Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact person during the exam:
Richard Williamson, (735) 90154

MA3002 General Topology

Saturday 1st June 2013
Time: 09:00 – 13:00
Examination aids: Code D
No written or handwritten support materials permitted.
Calculator: Hewlett Packard HP30S, Citizen SR-270X or Citizen SR-270X College.

Answer four of the five problems. If you answer all five problems, your four best answers will count.

Each problem is worth 25 marks. The apporoximate marks for each part are indicated in brackets. Your grade will be determined by your mark and by a qualitative overall assessment of your answers.

Problem 1

a) Which of the following sets define a topology on the set $X = \{a, b, c, d\}$?

(i) $\big\{\emptyset, \{a, b\}, \{c, d\}, \{b, c, d\}, X\big\}$

(ii) $\big\{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\big\}$

(iii) $\big\{\emptyset, \{b\}, \{c\}, \{a, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\big\}$

Give a reason for any which do not. [5]

b) Let $\mathcal{O}$ be the topology

$$\big\{\emptyset, \{b\}, \{a, b\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\big\}$$

on X. Is the set $\{c, d\}$ closed in $(X, \mathcal{O})$? Give a reason. [2]

c) Calculate the boundary of the set $\{a, b, c\}$ in $(X, \mathcal{O})$. [5]

d) Is $(X, \mathcal{O})$ compact? [3]

e) Let $X' = \{a', b', c', d'\}$, and let $\mathcal{O}'$ be the topology on X' with basis

$$\big\{\{a'\}, \{a', b'\}, \{a', c'\}, \{a', d'\}\big\}.$$

Let

$$X \xrightarrow{f} X'$$

be the map given by $a \mapsto b'$, $b \mapsto a'$, $c \mapsto d'$, $d \mapsto c'$. Is f continuous? Justify your answer. [5]

f) Let $Y = \{a, b, c, d, e\}$ and let $\mathcal{O}_Y$ be the topology

$$\big\{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{c, d, e\}, \{b, c, d, e\}, Y\big\}$$

on Y. Is $(Y, \mathcal{O}_Y)$ connected? Give a reason. [5]

Problem 2

a) Regard the letter A as a subset of $\mathbb{R}^2$ equipped with its subspace topology $\mathcal{O}_{\mathsf{A}}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

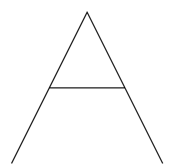
Explicitly, let A be the union of the sets

$$\big\{(x-1, y) \mid 0 \leq x \leq 1 \text{ and } y = x\big\},$$

$$\big\{(x, y+1) \mid 0 \leq x \leq 1 \text{ and } y = -x\big\},$$

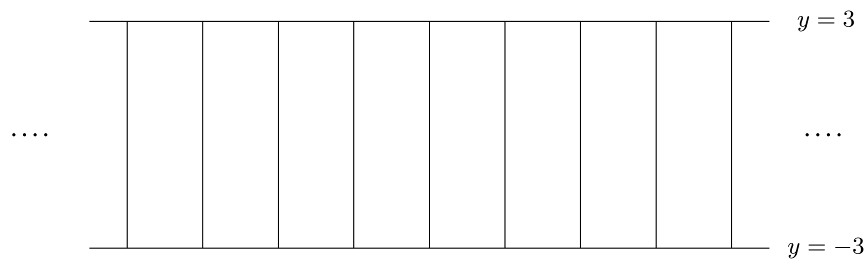
and

$$\big\{(x, y) \mid -\frac{1}{2} \leq x \leq \frac{1}{2} \text{ and } y = \frac{1}{2}\big\}.$$



Is $(\mathsf{A}, \mathcal{O}_{\mathsf{A}})$ compact? Give a reason. [7]

b) Let $X = \big\{(x, y) \in \mathbb{R}^2 \mid -3 \leq y \leq 3\big\}$ be equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Give an example of an open covering of X which does not admit a finite subcovering. [5]

c) Let X be a subset of $\mathbb{R}$, equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

The following assertion is incorrect: $(X, \mathcal{O}_X)$ is connected if and only if X is an open interval (a, b) or a closed interval $[a, b]$ for some $a, b \in \mathbb{R}$.

What is the correct characterisation of connected subsets of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$? [3]

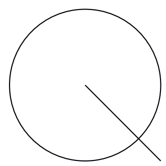
d) Regard the letter Q as a subset of $\mathbb{R}^2$, equipped with its subspace topology $\mathcal{O}_{\mathsf{Q}}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Explicitly, let Q be the union of the sets

$$\left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = 1\right\}$$

and

$$\left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ and } y = -x\right\}.$$



Show that $(\mathsf{A}, \mathcal{O}_{\mathsf{A}})$ is not homeomorphic to $(\mathsf{Q}, \mathcal{O}_{\mathsf{Q}})$.

State without proof the results from the course which you require. [10]

Problem 3

a) Show that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff.

You may not assume without proof any results from the course. [5]

b) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $X \times Y$ be equipped with the product topology $\mathcal{O}_{X \times Y}$. Prove that the map

$$X \times Y \xrightarrow{p} X$$

given by $(x, y) \mapsto x$ is continuous. [5]

c) Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that there are distinct elements $x, x' \in X$ such that the following hold.

(1) There is a path from x to x' in X.

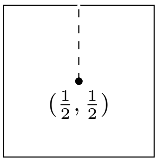
(2) For every $x'' \in X$ there is either a path from x to x'' in X or a path from x' to x'' in X.

Show that $(X, \mathcal{O}_X)$ is path connected.

You may quote without proof any results from the course. [5]

d) Let

$$Y = I^2 \setminus \{(x,y) \in I^2 \mid x = \frac{1}{2} \text{ and } y > \frac{1}{2}\}.$$



Let Y be equipped with its subspace topology $\mathcal{O}_Y$ with respect to $(I^2, \mathcal{O}_{I^2})$. Is $(Y, \mathcal{O}_Y)$ connected? Justify your answer.

You may quote without proof any result from the course. [5]

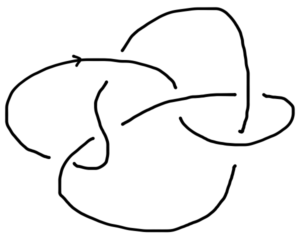
e) Is $(Y, \mathcal{O}_Y)$ locally compact? Justify your answer.

You may quote without proof any result from the course. [5]

Problem 4

a) Define a knot. [3]

b) Calculate the writhe of the oriented knot shown below. [5]



c) Do two isotopic knots necessarily have the same writhe? Give a proof or a counter example. [5]

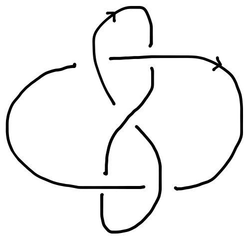
d) The skein relations satisfied by the Jones polynomial of an oriented link are as follows.

1) $V_{\bigcirc}(t) = 1$

unknot

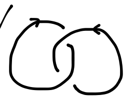
2) $t^{-1} V$  $(t) - t V$  $(t) = (t^{1/2} - t^{-1/2}) V$  (t)

Use the skein relations, or another method, to calculate the Jones polynomial of the oriented link shown below.

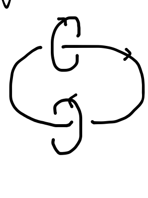


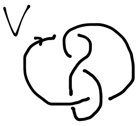
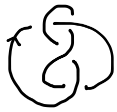
You may assume without proof the following Jones polynomials. [10]

$V_{\bigcirc\bigcirc}(t) = -t^{-1/2} - t^{1/2}$

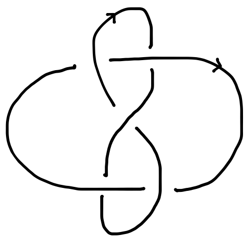
V  $(t) = -t^{5/2} - t^{1/2}$

V  $(t) = -t^{-5/2} - t^{-1/2}$

V  $(t) = t^{-2} + 2 + t^{2}$

$V_{\,}$  $(t) = V$  $(t) = t^{-2} - t^{-1} + 1 - t + t^{2}$

e) Is the oriented link

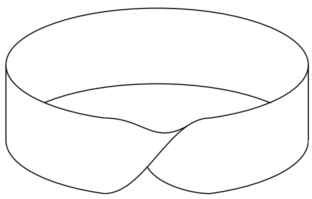


from d) isotopic to its mirror image?

You may quote without proof any results from the course. [2]

Problem 5

a) Define an equivalence relation $\sim$ on I^2 such that $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ is the Möbius band. [3]



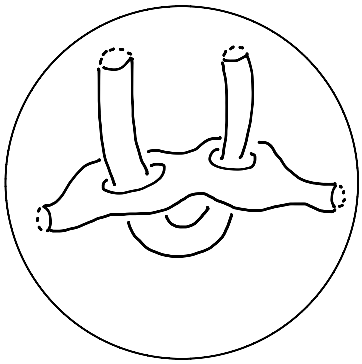
b) Show that the Möbius band is compact and connected.

You may assume that $(I, \mathcal{O}_I)$ is compact and connected, but must give a reason why any other topological spaces that appear in your argument are compact or connected.

You may quote without proof any results from the course, but may not assume unless you can rigorously prove it that the Möbius band can be viewed as a subset of $\mathbb{R}^n$ for some $n \geq 0$. [6]

c) Is the Möbius band a surface? [4]

d) Perform successive surgeries on the topological space $(X, \mathcal{O}_X)$ shown below — a 2-sphere with tunnels — until you obtain a topological space which is homeomorphic to the 2-sphere $(S^2, \mathcal{O}_{S^2})$.



Draw a picture for every surgery and give each picture a brief caption. [7]

e) Hence or otherwise find the Euler characteristic of $(X, \mathcal{O}_X)$.
You may quote without proof any results from the course. [5]