Norwegian University of Science and Technology
Department of Mathematical Sciences

MA3002 General Topology — Solutions to 2013 Exam

Problem 1

a) We have the following.

(i) Not a topology. For example, $\{a,b\} \cap \{b,c,d\} = \{b\}$, and $\{b\}$ does not belong to the given set.

(ii) Is a topology, by inspection.

(iii) Not a topology. For example, $\{b\} \cup \{a,d\} = \{a,b,d\}$, and $\{a,b,d\}$ does not belong to the given set.

b) Yes, the set $\{c,d\}$ is closed in $(X,\mathcal{O})$. To see this we can observe that $X \setminus \{c,d\} = \{a,b\}$ and $\{a,b\} \in \mathcal{O}$.

c) $\partial_X\{a,b,c\} = \{c,d\}$.

d) Yes, $(X,\mathcal{O}_X)$ is compact, since X is finite.

e) No, f is not continuous. For example, we have that $f^{-1}\big(\{a',d'\}\big) = \{b,c\} \notin \mathcal{O}$.

f) No, $(Y,\mathcal{O}_Y)$ is not connected. We have that $Y = \{a,b\} \sqcup \{c,d,e\}$. Both $\{a,b\}$ and $\{c,d,e\}$ belong to $\mathcal{O}_Y$.

Problem 2

a) Yes, $(\mathsf{A},\mathcal{O}_{\mathsf{A}})$ is compact. By Proposition 14.9 in the Lecture Notes, a subset of $\mathbb{R}^2$ is compact if it is closed in $(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})$ and bounded. We have that A is closed in $(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})$ and bounded.

b) An example of open covering of X which does not admit a finite subcovering is

$$\big\{(-n,n) \times [-3,3]\big\}_{n\in\mathbb{N}}.$$

c) $(X, \mathcal{O}_X)$ is connected if and only if X is an open interval (a, b), a closed interval $[a, b]$, a half open interval $[a, b)$, or a half open interval $(a, b]$, where $a, b \in \mathbb{R}$.

As an aside: the empty set may be added to this list or seen as a special case of any of these possibilities. It is not required that the empty set be mentioned at all.

d) Suppose that

$$\mathsf{Q} \xrightarrow{f} \mathsf{A}$$

is a homeomorphism.

Let $x \in \mathsf{Q}$. Let $\mathsf{Q} \setminus \{x\}$ be equipped with its subspace topology with respect to $(\mathsf{Q}, \mathcal{O}_\mathsf{Q})$, and let $\mathsf{A} \setminus \{f(x)\}$ be equipped with its subspace topology with respect to $(\mathsf{A}, \mathcal{O}_\mathsf{A})$.

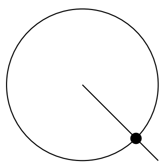
By Proposition 8.1 in the Lecture Notes we then have that the map

$$\mathsf{Q} \setminus \{x\} \longrightarrow \mathsf{A} \setminus \{f(x)\}$$

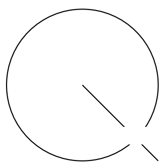
given by restricting f is also a homeomorphism for every $x \in \mathsf{Q}$.

We deduce by Proposition 8.14 in the Lecture Notes that $\mathsf{Q} \setminus \{x\}$ has the same number of connected components as $\mathsf{A} \setminus \{f(x)\}$.

Let $x \in \mathsf{Q}$ be the junction point pictured below.



Then $\mathsf{Q} \setminus \{x\}$ has three connected components.



There is no $y \in \mathsf{A}$ such that $\mathsf{A} \setminus \{y\}$ equipped with its subspace topology with respect to $(\mathsf{A}, \mathcal{O}_\mathsf{A})$ has three connected components. Depending on which y is chosen, $\mathsf{A} \setminus \{y\}$ has either one or two connected components.

Problem 3

a) Let $(x_0, x_1) \in \mathbb{R}^2$ and $(y_0, y_1) \in \mathbb{R}^2$ be distinct. At least one of the following holds.

(1) $x_0 \neq y_0$. Then let us define $\epsilon_0 \in \mathbb{R}$ to be $\frac{|y_0 - x_0|}{2}$, and define $\epsilon_1 > 0$ arbitrarily.

(2) $x_1 \neq y_1$. Then let us define $\epsilon_1 \in \mathbb{R}$ to be $\frac{|y_1 - x_1|}{2}$, and define $\epsilon_0 > 0$ arbitrarily.

If both (1) and (2) hold, we just pick one.

Let
$$U_0 = (x_0 - \epsilon_0, x_0 + \epsilon_0) \times (x_1 - \epsilon_1, x_1 + \epsilon_1).$$
Let
$$U_1 = (y_0 - \epsilon_0, y_0 + \epsilon_0) \times (y_1 - \epsilon_1, y_1 + \epsilon_1).$$

We make the following observations.

(1) $U_0, U_1 \in \mathcal{O}_{\mathbb{R}^2}$.

(2) $(x_0, x_1) \in U_0$ and $(y_0, y_1) \in U_1$.

(3) $U_0 \cap U_1 = \emptyset$.

A picture can be sufficient to obtain most of the marks.

b) Suppose that $U \in \mathcal{O}_X$. Then $p^{-1}(U) = U \times Y$. We have that $U \times Y \in \mathcal{O}_{X \times Y}$.

c) Let $x_0, x_1 \in X$. We make the following observations.

(i) By assumption (2) we can construct a path from either x or x' to x_0. Thus by Proposition 10.8 in the Lecture Notes we can construct a path from x_0 to either x or x'.

(ii) By assumption (1) we can construct a path from x to x'. By Proposition 10.8 in the Lecture Notes we can thus also construct a path from x' to x.

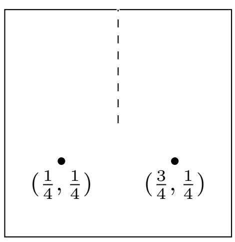
(iii) By (i), (ii) and Proposition 10.11 in the Lecture Notes we can construct a path from x_0 to both x and x'.

(iv) By assumption (2) we can construct a path from either x or x' to x_1.

(v) By (iii), (iv), and Proposition 10.11 in the Lecture Notes we can construct a path from x_0 to x_1.

A picture, with captions as needed, can be sufficient to obtain most of the marks, even all of the marks if it is clear that the idea of the proof is understood.

d) Yes, $(Y, \mathcal{O}_Y)$ is connected. We can join every point in Y to one of the two points depicted below by a straight line, and these two points can themselves be joined by a straight line.



Thus by part (c) we have that $(Y, \mathcal{O}_Y)$ is path connected.

By Proposition 10.17 in the Lecture Notes, we have that a path connected topological space is connected. We deduce that $(Y, \mathcal{O}_Y)$ is connected.

Note that due to the fact that $(\frac{1}{2}, \frac{1}{2}) \in Y$, it is not the case that $(Y, \mathcal{O}_Y)$ is homeomorphic to a product of a closed interval and a half open interval. An argument which asserts this but otherwise correctly quotes results from the course to deduce that $(Y, \mathcal{O}_Y)$ is connected should be awarded most of the marks.

e) No, $(Y, \mathcal{O}_Y)$ is not locally compact. Let U be any neighbourhood of $(\frac{1}{2}, \frac{1}{2})$. Then the closure $\overline{U}$ of U in $(Y, \mathcal{O}_Y)$ is not compact.

To obtain most of the marks, it is sufficient to assert this. To obtain full marks, some indication of the following argument — such as a picture — should be given. Full details are not required.

There exist $a, b, a', b' \in \mathbb{R}$ such that $Y \cap \big((a,b) \times (a',b')\big) \subset U$. The closure of $Y \cap \big((a,b) \times (a',b')\big)$ in $(Y, \mathcal{O}_Y)$ is

$$\big([a,b] \times [a',b']\big) \setminus \big\{(x,y) \in I^2 \mid x = \frac{1}{2} \text{ and } y > \frac{1}{2}\big\}.$$

We have that the closure of $Y \cap \big((a,b) \times (a',b')\big)$ in $(Y, \mathcal{O}_Y)$ is a subset of the closure of U in $(Y, \mathcal{O}_Y)$. Thus, by Proposition 13.7 in the Lecture Notes, if $\overline{U}$ were a compact subset of $(Y, \mathcal{O}_Y)$ we would have that

$$\big([a,b] \times [a',b']\big) \setminus \big\{(x,y) \in I^2 \mid x = \frac{1}{2} \text{ and } y > \frac{1}{2}\big\}$$

is a compact subset of $(Y, \mathcal{O}_Y)$. This is not the case, as can be seen in either of the following ways.

(1) An explicit open covering with no finite subcovering can be exhibited.

(2) A compact subset of Y is also a compact subset of $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Thus by Proposition 14.9 in the Lecture Notes we would have that

$$\big([a,b] \times [a',b']\big) \setminus \big\{(x,y) \in I^2 \mid x = \frac{1}{2} \text{ and } y > \frac{1}{2}\big\}$$

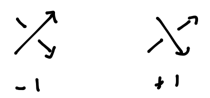
is closed in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. This is not the case.

Problem 4

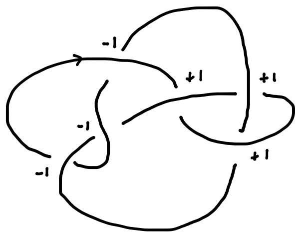
a) A knot is a subset K of $\mathbb{R}^3$ such that $(K, \mathcal{O}_K)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$, where $\mathcal{O}_K$ denotes the subspace topology on K with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$.

Complete precision is not necessary to obtain full marks.

b) Either convention for the sign of a crossing may be used. The convention adopted here is as follows.

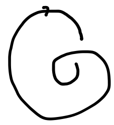


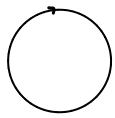
The signs of the crossings are then as follows.



Thus the writhe is 0.

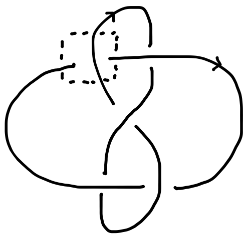
c) No, two isotopic knots do not necessarily have the same writhe. For example, the two knots below are isotopic since one is obtained from the other by an R1 Reidemeister move.



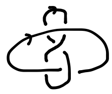


The writhe of the left knot is -1, and the writhe of the right knot is 0.

d) Let L be the given link, which is the Whitehead link. We work with the crossing indicated below.



We have the following.

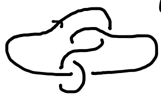
V  $(t) = V$  (t)

$= V$  (t)

$= V$  (t)

$= -t^{5/2} - t^{1/2}$

We also have the following.

V  $(t) = V$  (t)

$= t^{-2} - t^{-1} + 1 - t + t^{2}$

By skein relation 2) we thus have that

$$t^{-1}(-t^{\frac{5}{2}} - t^{\frac{1}{2}}) - tV_L(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(t^{-2} - t^{-1} + 1 - t + t^2).$$

Hence

$$-t^{\frac{3}{2}} - t^{-\frac{1}{2}} - tV_L(t) = t^{-\frac{3}{2}} - t^{-\frac{1}{2}} + t^{\frac{1}{2}} - t^{\frac{3}{2}} + t^{\frac{5}{2}} - t^{-\frac{5}{2}} + t^{-\frac{3}{2}} - t^{-\frac{1}{2}} + t^{\frac{1}{2}} - t^{\frac{3}{2}}.$$

Thus

$$-t^{\frac{3}{2}} - t^{-\frac{1}{2}} - tV_L(t) = -t^{-\frac{5}{2}} + 2t^{-\frac{3}{2}} - 2t^{-\frac{1}{2}} + 2t^{\frac{1}{2}} - 2t^{\frac{3}{2}} + t^{\frac{5}{2}}.$$

Hence

$$-tV_L(t) = -t^{-\frac{5}{2}} + 2t^{-\frac{3}{2}} - t^{-\frac{1}{2}} + 2t^{\frac{1}{2}} - t^{\frac{3}{2}} + t^{\frac{5}{2}}.$$

Thus

$$V_L(t) = t^{-\frac{7}{2}} - 2t^{-\frac{5}{2}} + t^{-\frac{3}{2}} - 2t^{-\frac{1}{2}} + t^{\frac{1}{2}} - t^{\frac{3}{2}}.$$

For the crossing to the right of the one we worked with above, the calculation is essentially the same.

Working at either of the bottom two crossings, one requires the Jones polynomials of the Hopf link with the other orientation and of the figure of eight knot.

Working at the middle crossing, one requires the other two Jones polynomials which were allowed to be assumed.

Half a mark is to be deducted for a minor error in the algebraic manipulations.

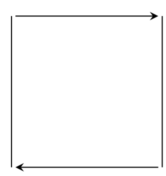
Between one mark and two marks are to be deducted for errors indicating a lack of understanding of how to work with Laurent polynomials.

One mark is to be deducted for an incorrect application of the Reidemeister moves.

e) No, the link L is not isotopic to its mirror image. By a result from the course, if a link is isotopic to its mirror image then its Jones polynomial is palindromic. By d) the Jones polynomial of L is not palindromic.

Problem 5

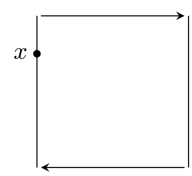
a) The equivalence relation $\sim$ can be defined for example to be that generated by the relation $(t, 0) \sim (1 - t, 1)$ for all $t \in I$.



b) By Proposition 13.2 in the Lecture Notes we have that $(I, \mathcal{O}_I)$ is compact. By Proposition 14.4 in the Lecture Notes a product of compact topological spaces is compact. Thus $(I^2, \mathcal{O}_{I^2})$ is compact. By Proposition 13.4 a quotient of a compact topological space is compact. We conclude that $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ is compact.

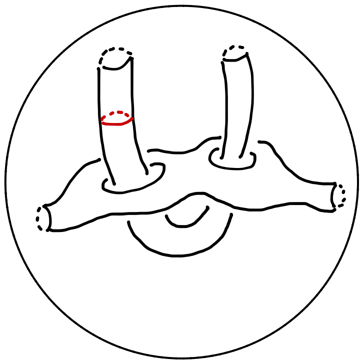
By Proposition 7.9 in the Lecture Notes we have that $(I, \mathcal{O}_I)$ is connected. By Proposition 7.13 in the Lecture Notes a product of connected topological spaces is connected. Thus $(I^2, \mathcal{O}_{I^2})$ is connected. By Proposition 7.15 a quotient of a connected topological space is connected. We conclude that $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ is connected.

c) No, the Möbius band is not a surface. Let $x \in I^2$ be such that $x = (0, t)$ or $x = (1, t)$ for some $t \in I$.

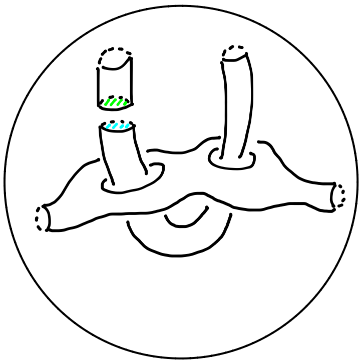


No neighbourhood of x is homeomorphic to an open disc.

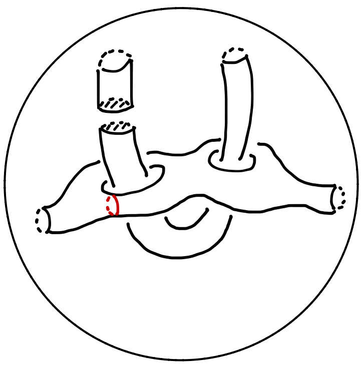
d) Four surgeries are needed. For example, first cut along the following curve.



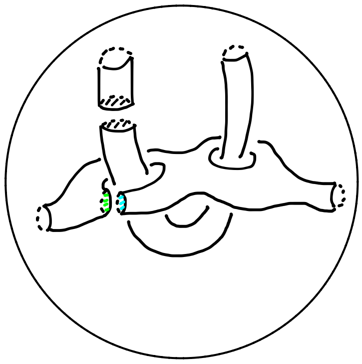
Glue in two discs along the resulting boundary circles.



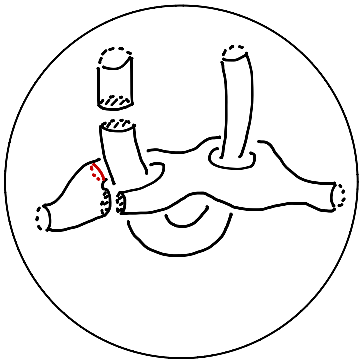
Next cut along the following curve.



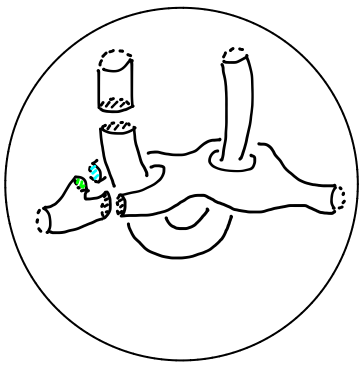
Glue in two discs along the resulting boundary circles.



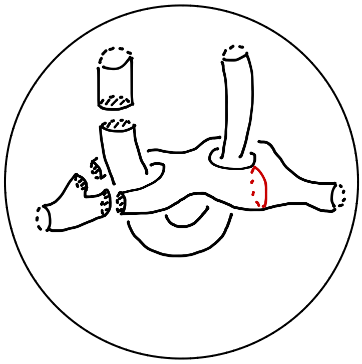
Next cut along the following curve.



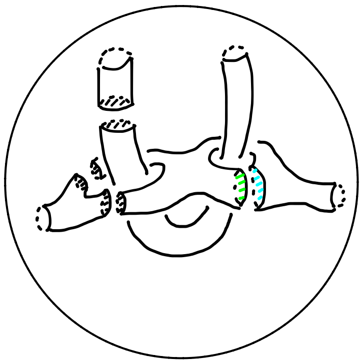
Glue in two discs along the resulting boundary circles.



Next cut along the following curve.



Glue in two discs along the resulting boundary circles.



e) By a result from the course, $\chi(S^2) = 2$. By another result from the course, glueing on a handle decreases Euler characteristic by 2. We conclude that $\chi(X) = 2 - (4 \cdot 2) = 2 - 8 = -6$.