suppose that $x_0 \sim x_1$. Then $x_1 \sim x_0$.

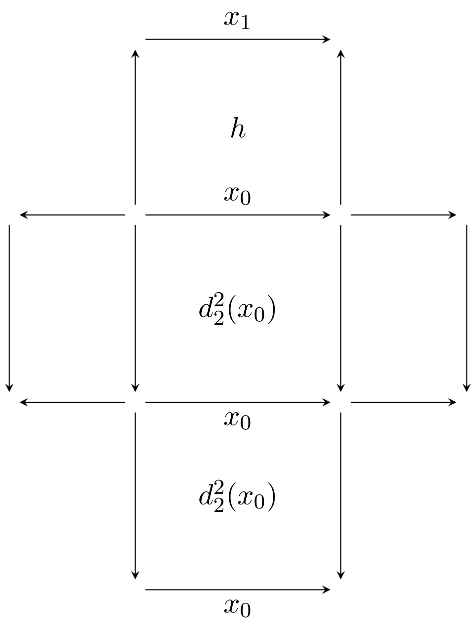
III.4.1.3. Before proving Proposition III.4.1.2, let us explore a proof in low dimensions. Suppose that x_0 and x_1 are 1-cubes of X as indicated below.



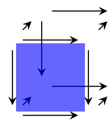
Suppose that h is a 2-cube of X with the following boundary, in which the vertical 1-cubes are degenerate.



The following then defines a 3-horn in X.

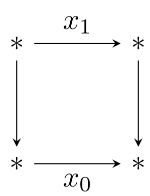


One can assemble this 3-horn by taking the front, shaded face of the 3-cube below to be the middle face of the 3-horn, namely $d_2^2(x_0)$.



Then the bottom face of the above 3-cube corresponds to other copy of $d_2^2(x_0)$ in the 3-horn, and the top face corresponds to the 2-cube h of the 3-horn. The back face of the 3-cube is the missing face of the 3-horn. For the remainder of this lecture, all 3-horns depicted as nets can be assembled in this way — we shall not draw further attention to this.

Since X is a Kan complex, this 3-horn extends to a 3-cube. The face of this 3-cube which is not part of the above horn has the following boundary.



Proof of Proposition III.4.1.2. Let us now give a proof in arbitrary dimensions. Let x_0 and x_1 be n-cubes of X belonging to $Z_n(X, *)$, and suppose that h defines a homotopy from x_0 to x_1. The following defines an $(n+1)$-horn

$$\sqcap^{n+1}_{n,1} \xrightarrow{g} X$$

of X.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} h & i = n+1, \\ d_n^n(x_0) & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} d_n^n(x_0) & i = n+1, \\ * & i < n. \end{cases}$$

Since X is a Kan complex, there is an $(n+1)$-cube

$$\square^{n+1} \xrightarrow{g'} X$$

of X such that the following diagram in $\mathsf{Set}^{\square^{op}}$ commutes.

$$\begin{array}{ccc} \sqcap^{n+1}_{n,1} & \hookrightarrow & \square^{n+1} \\ & {\scriptstyle g} \searrow & \downarrow {\scriptstyle g'} \\ & & X \end{array}$$

Then $f^{n+1}_{n,1}(g')$ defines a homotopy from x_1 to x_0. $\square$

III.4.1.4. Proposition Let X be a Kan complex, and let $*$ be a 0-cube of X. Let x_0, x_1, and x_2 be n-cubes of X belonging to $Z_n(X, *)$, namely with trivial boundary. Suppose that $x_0 \sim x_1$ and that $x_1 \sim x_2$. Then $x_0 \sim x_2$.

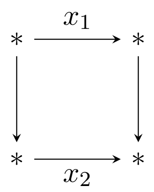
III.4.1.5. Again, let us explore a proof of Proposition III.4.1.4 in low dimensions. Suppose that x_0, x_1, and x_2 are 1-cubes of X with the following boundary.

$$* \longrightarrow *$$

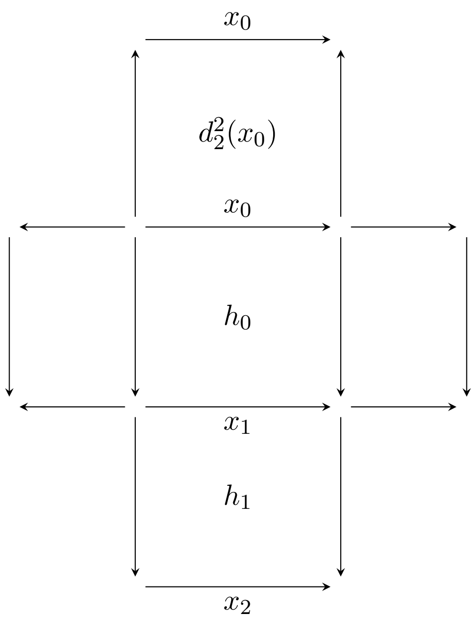
Suppose that h_0 is a 2-cube of X with the following boundary, in which the vertical 1-cubes are degenerate.

$$\begin{array}{ccc} * & \xrightarrow{x_0} & * \\ \downarrow & & \downarrow \\ * & \xrightarrow[x_1]{} & * \end{array}$$

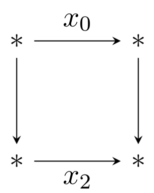
Suppose that h_1 is a 2-cube of X with the following boundary, in which the vertical 1-cubes are degenerate.



The following then defines a 3-horn in X.



Since X is a Kan complex, this 3-horn extends to a 3-cube. The face of this 3-cube which is not a part of the above horn has the following boundary



Proof of Proposition III.4.1.4. Let us now give a proof in arbitrary dimensions. Let x_0, x_1, and x_2 be n-cubes of X belonging to $Z_n(X, *)$. Suppose that h_0 defines a homotopy from x_0 to x_1, and that h_1 defines a homotopy from x_1 to x_2. The following defines an $(n+1)$-horn

$$\sqcap^{n+1}_{n+1,1} \xrightarrow{g} X$$

of X.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} h_0 & i = n+1, \\ d_n^n(x_0) & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} h_1 & i = n, \\ * & i < n. \end{cases}$$

Since X is a Kan complex, there is an $(n+1)$-cube

$$\square^{n+1} \xrightarrow{g'} X$$

of X such that the following diagram in $\mathsf{Set}^{\square^{op}}$ commutes.

$$\begin{array}{ccc} \sqcap^{n+1}_{n+1,1} & \hookrightarrow & \square^{n+1} \\ & \searrow^{g} & \downarrow{g'} \\ & & X \end{array}$$

Then $f^{n+1}_{n+1,1}(g')$ defines a homotopy from x_0 to x_2. $\square$

III.4.1.6. We have now established that if $(X, *)$ is a Kan complex, then $\sim$ defines an equivalence relation upon $Z_n(X, *)$. Thus, as a set, we can define $\pi_n(X, *)$ to be $Z_n(X, *)/\sim$, as we hoped to do in III.1.3.5. We now turn to equipping $\pi_n(X, *)$ with a group structure.

III.4.2. Group structure upon $\pi_1(X, *)$

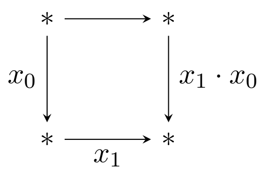
III.4.2.1. Let x_0 and x_1 be 1-cubes of a pointed Kan complex $(X, *)$ with the following boundary.

$$* \longrightarrow *$$

The following then defines a 2-horn of X.

$$\begin{array}{ccc} * & \longrightarrow & * \\ {\scriptstyle x_0}\downarrow & & \\ * & \xrightarrow[x_1]{} & * \end{array}$$

Since X is a Kan complex, we can extend this 2-horn to a 2-cube in X. We would like to define[17] $x_1 \cdot x_0$ to be the right vertical face of this 2-cube, as depicted below.

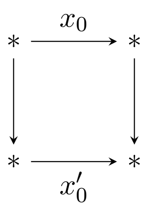


III.4.2.2. We must have that the following conditions are satisfied in order for this to work.

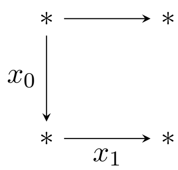
(i) If we have a 1-cube x_0' such that $x_0 \sim x_0'$, then $x_1 \circ x_0 \sim x_1 \circ x_0'$.

(ii) If we have a 1-cube x_1' such that $x_1 \sim x_1'$, then $x_1 \circ x_0 \sim x_1' \circ x_0$.

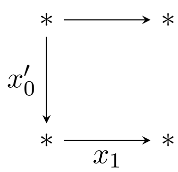
III.4.2.3. Let us verify that (i) holds. Let h be a 2-cube of X with the following boundary.



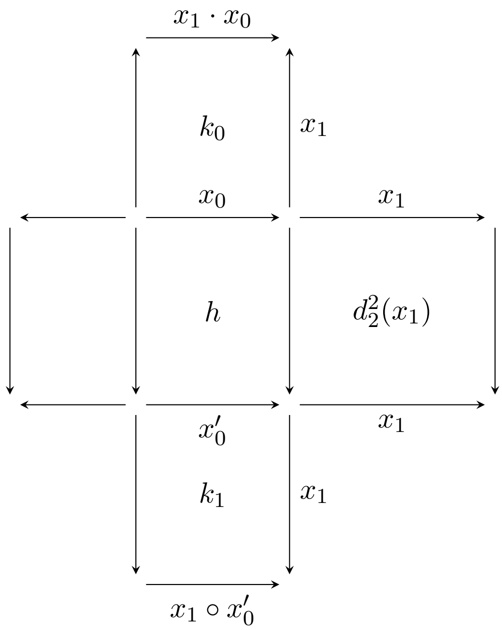
Let k_0 denote the 2-cube of X which extends the following 2-horn in X.



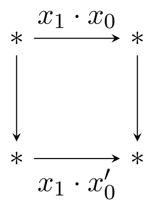
Let k_1 denote the 2-cube of X which extends the following 2-horn in X.



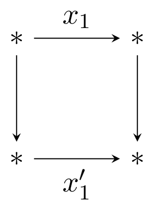
Since X is a Kan complex, the following 3-horn in X extends to a 3-cube in X.



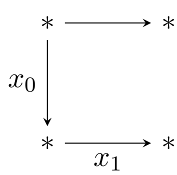
The face of this 3-cube which is not a part of the above 3-horn defines a homotopy with the following boundary.



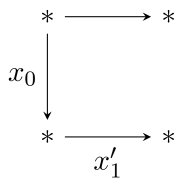
III.4.2.4. Let us now verify that (ii) holds. Let h be a 2-cube of X with the following boundary.



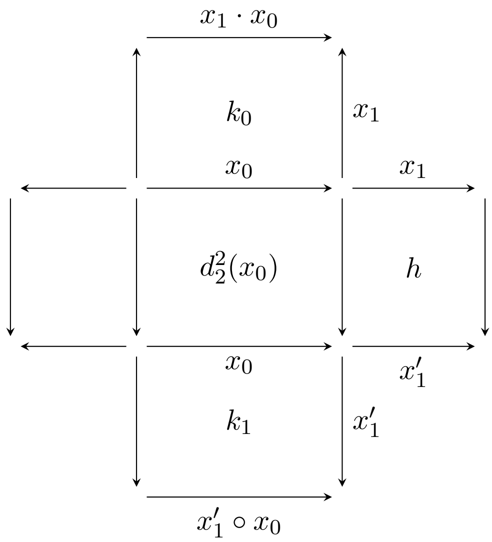
Let k_0 denote the 2-cube of X which extends the following 2-horn in X.



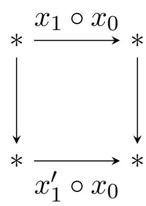
Let k_1 denote the 2-cube of X which extends the following 2-horn in X.



Since X is a Kan complex, the following 3-horn in X extends to a 3-cube in X.



The face of this 3-cube which is missing from the above 3-horn has the following boundary.



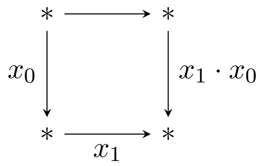
III.4.2.5. We have now shown that $- \cdot -$ gives rise to a map

$$\pi_1(X,*) \times \pi_1(X,*) \longrightarrow \pi_1(X,*).$$

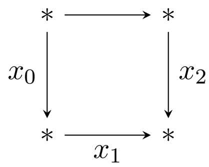
Let us prove that this map equips $\pi_1(X,*)$ with the structure of a group.

III.4.2.6. We begin with an observation. Suppose that x_0 and x_1 are 1-cube of X with the following boundary.

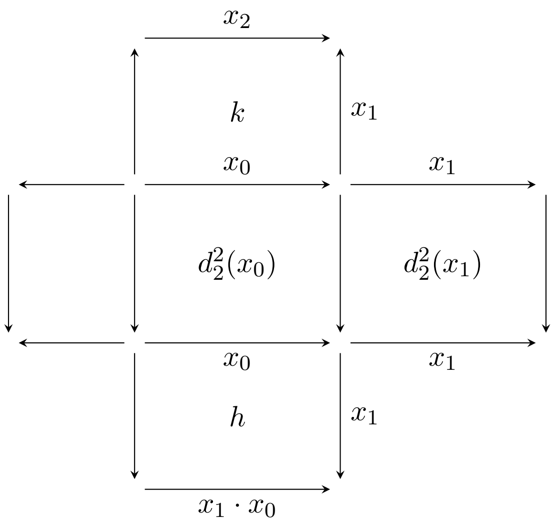
Let h denote a 2-cube of X with the following boundary.



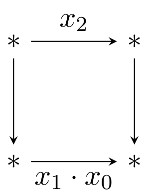
Suppose that we have a 2-cube k of X with the following boundary.



We claim that $x_2 \sim x_1 \cdot x_0$. To see this, note that since X is a Kan complex the following 3-horn in X extends to a 3-cube.



The face of this 3-cube corresponding to the missing face of the 3-horn has the following boundary.



III.4.2.7. Let us now turn to proving that the map

$$\pi_1(X,*) \times \pi_1(X,*) \longrightarrow \pi_1(X,*).$$

defined by $- \cdot -$ is associative. Suppose that x_0, x_1, and x_2 are 1-cubes of X with the following boundary.

$$* \longrightarrow *$$

Let h_0 be a 2-cube of X with the following boundary.

$$\begin{array}{ccc} * & \longrightarrow & * \\ {\scriptstyle x_0}\downarrow & & \downarrow{\scriptstyle x_1 \cdot x_0} \\ * & \underset{x_1}{\longrightarrow} & * \end{array}$$

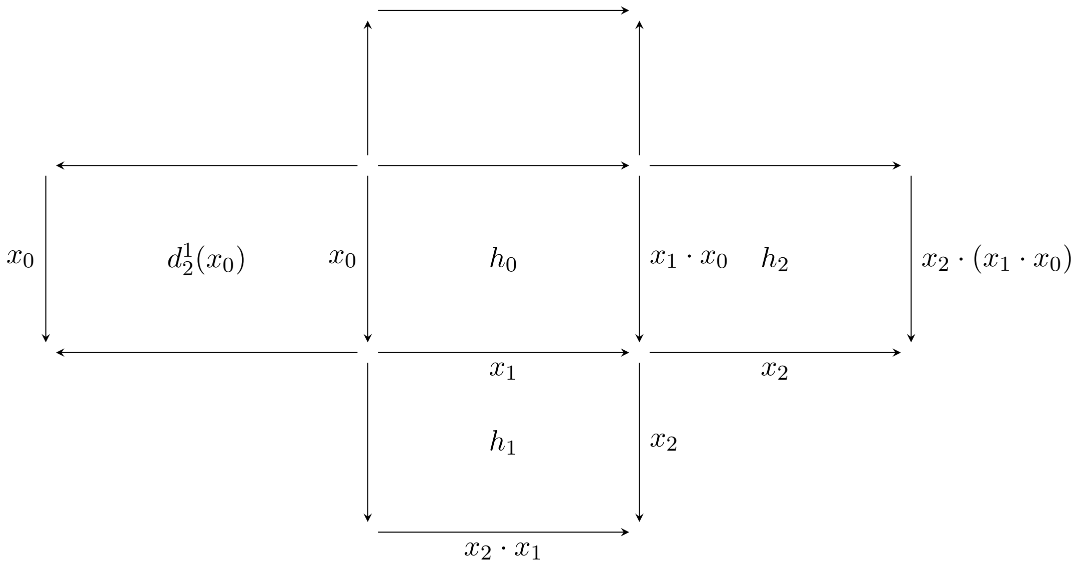
Let h_1 be a 2-cube of X with the following boundary.

$$\begin{array}{ccc} * & \longrightarrow & * \\ {\scriptstyle x_1}\downarrow & & \downarrow{\scriptstyle x_2 \cdot x_1} \\ * & \underset{x_2}{\longrightarrow} & * \end{array}$$

Let h_2 be a 2-cube of X with the following boundary.

$$\begin{array}{ccc} * & \longrightarrow & * \\ {\scriptstyle x_1 \cdot x_0}\downarrow & & \downarrow{\scriptstyle x_2 \cdot (x_1 \cdot x_0)} \\ * & \underset{x_2}{\longrightarrow} & * \end{array}$$

Since X is a Kan complex, the following 3-horn in X extends to a 3-cube in X.



The face of this 3-cube which is not a part of the above 3-horn has the following boundary.

$$\begin{array}{ccc} * & \longrightarrow & * \\ {\scriptstyle x_0}\downarrow & & \downarrow{\scriptstyle x_2 \cdot (x_1 \cdot x_0)} \\ * & \xrightarrow[x_2 \cdot x_1]{} & * \end{array}$$

By our observation of III.4.2.6, we deduce that $x_2 \cdot (x_1 \cdot x_0) \sim (x_2 \cdot x_1) \cdot x_0$.

III.4.2.8. Let us prove that the degenerate 1-cube $d^1_1(*)$ defines a right identity for the map

$$\pi_1(X, *) \times \pi_1(X, *) \longrightarrow \pi_1(X, *).$$

defined by $- \cdot -$. For any 1-cube

$$* \xrightarrow{x} *$$

of X, the degenerate 2-cube $d^1_2(x)$ of X has the following boundary.

$$\begin{array}{ccc} * & \longrightarrow & * \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x} \\ * & \xrightarrow[d^1_1(*)]{} & * \end{array}$$

It follows from our observation of III.4.2.6 that $x \cdot d^1_1(*) \sim x$.

III.4.2.9. If we are working with cubical sets with connection, we can also observe that $d_1^1(*)$ defines a left identity for the map

$$\pi_1(X,*) \times \pi_1(X,*) \longrightarrow \pi_1(X,*).$$

defined by $- \cdot -$. Indeed the 2-cube $\Gamma_2^{1,1}(x)$ of X has the following boundary.

$$\begin{array}{ccc} * & \longrightarrow & * \\ {\scriptstyle d_1^1(*)}\Big\downarrow & & \Big\downarrow{\scriptstyle x} \\ * & \underset{x}{\longrightarrow} & * \end{array}$$

It would then follow from our observation of III.4.2.6 once more that $d_1^1(*) \cdot x \sim x$.

III.4.2.10. If we do not assume that our cubical sets have connections, the author is not aware[18] of any straightforward geometric argument to show that $d_1^1(*)$ defines a left identity for the map

$$\pi_1(X,*) \times \pi_1(X,*) \longrightarrow \pi_1(X,*).$$

defined by $- \cdot -$. This can nevertheless be proven to be the case —indeed it suffices[19] to prove that every 1-cube of X admits a right inverse up to homotopy with respect to $- \cdot -$.

III.4.2.11. Let us turn to this. Let

$$* \xrightarrow{\;x\;} *$$

be a 1-cube of X. Since X is a Kan complex, the following 2-horn in X extends to a 2-cube h.

$$\begin{array}{ccc} * & \longrightarrow & * \\ {\scriptstyle x}\Big\downarrow & & \Big\downarrow \\ * & & * \end{array}$$

Let us denote the face of this 2-cube which is not part of the above 2-horn by x^{-1}, so that the boundary of h is as shown below.

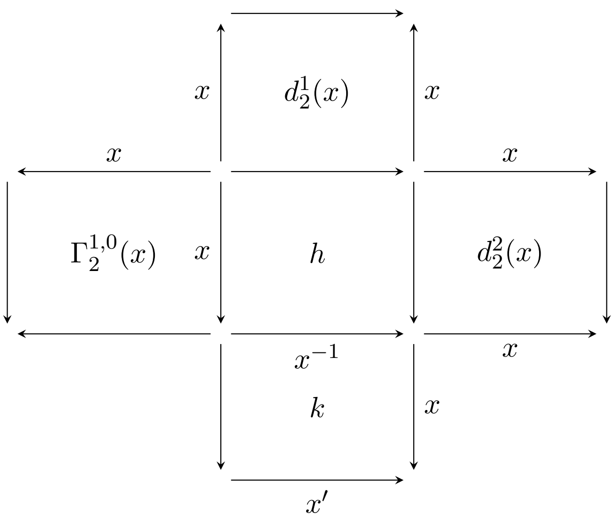
$$\begin{array}{ccc} * & \longrightarrow & * \\ {\scriptstyle x}\Big\downarrow & & \Big\downarrow \\ * & \underset{x^{-1}}{\longrightarrow} & * \end{array}$$

It follows immediately from our observation of III.4.2.6 that $x \cdot x^{-1} \sim d_1^1(*)$.

III.4.2.12. If we are working with cubical sets with connection, we can also observe that x^{-1} defines a left inverse of x up to homotopy. Indeed, let k be a 2-cube of X with the following boundary, for some 1-cube x' of X.

$$\begin{array}{ccc} * & \longrightarrow & * \\ {\scriptstyle x^{-1}}\downarrow & & \downarrow{\scriptstyle x'} \\ * & \underset{x}{\longrightarrow} & * \end{array}$$

Since X is a Kan complex, the following 3-horn in X extends to a 3-cube of X.



The face of this 3-cube which is not part of the above 3-horn has the following boundary.

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \underset{x}{\longrightarrow} & * \end{array}$$

It follows from our observation of III.4.2.6 that $x^{-1} \cdot x \sim d^1_1(*)$.

III.4.2.13. As in III.4.2.10, if we do not assume that our cubical sets have connections, then the author is not aware[20] of any straightforward geometric argument to show that x^{-1} defines a left inverse of x up to homotopy. Nevertheless, we have shown that $d^1_1(*)$ equips

$$\pi_1(X,*) \times \pi_1(X,*) \longrightarrow \pi_1(X,*).$$

defined by $- \cdot -$ with a right identity, and that every 1-cube x of X has a right inverse x^{-1} up to homotopy. As we mentioned in III.4.2.10, it follows formally from these two observations that $d_1^1(*)$ moreover defines a left identity, and that x^{-1} moreover is a left inverse of x.

III.4.2.14. This concludes our proof that the map

$$\pi_1(X,*) \times \pi_1(X,*) \longrightarrow \pi_1(X,*).$$

defined by $- \cdot -$ equips $\pi_1(X,*)$ with the structure of a group.

III.4.3. Group structure upon $\pi_n(X,*)$

III.4.3.1. Our construction of a group structure upon $\pi_1(X,*)$ for any pointed Kan complex $(X,*)$ can be carried out in any dimension. We now give the details.

III.4.3.2. Definition Let X be a Kan complex, and let $*$ be a 0-cube of X. Let x_0 and x_1 be n-cubes of X belonging to $Z_n(X,*)$, for some $n \geq 1$. The following defines an $(n+1)$-horn

$$\sqcap_{i,1}^{n+1} \xrightarrow{g} X$$

in X.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x_1 & i = n+1, \\ * & i < n. \end{cases}$$

Let

$$\square^n \xrightarrow{g'} X$$

denote the[21] extension of g to an $(n+1)$-cube of X. We denote by $x_1 \cdot x_0$ the n-cube $f_{n+1}^{n,1}(g)$ of X.

III.4.3.3. Proposition Let X be a Kan complex, and let $*$ be a 0-cube of X. Let x_0, x_0', and x_1 be n-cubes of X, and suppose that $x_0 \sim x_0'$. Then $x_1 \cdot x_0 \sim x_1 \cdot x_0'$.

Proof. Let h be an $(n+1)$-cube of X which defines a homotopy from x_0 to x_0'. Let k_0 be an $(n+1)$-cube of X with the following boundary.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x_1 & i = n+1, \\ x_1 \cdot x_0 & i = n, \\ * & i < n. \end{cases}$$

Let k_1 be an $(n+1)$-cube of X with the following boundary.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x'_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x_1 & i = n+1, \\ x_1 \cdot x'_0 & i = n, \\ * & i < n. \end{cases}$$

The following defines an $(n+2)$-horn

$$\sqcap^{n+2,1}_{n+2} \xrightarrow{g'} X$$

in X.

$$I^{i-1} \otimes i_0 \otimes I^{n+2-i} \mapsto \begin{cases} h & i = n+2, \\ * & i = n+1, \\ k_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} d^{n+1}_{n+1}(x_1) & i = n+1, \\ k_1 & i = n, \\ * & i < n. \end{cases}$$

Let g' denote the extension of g to an $(n+2)$-cube of X. Then the $(n+1)$-cube $f^{n+2,1}_{n+2}(g)$ defines a homotopy from $x_1 \cdot x_0$ to $x_1 \cdot x'_0$. □

III.4.3.4. Proposition Let X be a Kan complex, and let $*$ be a 0-cube of X. Let x_0, x_1, and x'_1 be n-cubes of X, and suppose that $x_1 \sim x'_1$. Then $x_1 \cdot x_0 \sim x'_1 \cdot x_0$.

Proof. Let h be an $(n+1)$-cube of X which defines a homotopy from x_1 to x'_1. Let k_0 be an $(n+1)$-cube of X with the following boundary.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x_1 & i = n+1, \\ x_1 \cdot x_0 & i = n, \\ * & i < n. \end{cases}$$

Let k_1 be an $(n+1)$-cube of X with the following boundary.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x_1' & i = n+1, \\ x_1' \cdot x_0 & i = n, \\ * & i < n. \end{cases}$$

The following defines an $(n+2)$-horn

$$\sqcap_{n+2}^{n+2,1} \xrightarrow{g'} X$$

in X.

$$I^{i-1} \otimes i_0 \otimes I^{n+2-i} \mapsto \begin{cases} d_2^2(x_0) & i = n+2, \\ * & i = n+1, \\ k_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} h & i = n+1, \\ k_1 & i = n, \\ * & i < n. \end{cases}$$

Let g' denote the extension of g to an $(n+2)$-cube of X. Then the $(n+1)$-cube $f_{n+2}^{n+2,1}(g)$ defines a homotopy from $x_1 \cdot x_0$ to $x_1' \cdot x_0$. □

III.4.3.5. Lemma Let X be a Kan complex, and let $*$ be a 0-cube of X. Let x_0 and x_1 be n-cubes of X belonging to $Z_n(X, *)$, for some $n \geq 1$. Suppose that h_0 is an $(n+1)$-cube of X with the following boundary, for some n-cube x_2 of X belonging to $Z_n(X, *)$.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x_1 & i = n+1, \\ x_2 & i = n, \\ * & i < n. \end{cases}$$

Suppose that h_1 is an $(n+1)$-cube of X with the following boundary, for an n-cube x_2' of X belonging to $Z_n(X,*)$.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x_1 & i = n+1, \\ x_2' & i = n, \\ * & i < n. \end{cases}$$

Then $x_2 \sim x_2'$.

Proof. The following defines an $(n+2)$-horn

$$\sqcap_{n+2}^{n+2,1} \xrightarrow{g} X$$

in X.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} d_{n+1}^{n+1}(x_0) & i = n+2, \\ * & i = n+1, \\ h_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} d_2^2(x_1) & i = n+1, \\ h_1 & i = n, \\ * & i < n. \end{cases}$$

Since X is a Kan complex, g extends to an $(n+2)$-cube g' in X. The $(n+1)$-cube $f_{n+2}^{n+2,1}(g')$ defines a homotopy from x_2 to x_2'. □

III.4.3.6. Proposition Let X be a Kan complex, and let $*$ be a 0-cube of X. The map

$$\pi_n(X,*) \times \pi_n(X,*) \longrightarrow \pi_n(X,*)$$

defined by $(x,y) \mapsto x \cdot y$ equips $\pi_n(X,*)$ with the structure of a group.

Proof. Firstly, let us show that this map is associative. Let x_0, x_1, and x_2 be n-cubes of X belonging to $Z_n(X,*)$. Let h_0 be an $(n+1)$-cube of X with the following boundary.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x_1 & i = n+1, \\ x_1 \cdot x_0 & i = n, \\ * & i < n. \end{cases}$$

Let h_1 be an $(n+1)$-cube of X with the following boundary.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x_1 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x_2 & i = n+1, \\ x_2 \cdot x_1 & i = n, \\ * & i < n. \end{cases}$$

Let h_2 be an $(n+1)$-cube of X with the following boundary.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x_1 \cdot x_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x_2 & i = n+1, \\ x_2 \cdot (x_1 \cdot x_0) & i = n, \\ * & i < n. \end{cases}$$

The following defines an $(n+2)$-horn

$$\sqcap^{n+2,1}_{n+2} \xrightarrow{g} X$$

in X.

$$I^{i-1} \otimes i_0 \otimes I^{n+2-i} \mapsto \begin{cases} h_0 & i = n+2, \\ d^1_2(x_0) & i = n+1, \\ * & i \leq n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+2-i} \mapsto \begin{cases} h_2 & i = n+1, \\ h_1 & i = n, \\ * & i < n. \end{cases}$$

Since X is a Kan complex, g extends to an $(n+2)$-cube g' in X. The $(n+1)$-cube $f^{n+2,1}_{n+2}(g')$ of X has the following boundary.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x_0 & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x_2 \cdot x_1 & i = n+1, \\ x_2 \cdot (x_1 \cdot x_0) & i = n, \\ * & i < n. \end{cases}$$

Appealing to Lemma III.4.3.5, we deduce that $x_2 \cdot (x_1 \cdot x_0) \sim (x_2 \cdot x_1) \cdot x_0$, as required.

Secondly, let us prove that $*$, regarded as a degenerate n-cube in the unique possible way as usual, defines a right identity for the map of the proposition when viewed as belonging to $\pi_n(X, *)$. Let x be an n-cube of X belonging to $Z_n(X, *)$. The $(n+1)$-cube $d^n_{n+1}(x)$ of X has the following boundary.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x & i = n, \\ * & i < n. \end{cases}$$

Appealing to Lemma III.4.3.5, we deduce that $x \cdot * \sim x$.

To complete the proof, it suffices[22] to show that for any n-cube x of X belonging to $Z_n(X, *)$, there is an n-cube x^{-1} of X belonging to $Z_n(X, *)$ such that $x \cdot x^{-1} \sim *$. To see this, note that the following defines an $(n+1)$-horn

$$\sqcap^{n+1,1}_{n+1} \xrightarrow{g} X$$

in X.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} * & i \leq n. \end{cases}$$

Since X is a Kan complex, g extends to an $(n+1)$-cube g' in X. Let x^{-1} denote the n-cube $f^{n+1,1}_{n+1}(g')$ of X. By Lemma III.4.3.5, $x \cdot x^{-1} \sim *$.

□

III.4.3.7. If we assume in Proposition III.4.3.6 that X is a cubical set with connections, we can see directly that $*$ defines a left identity up to homotopy for the map

$$\pi_n(X, *) \times \pi_n(X, *) \longrightarrow \pi_n(X, *)$$

defined by $(x, y) \mapsto x \cdot y$. Indeed the $(n+1)$-cube $\Gamma^{n,1}_{n+1}(x)$ of X has the following boundary.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ * & i \leq n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x & i = n+1, \\ x & i = n, \\ * & i < n. \end{cases}$$

Appealing to Lemma III.4.3.5 once more, we deduce that $* \cdot x \sim x$.

III.4.3.8. Let us again assume that X is a cubical set with connections in Proposition III.4.3.6, and let x be an n-cube of X belonging to $Z_n(X, *)$. We can see directly that the n-cube x^{-1} constructed in the proof of Proposition III.4.3.6 defines a left inverse up to homotopy for the map

$$\pi_n(X, *) \times \pi_n(X, *) \longrightarrow \pi_n(X, *)$$

defined by $(x, y) \mapsto x \cdot y$. Indeed, let h denote an $(n+1)$-cube of X with the following boundary.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x^{-1} i = n+1, & \\ * & i \leq n. \end{cases}$$

Let k denote an $(n+1)$-cube of X with the following boundary, for some n-cube x' of X belonging to $Z_n(X, *)$.

$$I^{i-1} \otimes i_0 \otimes I^{n+1-i} \mapsto \begin{cases} * & i = n+1, \\ x^{-1} & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+1-i} \mapsto \begin{cases} x i = n+1, & \\ x' & i \leq n. \end{cases}$$

The following defines an $(n+2)$-horn

$$\sqcap^{n+2,1}_{n+2} \xrightarrow{g} X$$

in X.

$$I^{i-1} \otimes i_0 \otimes I^{n+2-i} \mapsto \begin{cases} h & i = n+2, \\ d^n_{n+1}(x) & i = n+1, \\ \Gamma^{n,0}_{n+1}(x) & i = n, \\ * & i < n. \end{cases}$$

$$I^{i-1} \otimes i_1 \otimes I^{n+2-i} \mapsto \begin{cases} k & i = n+1, \\ d_{n+1}^{n+1}(x) & i = n, \\ * & i < n. \end{cases}$$

Since X is a Kan complex, g extends to an $(n+2)$-cube g' of X. The $(n+1)$-cube $f_{n+2}^{n+2,1}(g')$ of X defines a homotopy $* \sim x'$. Appealing to Lemma III.4.3.5, we deduce that $x^{-1} \cdot x \sim *$.

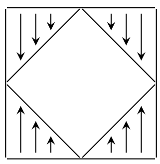
III.4.3.9. In a later lecture, we will prove that for any pointed Kan complex $(X, *)$, we have that $\pi_n(X, *)$ is abelian for $n \geq 2$.

Notes

12 We will see later that the requirement that $f_{n+1}^{i,\epsilon}(h)$ be degenerate for $i \neq n$ arises naturally from the conceptual definition of homotopy we touched upon in III.1.4.1 — for $n = 1$, it is akin to the commutativity requirement which a natural transformation must satisfy in category theory. Cubical sets are an algebraic notion, we should expect that 'everything must be specified'! Nevertheless, the reader may be puzzled by the fact that this requirement might not seem to be present in topology. In fact, however, there exists a homotopy between two paths in the usual sense in topology if and only if there exists a homotopy between these paths satisfying the extra condition we impose in the combinatorial setting. Indeed, let

$$I^2 \xrightarrow{l} I^2$$

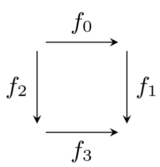
denote the map depicted below.



In words, l is the identity on the diamond, whilst the rest of the square retracts onto the boundary of the diamond. Then given a map

$$I^2 \xrightarrow{f} X$$

with the boundary



we have that the composite map

$$\begin{array}{ccc} I^2 & \xrightarrow{\;l\;} & I^2 \\ & \overset{f \circ l}{\searrow} & \downarrow f \\ & & X \end{array}$$

has the following boundary, where the vertical faces are constant maps.

$$\begin{array}{ccc} & \xrightarrow{\;f_1 \circ f_0\;} & \\ \downarrow & & \downarrow \\ & \xrightarrow[\;f_3 \circ f_2\;]{} & \end{array}$$

An analogous story holds in higher dimensions. In abstract homotopy theory, this idea has been pursued by Grandis, in [2] for example.

13 We will later explain formally how to cook up a cubical set from a recipe of this kind. Roughly speaking, one adds any degeneracies which one needs in order to obtain a cubical set, and nothing else!

14 We shall interpret the definition of a Kan complex constructively, though the reader happy with non-constructive foundations for mathematics may harmlessly ignore this if it is not to their taste! To be precise, we will think of a Kan complex as a cubical set equipped for every $1 \leq i \leq n$ and $0 \leq \epsilon \leq 1$ with a map $\Sigma^n_{i,\epsilon}$ from the set $\mathrm{Hom}_{\mathsf{Set}^{\square^{op}}}(\sqcap^n_{i,\epsilon}, X)$ to $\mathrm{Hom}_{\mathsf{Set}^{\square^{op}}}(\square^n, X)$ such that diagram

$$\begin{array}{ccc} \sqcap^n_{i,\epsilon} & \hookrightarrow & \square^n \\ & \overset{\Sigma^n_{i,\epsilon}(g)}{\searrow} & \downarrow g \\ & & X \end{array}$$

in $\mathsf{Set}^{\square^{op}}$ commutes for every g. A Kan complex in this constructive sense is referred to as an *algebraic Kan complex* in [6]. The reader who prefers not to implicitly work constructively in this way must appeal to the axiom of choice at various places later in this work, for example when we come to define the homotopy groups of a Kan complex.

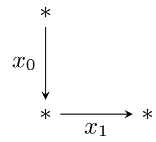
15 Indeed, the only possible 2-horns in Δ^1 are the following, which may be extended to degenerate 2-simplexes. The free-standing n-simplex Δ^n is not a Kan complex for $n \geq 2$ — the following 3-horn in Δ^2 cannot be extended to a 3-simplex, for example.

16 To be more precise, we think of $(\square^c)^{\leq 2}$ as the *free standing interval equipped with a contraction structure, an upper connection structure, and a lower connection structure, such that both connection structures are both compatible with contraction.*

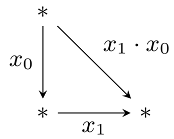
17 In the simplicial setting, the analogous definition of the group structure on $\pi_1(X, *)$ is perhaps more 'minimal', and more immediately clearly related to composition in category theory: given 1-simplexes

$$* \;\underset{x_1}{\overset{x_0}{\rightrightarrows}}\; *$$

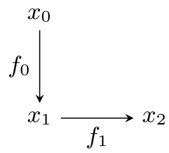
of a pointed simplicial Kan complex $(X, *)$ the following defines a 2-horn in X.



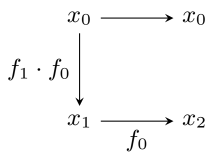
Since X is a Kan complex, this 2-horn extends to a 2-simplex. We define $x_1 \cdot x_0$ to be the diagonal face of this 2-simplex.



In higher dimensions, the $(n+2)$-horn giving rise to the group structure on $\pi_n(X, *)$ will contain degenerate $(n+1)$-simplexes, just as in the cubical setting — of course there are slightly fewer of them!
Note that the idea behind our notion of composition-up-to-homotopy in a Kan complex X — cubical or simplicial — more generally allows us to the composite of any pair of 1-cubes



in X to be the right vertical face of the 2-cube obtained by extending the following 2-horn in X.



This composite will be well-defined up to homotopy — this notion of homotopy is a slight generalisation of that we have seen so far, but is again a special case of the notion of the general notion of homotopy in $\mathsf{Set}^{\square^{op}}$ mentioned in III.1.4.1, which we will come to in the next lecture. More generally, the notion of composition-up-to-homotopy between n-cubes of X belonging to $Z_n(X, *)$ which we will give later extends to a notion of composition-up-to-homotopy of n-cubes of X with arbitrary compatible faces. These considerations are a route towards higher category theory.

18 In addition to being clear for cubical sets with connections, it is clear for simplicial sets.

19 It is an elementary algebraic observation that a set equipped with an associative binary relation is a group if and only if this binary relation admits a right identity, and if every element admits a right inverse. The same holds if we replace right by left in both cases (but not if we mix the two).

20 Again, it is straightforward to prove for simplicial sets.

21 Recall — see 14 — that we are thinking of our definition of a Kan complex as constructive, so that our n-cube g' is part of the structure of a Kan complex. Traditionally, one would only know that *an* n-cube extending g exists, and would then need the axiom of choice to make a simultaneous choice for every 2-horn, as mentioned in 14.

22 That this indeed suffices is for the same formal reasons as in 19.

A. A categorical miscellany

A.1. Free category upon a directed graph

A.1.1. Introduction

In the course, we will frequently construct a category as the free category upon a directed graph. We recall the details — it is possible to cook up the free category functor very quickly by a formal argument which we will see later in the course, but we will instead construct it explicitly here.

A.1.2. Free category

A.1.2.1. Definition Let Υ be a directed graph. The *free category* on Υ, which we will denote by $\mathcal{F}(\Upsilon)$ is defined as follows.

(i) The objects of $\mathcal{F}(\Upsilon)$ are the objects of Υ.

(ii) An arrow of $\mathcal{F}(\Upsilon)$ is a morphism from a directed graph

$$\underbrace{\bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet}_{n}$$

to Υ for some $n \geq 0$. The source of this arrow is the object of Υ to which the leftmost object of the directed graph is sent. The target of this arrow is the object of Υ to which the rightmost object of the directed graph is sent.

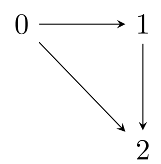
(iii) Composition is concatenation.

(iv) The identity arrow corresponding to an object of Υ is the morphism of directed graphs from $\bullet$ to Υ defined by this object.

A.1.2.2. Example The free category on the directed graph

$$0 \longrightarrow 1 \longrightarrow 2$$

is the directed graph



equipped with the unique possible structure of a category. In particular, we have that $(1 \to 2) \circ (0 \to 1) = (0 \to 2)$. This example captures the essence of the free category construction — we formally add in composites.

A.1.2.3. Definition Let

$$\Upsilon_0 \xrightarrow{F} \Upsilon_1$$

be a morphism of directed graphs. We denote by

$$\mathcal{F}(\Upsilon_0) \xrightarrow{\mathcal{F}(F)} \mathcal{F}(\Upsilon_1)$$

the functor defined as follows.

(i) To an object x of $\mathcal{F}(\Upsilon_0)$, we associate the object $F(x)$ of $\mathcal{F}(\Upsilon_1)$.

(ii) To an arrow

$$x_0 \xrightarrow{f_1} x_1 \longrightarrow \cdots\cdots \longrightarrow x_{n-1} \xrightarrow{f_n} x_n$$

of $\mathcal{F}(\Upsilon_0)$, we associate the arrow

$$F(x_0) \xrightarrow{F(f_1)} F(x_1) \longrightarrow \cdots\cdots \longrightarrow F(x_{n-1}) \xrightarrow{F(f_n)} F(x_n)$$

of $\mathcal{F}(\Upsilon_1)$.

A.1.2.4. Associating to a directed graph Υ the category $\mathcal{F}(\Upsilon)$, and associating to a morphism of directed graphs

$$\Upsilon_0 \xrightarrow{F} \Upsilon_1$$

the functor

$$\mathcal{F}(\Upsilon_0) \xrightarrow{\mathcal{F}(F)} \mathcal{F}(\Upsilon_1)$$

defines a functor from the category of directed graphs to the category of directed graphs. We refer to it as the *free category functor*.

A.1.3. A universal property of a free category

A.1.3.1. Proposition The free category functor $\mathcal{F}$ is left adjoint to the forgetful functor $\mathcal{U}$ from categories to directed graphs.

A.1.3.2. Proof This follows from the following observations.

(1) Let $\mathcal{A}$ be a category, and let Υ be a directed graph. Suppose that

$$\Upsilon \xrightarrow{F} \mathcal{U}(\mathcal{A})$$

defines a morphism of directed graphs. Then the following defines a functor

$$\mathcal{F}(\Upsilon) \xrightarrow{L(F)} \mathcal{A}.$$

(i) To an object x of $\mathcal{F}(\Upsilon)$ we associate the object $G(a)$ of $\mathcal{A}$.

(ii) To an arrow

$$x_0 \xrightarrow{f_1} x_1 \longrightarrow \cdots\cdots \longrightarrow x_{n-1} \xrightarrow{f_n} x_n$$

of $\mathcal{F}(\Upsilon)$ we associate the — unique, by associativity! — composite of the following arrows in $\mathcal{A}$.

$$F(x_0) \xrightarrow{F(f_1)} F(x_1) \longrightarrow \cdots\cdots \longrightarrow F(x_{n-1}) \xrightarrow{F(f_n)} F(x_n)$$

(2) Let $\mathcal{A}$ be a category, and let Υ be a directed graph. Suppose that

$$\mathcal{F}(\Upsilon) \xrightarrow{F} \mathcal{A}$$

defines a functor. Then the following defines a morphism

$$\Upsilon \xrightarrow{R(F)} \mathcal{U}(\mathcal{A})$$

of directed graphs.

(i) To an object x of Υ, we associate the object $F(x)$ of $\mathcal{U}(\mathcal{A})$.

(ii) To an arrow

$$x_0 \xrightarrow{f} x_1$$

of Υ, we associate the arrow

$$F(x_0) \xrightarrow{F(f)} F(x_1)$$

of $\mathcal{A}$.

(3) Let $\mathcal{A}$ be a category, and let Υ be a directed graph. It is immediately verified that $L(-)$ defines a natural transformation from the functor $\mathrm{Hom}_{\mathsf{Cat}}\big(\mathcal{F}(-), -\big)$ to the functor $\mathrm{Hom}_{\mathsf{dGraphs}}\big(-, \mathcal{U}(-)\big)$, and that $R(-)$ defines a natural transformation in the other direction. Here Cat denotes the category of categories, and $\mathsf{dGraphs}$ denotes the category of directed graphs.

(4) It is clear that $L(-)$ and $R(-)$ are inverse to one another.

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