Generell Topologi — Exercise Sheet 2

Richard Williamson

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Guide

To help you to decide which questions to focus on, I have made a few remarks below. A question which is important for one of you may however be less important for another of you — if you need to work on your geometric intuition, for example, prioritise Question 7.

I encourage you to attempt all the questions if you have time — they have all been included for different reasons, to help your understanding. When you come to revise, you should check that you understand all of the solutions that I give.

(1) Questions 5 and 6 are essential, concerning constructions that we will make use of throughout the course.

(2) Questions 1 and 2 will help familiarise you with the axioms of a topological space.

(3) Question 3 tests your understanding of the part of Lecture 1 which approached the construction of a topology on $\mathbb{R}$, and the role of the completeness of $\mathbb{R}$ in this. It also motivates Question 1.4.

(4) Question 4 allows you to practise writing a proof which directly appeals to the axioms of a topological space. The argument is very typical of proofs in this early part of the course.

(5) Question 7 will help develop your geometric intuition, which is a vital aspect of the course. It will also help improve your understanding of subspace and product topologies.

(6) Question 8 and Question 9 give constructions of topological spaces different from those we have met in the lectures so far. Both will help with deepening your understanding of the axioms of a topological space. Both questions are also of wider significance. Questions on future Exercise Sheets will build upon Question 8.
Questions

1

**Question.** Let \( X := \{a, b, c, d\} \) be a set with four elements. Which of the following sets \( \mathcal{O} \) of subsets of \( X \) define a topology on \( X \)?

1. \( \mathcal{O}_1 := \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, d\}, X\} \).
2. \( \mathcal{O}_2 := \{\emptyset, \{a, c\}, \{d\}, \{b, d\}, \{a, c, d\}, X\} \).
3. \( \mathcal{O}_3 := \{\emptyset, \{a\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, X\} \).

2

**Question.**

(a) Find five topologies on \( X := \{a, b, c\} \).

(b) Check whether any of these topologies are the same up to a bijection \( X \rightarrow X \), namely a relabelling of the elements of \( X \). If so, replace it by a different topology. Repeat until you end up with five topologies which are all distinct from each other up to a bijection \( X \rightarrow X \).

(c) Let \( \mathcal{O} \) be one of the five topologies that you ended up with in (b). Find all of the subsets of \( X \) which are closed with respect to \( \mathcal{O} \). Do this for each of the five topologies that you ended up with in (b).

3

**Question.**

(a) Let \( \{[a_j, b_j]\}_{j \in J} \) be a set of (possibly infinitely many) closed intervals in \( \mathbb{R} \). Prove that \( \bigcap_{j \in J} [a_j, b_j] \) is either a closed interval in \( \mathbb{R} \) or \( \emptyset \).

(b) Let \( a, a', b, b' \in \mathbb{R} \). Find a condition to express exactly when \([a, b] \cup [a', b']\) is disjoint, namely when \([a, b] \cap [a', b'] = \emptyset\). Suppose that \([a, b] \cup [a', b']\) is not disjoint. Prove that in this case \([a, b] \cap [a', b']\) is a closed interval in \( \mathbb{R} \).

(c) Let \( \{[a_j, b_j]\}_{j \in J} \) be a set of (possibly infinitely many) closed intervals in \( \mathbb{R} \). Give an example to show that \( \bigcup_{j \in J} [a_j, b_j] \) need not be a closed interval, even if this union cannot be expressed as a disjoint union of a pair of subsets of \( \mathbb{R} \).
4

**Question.** Motivated by Question 3, consider a pair \((X, C)\) of a set \(X\) and a set \(C\) of subsets of \(X\) such that the following conditions are satisfied.

1. \(\emptyset\) belongs to \(C\).
2. \(X\) belongs to \(C\).
3. An intersection of (possibly infinitely many) subsets of \(X\) belonging to \(C\) belongs to \(C\).
4. Let \(V\) and \(V'\) be subsets of \(X\) belonging to \(C\). Then \(V \cup V'\) belongs to \(C\).

Then:

(i) Let \((X, \mathcal{O})\) be a topological space. Let \(C\) denote the set of closed subsets of \(X\) with respect to \(\mathcal{O}\). Prove that \((X, C)\) satisfies the four conditions above.

(ii) Suppose that \((X, C)\) satisfies the four conditions above. Let \(\mathcal{O}\) denote the set of subsets \(U\) of \(X\) such that \(X \setminus U\) belongs to \(C\). Prove that \((X, \mathcal{O})\) defines a topological space.

5

Let \((Y, \mathcal{O}_Y)\) be a topological space, and let \(X\) be a subset of \(Y\). Prove that \((X, \mathcal{O}_X)\) defines a topological space, where

\[
\mathcal{O}_X := \{X \cap U \mid U \in \mathcal{O}_Y\}.
\]

6

Let \((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\) be topological spaces. Prove that \((X \times Y, \mathcal{O}_{X \times Y})\) defines a topological space, where \(\mathcal{O}_{X \times Y}\) denote the set of subsets \(W\) of \(X \times Y\) such that for every \((x, y) \in W\) there exist \(U \in \mathcal{O}_X\) and \(U' \in \mathcal{O}_Y\) with \(x \in U\), \(y \in U'\), and \(U \times U' \subset W\).

7

**Question.**

(a) Equip the subset \(X := [1, 2] \cup [4, 5]\) of \(\mathbb{R}\) with the subspace topology \(\mathcal{O}_X\) with respect to \(\mathcal{O}_\mathbb{R}\). Give and draw an example of a subset \(U\) of \(X\) which belongs to \(\mathcal{O}_X\) in each of the following cases.

1. \(U \cap [4, 5] = \emptyset\), and neither 1 nor 2 belongs to \(U\).
2. \(U \cap [1, 2] = \emptyset\), and 4 does not belong to \(U\).
3. \(U \cap [4, 5] = \emptyset\), and 1 belongs to \(U\).
4. \(U \cap [1, 2] = \emptyset\), and 4 belongs to \(U\).
(v) 2 and 4 both belong to $U$.
(vi) $U \cap [1, 2] \neq \emptyset$, $U \cap [4, 5] \neq \emptyset$, and 1, 2, and 4 all do not belong to $U$.

(b) Let $0 < k < 1$ be a real number. Recall the topological spaces $(A_k, O_{A_k})$ and $(I, O_I)$ from Lecture 1. Equip $A_k \times I$ with the product topology $(A_k \times I, O_{A_k \times I})$. Draw $A_k \times I$, and visualise (draw if you can!) some subsets of $A_k \times I$ belonging to $O_{A_k \times I}$.

(c) Let $X$ be a subset of $\mathbb{R}^2$ consisting of the red and blue parts of the Norwegian flag shown below.

![Norwegian Flag](image)

Equip $X$ with the subspace topology $O_X$ with respect to $(\mathbb{R}^2, O_{\mathbb{R} \times \mathbb{R}})$. Draw an example of a subset $U$ of $X$ belonging to $O_X$ in each of the following cases.

(i) $U$ is contained in the upper right red rectangle.
(ii) $U$ intersects all four red rectangles, and both of the blue rectangles.
(iii) $U$ intersects both of the blue rectangles, but none of the red rectangles.
(iv) $U$ intersects only the horizontal blue rectangle, the upper left red rectangle, and the lower left red rectangle.
(v) $U$ intersects only the vertical blue rectangle and the two upper red rectangles.

(d) Let $X$ be a subset of $\mathbb{R}^2$ as shown below, a ‘blob’ with two open discs cut out.

![Blob with Discs](image)

Equip $X$ with the subspace topology $O_X$ with respect to $(\mathbb{R}^2, O_{\mathbb{R} \times \mathbb{R}})$. Draw an example of a subset $U$ of $X$ belonging to $O_X$ in each of the following cases.

(i) $U$ intersects part but not all of one of the circles, and does not intersect the other circle.
(ii) $U$ intersects all of one circle, and part but not all of the other.
(iii) $U$ intersects part but not all of both circles.
(iv) $U$ intersects neither of the two circles.
(v) $U$ intersects all of both circles, but not all of $X$.

8

A pre-order on a set $X$ consists for every ordered pair $(x, x')$ of distinct elements of $X$ of either one or zero arrows from $x$ to $x'$. We require that for any ordered triple $(x, x', x'')$ of distinct elements of $X$, the following condition is satisfied: if there is an arrow from $x$ to $x'$, and an arrow from $x'$ to $x''$, then there is an arrow from $x$ to $x''$.

Examples.
(1) Let $X = \{0, 1\}$. There are four pre-orders on $X$, pictured below.

\[
\begin{array}{cccccc}
    & 0 & \rightarrow & 1 & \leftarrow & 1 & \rightarrow & 1 & 0 & 1 \\
    & 2 & & & & & & & & \\
\end{array}
\]

The rightmost pre-order should be interpreted as the case that there zero arrows from 0 to 1 and from 1 to 0.

(2) Let $X := \{0, 1, 2\}$. There are 29 possible pre-orders on $X$. A few of them are pictured below.

\[
\begin{array}{cccccc}
    & 0 & \rightarrow & 1 & \leftarrow & 1 & \rightarrow & 1 & 0 & 1 \\
    & 2 & & & & & & & & \\
    & 2 & & & & & & & & \\
\end{array}
\]

The following are not examples of pre-orders on $X$. Check that you understand why!

\[
\begin{array}{cccccc}
    & 0 & \rightarrow & 1 & \leftarrow & 1 & \rightarrow & 1 & 0 & 1 \\
    & 2 & & & & & & & & \\
\end{array}
\]
Let \( X := \mathbb{N} \), the set of natural numbers. The following defines a pre-order on \( X \).

\[
\begin{array}{ccccccccc}
0 & \rightarrow & 1 & \leftarrow & 2 & \rightarrow & 3 & \leftarrow & 4 & \rightarrow & 5 & \leftarrow & 6 & \rightarrow & \ldots \ldots
\end{array}
\]

**Question.** Let \( X \) be a set equipped with a pre-order. For any pair \((x, x')\) of elements of \( X \), we write \( x < x' \) if there is an arrow from \( x \) to \( x' \) or if \( x = x' \). Let \( O_X \) denote the set consisting of the subsets \( U \) of \( X \) with the property that if \( x \in U \) and \( x' \) has the property that \( x < x' \), then \( x' \in U \).

(a) Prove that \((X, O_X)\) defines a topological space.

(b) Which of the four pre-orders on \( X := \{0, 1\} \) corresponds to the topology defining the Sierpiński interval? Which corresponds to the discrete topology? Which to the indiscrete topology?

(c) Find a pre-order on \( X := \{a, b, c\} \) which corresponds to the topology

\[
O := \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}
\]

on \( X \).

(d) List all of the subsets of \( X := \{a, b, c, d\} \) which belong to the topology \( O \) on \( X \) corresponding to the following pre-order.

\[
\begin{array}{ccc}
& a & \\
| & & \\
b & \leftarrow & c \\
| & & \\
d & \rightarrow & \\
\end{array}
\]

The topological space \((X, O)\) is sometimes known as the *pseudo-circle*.

(e) Let \((X, <)\) be a set equipped with a pre-order, and let \( O_X \) denote the corresponding topology on \( X \). Prove that for any set \( \{U_j\}_{j \in J} \) of subsets of \( X \) belonging to \( O_X \) we have that \( \bigcap_{j \in J} U_j \in O_X \). In particular, this holds even if \( J \) is infinite.

**Definition.** A topological space \((X, O)\) is an *Alexandroff space* if for any set \( \{U_j\}_{j \in J} \) of subsets of \( X \) belonging to \( O \) we have that \( \bigcap_{j \in J} U_j \in O \). In particular this holds even if \( J \) is infinite.

**Observation.** Every finite space is an Alexandroff space.
By (a) and (e) we may cook up an Alexandroff space from a pre-order \((X, <)\). We now proceed to establish a converse.

Let \((X, \mathcal{O})\) be an Alexandroff space. For any \(x \in X\), let \(U_x\) denote the intersection of all subsets of \(X\) which contain \(x\) and which belong to \(\mathcal{O}\). For any \(x' \in X\), define \(x < x'\) if \(U_x \subset U_{x'}\).

**Question (continued).**

(f) Prove that \(<\) defines a pre-order on \(X\).

(g) Draw the pre-order corresponding to the topology on \(X := \{a, b, c, d, e\}\) given by

\[
\mathcal{O} := \{\emptyset, \{a, b\}, \{c\}, \{d, e\}, \{a, b, c\}, \{c, d, e\}, \{a, b, d, e\}, X\}.
\]

**Question.** Let \(\mathbb{Z}\) denote the set of integers. Let us denote the set of prime numbers by \(\text{Spec}(\mathbb{Z})\). For any integer \(n\), let

\[
V(n) := \{p \in \mathbb{Z} \mid p \text{ is prime, and } p \mid n\}.
\]

Prove that

\[
\mathcal{O} := \{\text{Spec}(\mathbb{Z}) \setminus V(n) \mid n \in \mathbb{Z}\}
\]

defines a topology on \(\text{Spec}(\mathbb{Z})\). If you wish you may appeal to Question 4.

**Remark.** This topology is known as the Zariski topology on \(\mathbb{Z}\). A generalisation defines a topology on the set of prime ideals in any commutative ring, which is the starting point of algebraic geometry.