Generell Topologi — Exercise Sheet 2

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Guide

To help you to decide which questions to focus on, I have made a few remarks below. A question which is important for one of you may however be less important for another of you — if you need to work on your geometric intuition, for example, prioritise Question 7.

I encourage you to attempt all the questions if you have time — they have all been included for different reasons, to help your understanding. When you come to revise, you should check that you understand all of the solutions that I give.

(1) Questions 5 and 6 are essential, concerning constructions that we will make use of throughout the course.

(2) Questions 1 and 2 will help familiarise you with the axioms of a topological space.

(3) Question 3 tests your understanding of the part of Lecture 1 which approached the construction of a topology on $\mathbb{R}$, and the role of the completeness of $\mathbb{R}$ in this. It also motivates Question 1.4.

(4) Question 4 allows you to practise writing a proof which directly appeals to the axioms of a topological space. The argument is very typical of proofs in this early part of the course.

(5) Question 7 will help develop your geometric intuition, which is a vital aspect of the course. It will also help improve your understanding of subspace and product topologies.

(6) Question 8 and Question 9 give constructions of topological spaces different from those we have met in the lectures so far. Both will help with deepening your understanding of the axioms of a topological space. Both questions are also of wider significance. Questions on future Exercise Sheets will build upon Qeustion 8.

Questions

1

Question. Let $X := \{a, b, c, d\}$ be a set with four elements. Which of the following sets $\mathcal{O}$ of subsets of X define a topology on X?

(1) $\mathcal{O}_1 := \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, d\}, X\}$.

(2) $\mathcal{O}_2 := \{\emptyset, \{a, c\}, \{d\}, \{b, d\}, \{a, c, d\}, X\}$.

(3) $\mathcal{O}_3 := \{\emptyset, \{a\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, X\}$.

2

Question.

(a) Find five topologies on $X := \{a, b, c\}$.

(b) Check whether any of these topologies are the same up to a bijection

$$X \longrightarrow X,$$

namely a relabelling of the elements of X. If so, replace it by a different topology. Repeat until you end up with five topologies which are all distinct from each other up to a bijection

$$X \longrightarrow X.$$

(c) Let $\mathcal{O}$ be one of the five topologies that you ended up with in (b). Find all of the subsets of X which are closed with respect to $\mathcal{O}$. Do this for each of the five topologies that you ended up with in (b).

3

Question.

(a) Let $\left\{ [a_j, b_j] \right\}_{j \in J}$ be a set of (possibly infinitely many) closed intervals in $\mathbb{R}$. Prove that $\bigcap_{j \in J} [a_j, b_j]$ is either a closed interval in $\mathbb{R}$ or $\emptyset$.

(b) Let $a, a', b, b' \in \mathbb{R}$. Find a condition to express exactly when $[a, b] \cup [a', b']$ is disjoint, namely when $[a, b] \cap [a', b'] = \emptyset$. Suppose that $[a, b] \cup [a', b']$ is not disjoint. Prove that in this case $[a, b] \cap [a', b']$ is a closed interval in $\mathbb{R}$.

(c) Let $\left\{ [a_j, b_j] \right\}_{j \in J}$ be a set of (possibly infinitely many) closed intervals in $\mathbb{R}$. Give an example to show that $\bigcup_{j \in J} [a_j, b_j]$ need not be a closed interval, even if this union cannot be expressed as a disjoint union of a pair of subsets of $\mathbb{R}$.

4

Question. Motivated by Question 3, consider a pair $(X, \mathcal{C})$ of a set X and a set $\mathcal{C}$ of subsets of X such that the following conditions are satisfied.

(1) $\emptyset$ belongs to $\mathcal{C}$.

(2) X belongs to $\mathcal{C}$.

(3) An intersection of (possibly infinitely many) subsets of X belonging to $\mathcal{C}$ belongs to $\mathcal{C}$.

(4) Let V and V' be subsets of X belonging to $\mathcal{C}$. Then $V \cup V'$ belongs to $\mathcal{C}$.

Then:

(i) Let $(X, \mathcal{O})$ be a topological space. Let $\mathcal{C}$ denote the set of closed subsets of X with respect to $\mathcal{O}$. Prove that $(X, \mathcal{C})$ satisfies the four conditions above.

(ii) Suppose that $(X, \mathcal{C})$ satisfies the four conditions above. Let $\mathcal{O}$ denote the set of subsets U of X such that $X \setminus U$ belongs to $\mathcal{C}$. Prove that $(X, \mathcal{O})$ defines a topological space.

5

Let $(Y, \mathcal{O}_Y)$ be a topological space, and let X be a subset of Y. Prove that $(X, \mathcal{O}_X)$ defines a topological space, where

$$\mathcal{O}_X := \{X \cap U \mid U \in \mathcal{O}_Y\}.$$

6

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that $(X \times Y, \mathcal{O}_{X \times Y})$ defines a topological space, where $\mathcal{O}_{X \times Y}$ denote the set of subsets W of $X \times Y$ such that for every $(x, y) \in W$ there exist $U \in \mathcal{O}_X$ and $U' \in \mathcal{O}_Y$ with $x \in U$, $y \in U'$, and $U \times U' \subset W$.

7

Question.

(a) Equip the subset $X := [1, 2] \cup [4, 5)$ of $\mathbb{R}$ with the subspace topology $\mathcal{O}_X$ with respect to $\mathcal{O}_{\mathbb{R}}$. Give and draw an example of a subset U of X which belongs to $\mathcal{O}_X$ in each of the following cases.

   (i) $U \cap [4, 5) = \emptyset$, and neither 1 nor 2 belongs to U.

   (ii) $U \cap [1, 2] = \emptyset$, and 4 does not belong to U.

   (iii) $U \cap [4, 5) = \emptyset$, and 1 belongs to U.

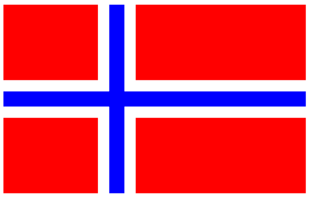
   (iv) $U \cap [1, 2] = \emptyset$, and 4 belongs to U.

(v) 2 and 4 both belong to U.

(vi) $U \cap [1,2] \neq \emptyset$, $U \cap [4,5] \neq \emptyset$, and 1, 2, and 4 all do not belong to U.

(b) Let $0 < k < 1$ be a real number. Recall the topological spaces $(A_k, \mathcal{O}_{A_k})$ and $(I, \mathcal{O}_I)$ from Lecture 1. Equip $A_k \times I$ with the product topology $(A_k \times I, \mathcal{O}_{A_k \times I})$. Draw $A_k \times I$, and visualise (draw if you can!) some subsets of $A_k \times I$ belonging to $\mathcal{O}_{A_k \times I}$.

(c) Let X be a subset of $\mathbb{R}^2$ consisting of the red and blue parts of the Norwegian flag shown below.



Equip X with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R} \times \mathbb{R}})$. Draw an example of a subset U of X belonging to $\mathcal{O}_X$ in each of the following cases.

(i) U is contained in the upper right red rectangle.

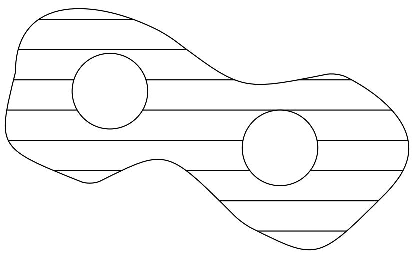
(ii) U intersects all four red rectangles, and both of the blue rectangles.

(iii) U intersects both of the blue rectangles, but none of the red rectangles.

(iv) U intersects only the horizontal blue rectangle, the upper left red rectangle, and the lower left red rectangle.

(v) U intersects only the vertical blue rectangle and the two upper red rectangles.

(d) Let X be a subset of $\mathbb{R}^2$ as shown below, a 'blob' with two open discs cut out.



Equip X with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R} \times \mathbb{R}})$. Draw an example of a subset U of X belonging to $\mathcal{O}_X$ in each of the following cases.

(i) U intersects part but not all of one of the circles, and does not intersect the other circle.

(ii) U intersects all of one circle, and part but not all of the other.

(iii) U intersects part but not all of both circles.

(iv) U intersects neither of the two circles.

(v) U intersects all of both circles, but not all of X.

8

A *pre-order* on a set X consists for every ordered pair (x, x') of distinct elements of X of either one or zero arrows from x to x'. We require that for any ordered triple (x, x', x'') of distinct elements of X, the following condition is satisfied: if there is an arrow from x to x', and an arrow from x' to x'', then there is an arrow from x to x''.

Examples.

(1) Let $X = \{0, 1\}$. There are four pre-orders on X, pictured below.

$$0 \longrightarrow 1 \qquad 0 \longleftarrow 1 \qquad 0 \rightleftarrows 1 \qquad 0 \quad\quad 1$$

The rightmost pre-order should be interpreted as the case that there zero arrows from 0 to 1 and from 1 to 0.

(2) Let $X := \{0, 1, 2\}$. There are 29 possible pre-orders on X. A few of them are pictured below.

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \\ & 2 \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, shift left] & 1 \arrow[l, shift left] \\ & 2 \arrow[ul] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 1 \\ & 2 \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 & 1 \arrow[d] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 & 1 \\ & 2 \end{tikzcd}$$

The following are not examples of pre-orders on X. Check that you understand why!

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[d] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, shift left] \arrow[dr] & 1 \arrow[l, shift left] \arrow[d, shift left] \\ & 2 \arrow[u, shift left] \end{tikzcd}$$

(3) Let $X := \mathbb{N}$, the set of natural numbers. The following defines a pre-order on X.

$$0 \longrightarrow 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5 \longleftarrow 6 \longrightarrow \quad \cdots\cdots\cdots$$

Question. Let X be a set equipped with a pre-order. For any pair (x, x') of elements of X, we write $x < x'$ if there is an arrow from x to x' or if $x = x'$. Let $\mathcal{O}_X$ denote the set consisting of the subsets U of X with the property that if $x \in U$ and x' has the property that $x < x'$, then $x' \in U$.

(a) Prove that $(X, \mathcal{O}_X)$ defines a topological space.

(b) Which of the four pre-orders on $X := \{0, 1\}$ corresponds to the topology defining the Sierpiński interval? Which corresponds to the discrete topology? Which to the indiscrete topology?

(c) Find a pre-order on $X := \{a, b, c\}$ which corresponds to the topology

$$\mathcal{O} := \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$$

on X.

(d) List all of the subsets of $X := \{a, b, c, d\}$ which belong to the topology $\mathcal{O}$ on X corresponding to the following pre-order.

$$\begin{tikzcd}
& a \arrow[dl] \arrow[dr] & \\
b & & c \\
& d \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The topological space $(X, \mathcal{O})$ is sometimes known as the *pseudo-circle*.

(e) Let $(X, <)$ be a set equipped with a pre-order, and let $\mathcal{O}_X$ denote the corresponding topology on X. Prove that for any set $\{U_j\}_{j \in J}$ of subsets of X belonging to $\mathcal{O}_X$ we have that $\bigcap_{j \in J} U_j \in \mathcal{O}_X$. In particular, this holds even if J is infinite.

Definition. A topological space $(X, \mathcal{O})$ is an *Alexandroff space* if for any set $\{U_j\}_{j \in J}$ if subsets of X belonging to $\mathcal{O}$ we have that $\bigcap_{j \in J} U_j \in \mathcal{O}$. In particular this holds even if J is infinite.

Observation. Every finite space is an Alexandroff space.

By (a) and (e) we may cook up an Alexandroff space from a pre-order $(X, <)$. We now proceed to establish a converse.

Let $(X, \mathcal{O})$ be an Alexandroff space. For any $x \in X$, let U_x denote the intersection of all subsets of X which contain x and which belong to $\mathcal{O}$. For any $x' \in X$, define $x < x'$ if $U_x \subset U_{x'}$.

Question (continued).

(f) Prove that $<$ defines a pre-order on X.

(g) Draw the pre-order corresponding to the topology on $X := \{a, b, c, d, e\}$ given by

$$\mathcal{O} := \big\{\emptyset, \{a, b\}, \{c\}, \{d, e\}, \{a, b, c\}, \{c, d, e\}, \{a, b, d, e\}, X\big\}.$$

9

Question. Let $\mathbb{Z}$ denote the set of integers. Let us denote the set of prime numbers by $\mathsf{Spec}(\mathbb{Z})$. For any integer n, let

$$V(n) := \{p \in \mathbb{Z} \mid p \text{ is prime, and } p \mid n\}.$$

Prove that

$$\mathcal{O} := \{\mathsf{Spec}(\mathbb{Z}) \setminus V(n) \mid n \in \mathbb{Z}\}$$

defines a topology on $\mathsf{Spec}(\mathbb{Z})$. If you wish you may appeal to Question 4.

Remark. This topology is known as the *Zariski topology* on $\mathbb{Z}$. A generalisation defines a topology on the set of prime ideas in any commutative ring, which is the starting point of algebraic geometry.