Generell Topologi

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1 Tuesday 15th January

1.1 Topological spaces — definition, terminology, finite examples

Definition 1.1. A *topological space* is a pair $(X, \mathcal{O})$ of a set X and a set $\mathcal{O}_X$ of subsets of X, such that the following conditions are satisfied.

(1) The empty set $\emptyset$ belongs to $\mathcal{O}$.

(2) The set X itself belongs to $\mathcal{O}$.

(3) Let U be a (possibly infinite) union of subsets of X belonging to $\mathcal{O}$. Then U belongs to $\mathcal{O}$.

(4) Let U and U' be subsets of X belonging to $\mathcal{O}$. Then the set $U \cap U'$ belongs to $\mathcal{O}$.

Remark 1.2. By induction, the following condition is equivalent to (4).

(4') Let $\{U_j\}_{j \in J}$ be a finite set of subsets of X belonging to $\mathcal{O}$. Then $\bigcap_j U_j$ belongs to $\mathcal{O}$.

Terminology 1.3. Let $(X, \mathcal{O})$ be a topological space. We refer to $\mathcal{O}$ as a *topology* on X.

A set may be able to be equipped with many different topologies! See Examples 1.7.

Convention 1.4. Nevertheless, a topological space $(X, \mathcal{O})$ is often denoted simply by X. To avoid confusion, we will not make use of this convention, at least in the early part of the course.

Notation 1.5. Let X be a set. We will write $A \subset X$ to mean that A is a subset of X, allowing that A may be equal to X. In the past you may instead have written $A \subseteq X$.

Terminology 1.6. Let $(X, \mathcal{O})$ be a topological space. If $U \subset X$ belongs to $\mathcal{O}$, we say that U is an *open subset* of X with respect to $\mathcal{O}$, or simply that U is *open* in X with respect to $\mathcal{O}$.

If $V \subset X$ has the property that $X \setminus V$ is an open subset of X with respect to $\mathcal{O}$, we say that V is a *closed subset* of X with respect to $\mathcal{O}$, or simply that V is *closed* in X with respect to $\mathcal{O}$.

Examples 1.7.

(1) We can equip any set X with the following two topologies.

  (i) *Discrete topology.* Here we define $\mathcal{O}$ to be the set of all subsets of X. In other words, $\mathcal{O}$ is the power set of X.

(ii) *Indiscrete topology.* Here we define $\mathcal{O}$ to be the set $\{\emptyset, X\}$. By conditions (1) and (2) of Definition 1.1, any topology on X must include both $\emptyset$ and X. Thus $\mathcal{O}$ is the smallest topology with which X may be equipped.

(2) Let $X = \{a\}$ be a set with one element. Then X can be equipped with exactly one topology, $\mathcal{O} = \{\emptyset, X\}$. In particular, the discrete topology on X is the same as the indiscrete topology on X.

The topological space $(X, \mathcal{O})$ is important! It is known as the *point*.

(3) Let $X = \{a, b\}$ be a set with two elements. We can define exactly four topologies upon X.

(i) Discrete topology. $\mathcal{O} := \big\{\emptyset, \{a\}, \{b\}, X\big\}$.

(ii) $\mathcal{O} := \big\{\emptyset, \{a\}, X\big\}$.

(iii) $\mathcal{O} := \big\{\emptyset, \{b\}, X\big\}$.

(iv) Indiscrete topology. $\mathcal{O} := \big\{\emptyset, X\big\}$.

Up to the bijection

$$X \xrightarrow{f} X$$

given by $a \mapsto b$ and $b \mapsto a$, or in other words up to relabelling the elements of X, the topologies of (ii) and (iii) are the same.

The topological space $(X, \mathcal{O})$ where $\mathcal{O}$ is defined as in (ii) or (iii) is known as the *Sierpiński interval* or *Sierpiński space*.

(4) Let $X = \{a, b, c\}$ be a set with three elements. We can define exactly 29 topologies upon X! Again, up to relabelling, many of these topologies are the same.

(i) For instance,

$$\mathcal{O} := \big\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\big\}$$

defines a topology on X.

(ii) But $\mathcal{O} := \big\{\emptyset, \{a\}, \{c\}, X\big\}$ does not define a topology on X. This is because

$$\{a\} \cup \{c\} = \{a, c\}$$

does not belong to $\mathcal{O}$, so condition (3) of Definition 1.1 is not satisfied.

(iii) Also, $\mathcal{O} := \big\{\emptyset, \{a, b\}, \{a, c\}, X\big\}$ does not define a topology on X. This is because $\{a, b\} \cap \{b, c\} = \{b\}$ does not belong to $\mathcal{O}$, so condition (4) of Definition 1.1 is not satisfied.

There are quite a few more 'non-topologies' on X.

1.2 Towards a topology on $\mathbb{R}$ — recollections on completeness of $\mathbb{R}$

Notation 1.8. Let $a, b \in \mathbb{R}$.

(1) We refer to a subset of $\mathbb{R}$ of one of the following four kinds as an *open interval*.

  (i) $(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$.

  (ii) $(a, \infty) := \{x \in \mathbb{R} \mid x > a\}$.

  (iii) $(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$.

  (iv) $\mathbb{R}$, which we may sometimes also denote by $(-\infty, \infty)$.

(2) We refer to a subset of $\mathbb{R}$ of the following kind as a *closed interval*.

$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

(3) We refer to a subset of $\mathbb{R}$ of one of the following four kinds as a *half open interval*.

  (i) $[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}$.

  (ii) $(a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}$.

  (iii) $[a, \infty) := \{x \in \mathbb{R} \mid x \geq a\}$.

  (iv) $(-\infty, b] := \{x \in \mathbb{R} \mid x \leq b\}$.

Recollection 1.9. The key property of $\mathbb{R}$ is *completeness*. There are many equivalent characterisations of this property — Theorem 1.10 and Theorem 1.15 are the two characterisations that are of importance to us here.

Theorem 1.10. Let $\{x_j\}_{j \in J}$ be a (possibly infinite) set of real numbers. Suppose that there exists a $b \in \mathbb{R}$ such that $x_j \leq b$ for all $j \in J$. Then there exists a $b' \in \mathbb{R}$ such that:

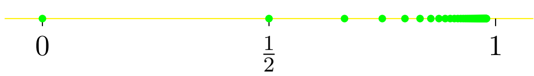
(i) $x_j \leq b'$ for all $j \in J$,

(ii) if $b'' \in \mathbb{R}$ has the property that $x_j \leq b''$ for all $j \in J$, then $b'' \geq b'$.

Remark 1.11. In other words, if $\{x_j\}_{j \in J}$ has an upper bound b, then $\{x_j\}_{j \in J}$ has an upper bound b' which is less than or equal to any upper bound b'' of $\{x_j\}_{j \in J}$.

Terminology 1.12. Let $\{x_j\}_{j \in J}$ be a set of real numbers which admits an upper bound. We refer to the corresponding least upper bound b' of $\{x_j\}_{j \in J}$ that the completeness of $\mathbb{R}$ in the form of Theorem 1.10 gives us as the *supremum* of $\{x_j\}_{j \in J}$. We denote it by $\sup x_j$.

Recollection 1.13. Recall from your early courses in real analysis some examples of a supremum. For instance, the supremum of the set $\{1 - \frac{1}{n}\}_{n \in \mathbb{N}}$ is 1.



The picture shows the elements of $\{1 - \frac{1}{n}\}_{n \in \mathbb{N}}$ for $1 \leq n \leq 50$, getting closer and closer to 1 without reaching it!

Notation 1.14. Let $\{x_j\}_{j\in J}$ be a set of real numbers such that for every $b \in \mathbb{R}$ there is a $k \in J$ with the property that $x_k > b$. In other words, we assume that $\{x_j\}_{j\in J}$ is not bounded above. In this case, we write $\sup x_j = \infty$.

Theorem 1.15. Let $\{x_j\}_{j\in J}$ be a (possibly infinite) set of real numbers. Suppose that there exists a $b \in \mathbb{R}$ such that $x_j \geq b$ for all $j \in J$. Then there exists a $b' \in \mathbb{R}$ such that:

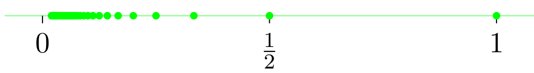
(i) $x_j \geq b'$ for all $j \in J$,

(ii) if $b'' \in \mathbb{R}$ has the property that $x_j \geq b''$ for all $j \in J$, then $b'' \leq b'$.

Remark 1.16. In other words, if $\{x_j\}_{j\in J}$ has a lower bound b, then $\{x_j\}_{j\in J}$ has a lower bound b' which is greater than or equal to any lower bound b'' of $\{x_j\}_{j\in J}$.

Terminology 1.17. Let $\{x_j\}_{j\in J}$ be a set of real numbers which admits a lower bound. We refer to the corresponding greatest upper bound b' of $\{x_j\}_{j\in J}$ that the completeness of $\mathbb{R}$ in the form of Theorem 1.15 gives us as the *infimum* of $\{x_j\}_{j\in J}$. We denote it by $\inf x_j$.

Recollection 1.18. Recall from your early courses in real analysis some examples of an infimum. For instance, the infimum of the set $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is 0.



The picture shows the elements of $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ for $1 \leq n \leq 50$, getting closer and closer to 0 without reaching it!

Notation 1.19. Let $\{x_j\}_{j\in J}$ be a set of real numbers such that for every $b \in \mathbb{R}$ there is a $k \in J$ with the property that $x_k < b$. In other words, we assume that $\{x_j\}_{j\in J}$ is not bounded below. In this case, we write $\inf x_j = -\infty$.

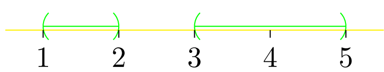
Goal 1.20. To equip $\mathbb{R}$ with a a topology to which the open intervals in $\mathbb{R}$ belong.

Observation 1.21. Let $a, b, a', b' \in \mathbb{R}$. Then

$$(a,b) \cap (a',b') = \begin{cases} \big(\sup\{a,a'\}, \inf\{b,b'\}\big) & \text{if } \sup\{a,a'\} < \inf\{b,b'\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Remark 1.22. Thus condition (4) of Definition 1.1 is satisfied for $\mathcal{O}' := \{\text{open intervals in } \mathbb{R}\}$.

However, condition (3) of Definition 1.1 is not satisfied for $\mathcal{O}' := \{\text{open intervals in } \mathbb{R}\}$. Indeed, take any two open intervals in $\mathbb{R}$ which do not intersect. For example, $(1,2)$ and $(3,5)$. The union of these two open intervals is disjoint, and in particular is not an open interval.



Idea 1.23. Observing this, we might try to enlarge $\mathcal{O}'$ to include disjoint unions of (possibly infinitely many) open intervals in $\mathbb{R}$. This works! The set

$$\mathcal{O} := \{\bigsqcup_{j \in J} U_j \mid U_j \text{ is an open interval in } \mathbb{R}\}$$

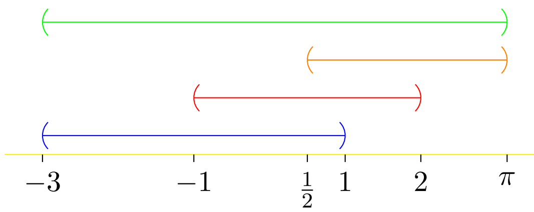
does equip $\mathbb{R}$ with a topology.

We will not prove this now. It will be more convenient for us to build a topology on $\mathbb{R}$ by a formal procedure — the topology 'generated by' open intervals in $\mathbb{R}$. We will see this in the next lecture, as Definition 2.5. Later on, we will prove that this topology is exactly $\mathcal{O}$.

Observation 1.24. However, we can already appreciate one of the two key aspects of the proof. Suppose that we have a set $\{(a_j, b_j)\}_{j \in J}$ of (possibly infinitely many) open intervals in $\mathbb{R}$. Suppose that $\bigcup_{j \in J}(a_j, b_j)$ cannot be obtained as a disjoint union of any pair of subsets of $\mathbb{R}$. Then

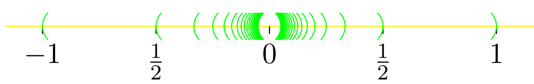
$$\bigcup_{j \in J}(a_j, b_j) = (\inf a_j, \sup b_j).$$

Remark 1.25. Observation 1.24 expresses the intuition that a 'chain of overlapping open intervals' is an open interval. For instance, the union of $\{(-3, 1), (-1, 2), (\frac{1}{2}, \pi)\}$ is $(-3, \pi)$.



Remark 1.26. By contrast with Observation 1.21, Observation 1.24 relies on the full strength of the completeness of $\mathbb{R}$ as expressed in Theorem 1.10 and Theorem 1.15.

An intersection of open intervals, even a 'chain of overlapping open intervals', need not be an open interval. For instance, $\bigcap_{n \in \mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$, and the set $\{0\}$ is not an open interval in $\mathbb{R}$!



The picture shows the suprema and infima of the intervals $(-\frac{1}{n}, \frac{1}{n})$ for $1 \leq n \leq 20$.

Summary 1.27.

(1) A union of (possibly infinitely many) open intervals in $\mathbb{R}$ is an open interval, if these open intervals 'overlap sufficiently nicely'.

(2) An intersection of a pair of open intervals in $\mathbb{R}$ which overlap is an open interval.

(3) An intersection of infinitely many open intervals in $\mathbb{R}$ need not be an open interval, even if these open intervals 'overlap sufficiently nicely'.

Remark 1.28. These three facts together motivate the requirement in condition (3) of Definition 1.1 that unions of possibly infinitely many subsets of X belonging to $\mathcal{O}$ belong to $\mathcal{O}$, by contrast with condition (4) of Definition 1.1, in which an intersection of only a pair of subsets of X belonging to $\mathcal{O}$ is required to belong to $\mathcal{O}$.

Remark 1.29. In Exercise Sheet 1 we will explore topological spaces $(X, \mathcal{O})$ with the property that an intersection of any set of subsets of X, possibly infinitely many, belonging to $\mathcal{O}$ belongs to $\mathcal{O}$. These topological spaces are known as *Alexandroff spaces*.

1.3 Canonical constructions of topological spaces — subspace topologies, product topologies, examples

Assumption 1.30. For now let us assume that we have equipped $\mathbb{R}$ with a topology $\mathcal{O}_{\mathbb{R}}$ to which every open interval in $\mathbb{R}$ belongs. As indicated in Idea 1.23, will construct $\mathcal{O}_{\mathbb{R}}$ in the next lecture.

Theme 1.31. Given $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, we can construct many topological spaces in a 'canonical way'.

Preview 1.32. Over the next few lectures, we will become acquainted with four tools:

(1) subspace topologies,

(2) product topologies,

(3) quotient topologies,

(4) coproduct topologies.

We will investigate (1) and (2) now. In Lecture 3, we will investigate (3). Later, we will investigate (4).

Proposition 1.33. Let $(Y, \mathcal{O}_Y)$ be a topological space. Let X be a subset of Y. Then

$$\mathcal{O}_X := \{X \cap U \mid U \in \mathcal{O}_Y\}$$

defines a topology on X.

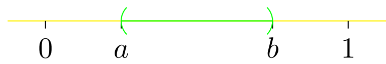
Proof. Exercise Sheet 1. □

Terminology 1.34. Let $(Y, \mathcal{O}_Y)$ be a topological space. Let X be a subset of Y. We refer to the topology $\mathcal{O}_X$ on X defined in Proposition 1.33 as the *subspace topology* on X.

Example 1.35. Let I denote the closed interval $[0, 1]$ in $\mathbb{R}$. Let $\mathcal{O}_I$ denote the subspace topology on I with respect to the topological space $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. We refer to the topological space $(I, \mathcal{O}_I)$ as the *unit interval*.

Explicitly, $\mathcal{O}_I$ consists of subsets of I of the following three kinds, in addition to $\emptyset$ and I itself.

(1) Open intervals (a, b) with $a, b \in \mathbb{R}$, $a > 0$, and $b < 1$.



(2) Half open intervals $[0, b)$ with $0 < b < 1$.

(3) Half open intervals $(a, 1]$ with $0 < a < 1$.

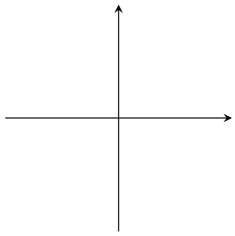
Proposition 1.36. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\mathcal{O}_{X \times Y}$ denote the set of subsets W of $X \times Y$ such that for every $(x, y) \in W$ there exists $U \in \mathcal{O}_X$ and $U' \in \mathcal{O}_Y$ with $x \in U$, $y \in U'$, and $U \times U' \subset W$. Then $\mathcal{O}_{X \times Y}$ defines a topology on $X \times Y$.

Proof. Exercise Sheet 1. $\square$

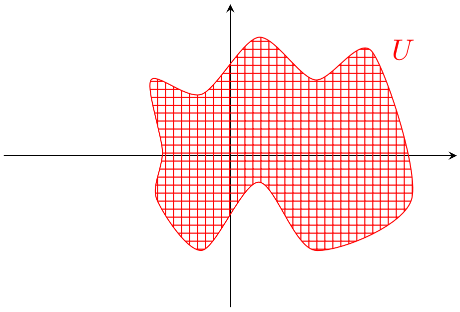
Terminology 1.37. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. We refer to the topology $\mathcal{O}_{X \times Y}$ on $X \times Y$ defined in Proposition 1.36 as the *product topology* on $X \times Y$.

Examples 1.38.

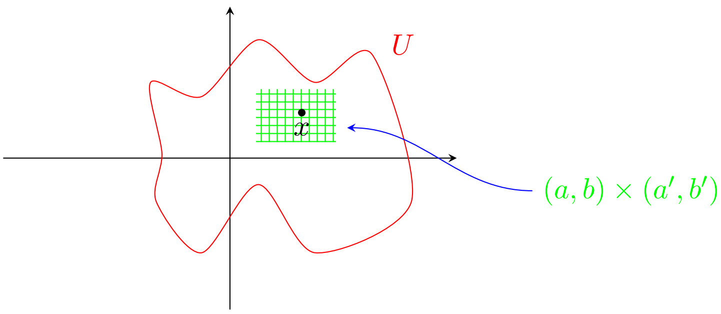
(1) $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, equipped with the product topology $\mathcal{O}_{\mathbb{R} \times \mathbb{R}}$.



A typical example of a subset of $\mathbb{R}^2$ belonging to $\mathcal{O}_{\mathbb{R}\times\mathbb{R}}$ is an 'open blob' U.

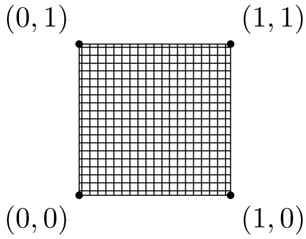


Indeed by the completeness of $\mathbb{R}$ we have that for any $x \in \mathbb{R}$ belonging to U there is an 'open rectangle' contained in U to which x belongs. By an 'open rectangle' we mean a product of an open interval (a, b) with an open interval (a', b'), for some $a, b, a', b' \in \mathbb{R}$.

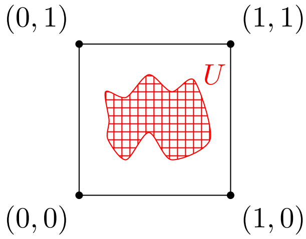


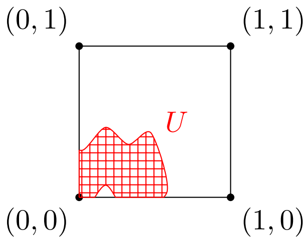
The boundary of U in the last two pictures is not to be thought of as belonging to U.

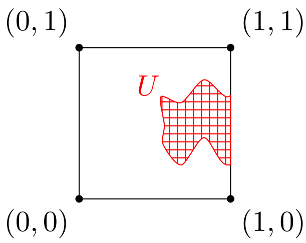
(2) $I^2 := I \times I$, equipped with the product topology $\mathcal{O}_{I\times I}$. We refer to the topological space $(I^2, \mathcal{O}_{I\times I})$ as the *unit square*.

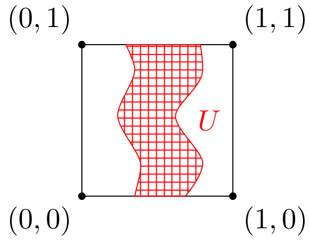


A typical example of a subset U of I^2 belonging to $\mathcal{O}_{I\times I}$ is an intersection with I^2 of an 'open blob' in $\mathbb{R}^2$.



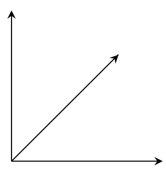






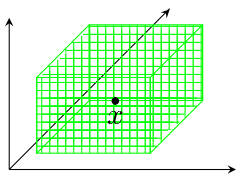
In the first figure, the boundary of U is not to be thought of as belonging to U. In the last three figures, the part of the boundary of U which intersects the boundary of the square belongs to U, but the remainder of the boundary of U is not to be thought of as belonging to U.

(3) $\mathbb{R}^3 := \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.



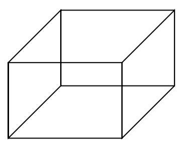
A typical example of a subset U of $\mathbb{R}^3$ belonging to $\mathcal{O}_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}}$ is a '3-dimensional open blob'. I leave it to your imagination to visualise one of these!

By the completeness of $\mathbb{R}$, for any $x \in U$ there is an 'open rectangular cuboid' contained in U to which x belongs.



Our notation $\mathcal{O}_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}}$ is potentially ambiguous, since we may cook up a product topology on $\mathbb{R}^3$ either by viewing $\mathbb{R}^3$ as $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ or by viewing $\mathbb{R}^3$ as $\mathbb{R}\times(\mathbb{R}\times\mathbb{R})$. However, these two topologies coincide, and the same is true in general.

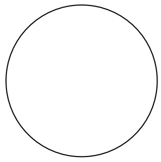
(4) $I^3 := I \times I \times I$, equipped with the product topology $\mathcal{O}_{I\times I\times I}$. We refer to the topological space $(I^3, \mathcal{O}_{I\times I\times I})$ as the *unit cube*.



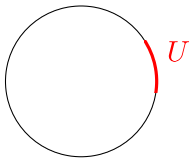
A typical example of a subset of I^3 belonging to $\mathcal{O}_{I\times I\times I}$ is the intersection of a '3-dimensional open blob' in $\mathbb{R}^3$ with I^3. Again I leave the visualisation of such a subset to your imagination!

(5) Examples (1) and (3) generalise to a product topology upon $\mathbb{R}^n := \underbrace{\mathbb{R}\times\ldots\times\mathbb{R}}_{n}$ for any $n \in \mathbb{N}$. Examples (2) and (4) generalise to a product topology upon $I^n := \underbrace{I\times\ldots\times I}_{n}$ for any $n \in \mathbb{N}$.

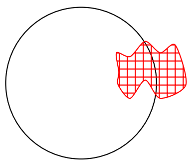
(6) $S^1 := \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\| = 1\}$, equipped with the subspace topology $\mathcal{O}_{S^1}$ with respect to the topological space $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. We refer to $(S^1, \mathcal{O}_{S^1})$ as the *circle*.



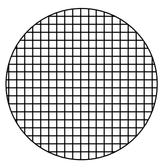
A typical subset of S^1 belonging to $\mathcal{O}_{S^1}$ is the intersection of an 'open blob' in $\mathbb{R}^2$ with S^1. For instance, the subset U of S^1 pictured below belongs to $\mathcal{O}_{S^1}$.



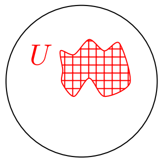
Indeed, U is the intersection with S^1 of the 'open blob' in the picture below.

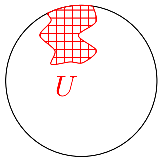


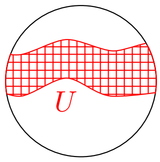
(7) $D^2 := \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\| \leq 1\}$, equipped with the subspace topology $\mathcal{O}_{D^2}$ with respect to the topological space $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. We refer to $(D^2, \mathcal{O}_{D^2})$ as the *disc*.



A typical example of a subset of D^2 belonging to $\mathcal{O}_{D^2}$ is an intersection of an 'open blob' in $\mathbb{R}^2$ with D^2.

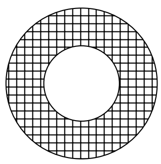




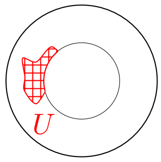


In the first figure, the boundary of U is not to be thought of as belonging to U. In the last two figures, the part of the boundary of U which intersects the boundary of the disc belongs to U, but the remainder of the boundary of U is not to be thought of as belonging to U.

(8) For any $k \in \mathbb{R}$ with $0 < k < 1$, $A_k := \{(x, y) \in \mathbb{R}^2 \mid k \leq \|(x, y)\| \leq 1\}$, equipped with the subspace topology $\mathcal{O}_{A_k}$ with respect to the topological space $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. We refer to $(A_k, \mathcal{O}_{A_k})$ as an *annulus*.



A typical example of a subset of A_k belonging to $\mathcal{O}_{A_k}$ is an intersection of an 'open blob' in $\mathbb{R}^2$ with A_k.



(9) $S^1 \times I$, equipped with the product topology $\mathcal{O}_{S^1\times I}$. We refer to $(S^1 \times I, \mathcal{O}_{S^1\times I})$ as the *cylinder*.

(10) $D^2 \times I$, equipped with product topology $\mathcal{O}_{D^2\times I}$. We refer to $(D^2 \times I, \mathcal{O}_{D^2\times I})$ as the *solid cylinder*.