Generell Topologi

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2 Thursday 17th January

2.1 Basis of a topological space — generating a topology with a specified basis — standard topology on $\mathbb{R}$ — examples

Definition 2.1. Let $(X, \mathcal{O})$ be a topological space. A *basis* for $(X, \mathcal{O})$ is a set $\mathcal{O}'$ of subsets of X belonging to $\mathcal{O}$ such that every subset U of X belonging to $\mathcal{O}$ may be obtained as a union of subsets of X belonging to $\mathcal{O}'$.

Proposition 2.2. Let X be a set, and let $\mathcal{O}'$ be a set of subsets of X such that the following conditions are satisfied.

(1) X can be obtained as a union of (possibly infinitely many) subsets of X belonging to $\mathcal{O}'$.

(2) Let U and U' be subsets of X belonging to $\mathcal{O}'$. Then $U \cap U'$ belongs to $\mathcal{O}'$.

Let $\mathcal{O}$ denote the set of subsets U of X which may be obtained as a union of (possibly infinitely many) subsets of $\mathcal{O}'$. Then $(X, \mathcal{O})$ is a topological space, with basis $\mathcal{O}'$.

Proof. We verify conditions (1)-(4) of Definition 1.1.

(1) We think of $\emptyset$ an an 'empty union' of subsets of X belonging to $\mathcal{O}'$, so that $\emptyset \in \mathcal{O}$. If you are not comfortable with this, just change the definition of $\mathcal{O}$ to include $\emptyset$ as well.

(2) We have that $X \in \mathcal{O}$ by definition of $\mathcal{O}$ together with the fact that $\mathcal{O}'$ atsifies condition (1) in the statement of the proposition.

(3) Let $\{U_j\}_{j \in J}$ be a set of subsets of X belonging to $\mathcal{O}$. For every $j \in J$, by definition of $\mathcal{O}$ we have that $X = \bigcup_{k \in K_j} U'_k$ for a set K_j, where $U'_k \in \mathcal{O}'$. Then

$$\begin{aligned}\bigcup_{j \in J} U_j &= \bigcup_{j \in J} \Big(\bigcup_{k \in K_j} U'_k \Big) \\ &= \bigcup_{r \in \left(\bigcup_{j \in J} K_j\right)} U'_r.\end{aligned}$$

Thus $\bigcup_{j \in J} U_j$ is a union of subsets of X belonging to $\mathcal{O}'$, and hence $\bigcup_{j \in J} U_j$ belongs to $\mathcal{O}$.

(4) Let U and U' be subsets of X which belong to $\mathcal{O}$. By definition of $\mathcal{O}$, we have that $U = \bigcup_{j \in J} U_j$ where $U_j \in \mathcal{O}'$ for all $j \in J$, and that $U' = \bigcup_{j' \in J'} U'_j$, where $U'_j \in \mathcal{O}'$ for all $j' \in J'$. Then

$$\begin{aligned}U \cap U' &= \Big(\bigcup_{j \in J} U_j \Big) \cap \Big(\bigcup_{j' \in J'} U'_{j'} \Big) \\ &= \bigcup_{(j,j') \in J \times J'} U_j \cap U_{j'}.\end{aligned}$$

Since $\mathcal{O}'$ satisfies condition (2) of the proposition, we have that $U_j \cap U_{j'}$ belongs to $\mathcal{O}'$ for every $(j, j') \in J \times J'$. Thus $U \cap U'$ belongs to $\mathcal{O}'$.

By construction of $\mathcal{O}$, we have that $\mathcal{O}'$ is a basis for $(X, \mathcal{O})$. □

Terminology 2.3. Let X be a set, and let $\mathcal{O}'$ be a set of subsets of X satisfying conditions (1) and (2) of Proposition 2.2. Let $\mathcal{O}$ denote the set of unions of subsets of X belonging to $\mathcal{O}'$, which by Proposition 2.2 defines a topology on X. We refer to $\mathcal{O}$ as the topology on X which is *generated* by $\mathcal{O}'$.

Observation 2.4. Let $\mathcal{O}' := \{(a, b) \mid a, b \in \mathbb{R}\}$. Then $\mathcal{O}'$ satisfies condition (1) of Proposition 2.2 with respect to $\mathbb{R}$, since for example $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$. By Observation 1.21, we have that $\mathcal{O}'$ satisfies condition (2) of Proposition 2.2.

Definition 2.5. The *standard topology* on $\mathbb{R}$ is the topology $\mathcal{O}_{\mathbb{R}}$ generated by $\mathcal{O}'$.

Observation 2.6. All open intervals in $\mathbb{R}$ belong to $\mathcal{O}_{\mathbb{R}}$. We have the following cases.

(1) If $a, b \in \mathbb{R}$, then by definition of $\mathcal{O}$ and $\mathcal{O}'$ we have that $(a, b) \in \mathcal{O}_{\mathbb{R}}$.

(2) If $a \in \mathbb{R}$, we have that $(a, \inf) = \bigcup_{n \in \mathbb{N}} (a, a + n)$. Since $(a, a + n)$ belongs to $\mathcal{O}'$ for every $n \in \mathbb{N}$, we deduce that $(a, \inf) \in \mathcal{O}_{\mathbb{R}}$.

(3) If $b \in \mathbb{R}$, we have that $(-\inf, b) = \bigcup_{n \in \mathbb{N}} (b - n, b)$. Since $(b - n, b)$ belongs to $\mathcal{O}'$ for every $n \in \mathbb{N}$, we deduce that $(-\inf, b) \in \mathcal{O}_{\mathbb{R}}$.

(4) We noted in Observation 2.4 that $\mathbb{R} \in \mathcal{O}_{\mathbb{R}}$.

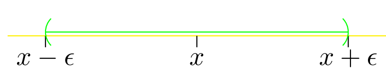
Remark 2.7. As mentioned in Idea 1.23, we will prove later in the course that $\mathcal{O}_{\mathbb{R}}$ consists exactly of disjoint unions of (possibly infinitely many) open intervals.

Observation 2.8. Let $(X, \mathcal{O})$ be a topological space, and let $\mathcal{O}'$ be a basis for $(X, \mathcal{O})$. Let $\mathcal{O}''$ be a set of subsets of X. If every $U \subset X$ such that $U \in \mathcal{O}'$ can be obtained as a union of subsets of $\mathcal{O}''$, then $\mathcal{O}''$ defines a basis for $(X, \mathcal{O})$.

Examples 2.9.

(1) For $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$, and for any $x \in \mathbb{R}$, let

$$B_\epsilon(x) := \{y \in \mathbb{R} \mid x - \epsilon < y < x + \epsilon\}.$$



In other words, $B_\epsilon(x)$ is the open interval $(x - \epsilon, x + \epsilon)$. Then

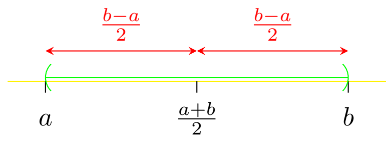
$$\mathcal{O}'' := \{B_\epsilon(x) \mid \epsilon \in \mathbb{R} \text{ and } \epsilon > 0, \text{ and } x \in \mathbb{R}\}$$

is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Proof. By Observation 2.8, it suffices to prove that for every $a, b \in \mathbb{R}$ we can obtain the open interval (a, b) as a union of subsets of $\mathbb{R}$ belonging to $\mathcal{O}''$. In fact, (a, b) itself already belongs to $\mathcal{O}''$. Indeed

$$(a, b) = B_{\frac{b-a}{2}}\Big(\frac{a+b}{2}\Big).$$

□



In particular, we see that a topological space may admit more than one basis.

(2) Let $X = \{a, b, c, d, e\}$, and let $\mathcal{O}$ denote the topology on X given by

$$\big\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{d, e\}, \{a, b, c\}, \{b, d, e\}, \{a, b, d, e\}, \{b, c, d, e\}, X\big\}.$$

Then

$$\mathcal{O}_1 := \big\{\{b\}, \{a, b\}, \{b, c\}, \{d, e\}\big\}$$

is a basis for $(X, \mathcal{O})$.

The same holds for any set $\mathcal{O}_2$ of subsets of X such that $\mathcal{O}_1 \subset \mathcal{O}_2$. No other set of subsets of X is a basis for $(X, \mathcal{O})$. For example,

$$\mathcal{O}_3 := \big\{\{a, b\}, \{b, c\}, \{d, e\}\big\}$$

is not a basis for $(X, \mathcal{O})$, since $\{b\}$ cannot be obtained as a union of subsets of X belonging to $\mathcal{O}''$. Similarly

$$\mathcal{O}_4 := \big\{\{b\}, \{a, b\}, \{d, e\}\big\}$$

is not a basis for $\mathcal{O}''$, since $\{b, c\}$ cannot be obtained as a union of subsets of X belonging to $\mathcal{O}_4$.

(3) Let

$$\mathcal{O}' := \{(-\infty, b) \mid b \in \mathbb{R}\}.$$

Then $\mathcal{O}'$ is not a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, since for example we cannot obtain the open interval $(0, 1)$ as a union of open intervals of the form $(-\infty, b)$.

But $\mathcal{O}'$ satisfies the conditions of Proposition 2.2, and thus generates a topology $\mathcal{O}$ on $\mathbb{R}$. In the manner of Observation 1.24, one can prove that $\mathcal{O} = \mathcal{O}' \cup \{\emptyset, \mathbb{R}\}$.

2.2 Continuous maps — examples — continuity of inclusion maps, compositions of continuous maps, and constant maps

Notation 2.10. Let X and Y be sets, and let

$$X \xrightarrow{f} Y$$

be a map. Let U be a subset of Y. We define $f^{-1}(U)$ to be $\{x \in X \mid f(x) \in U\}$.

Definition 2.11. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is *continuous* if for every $U \in \mathcal{O}_Y$ we have that $f^{-1}(U)$ belongs to $\mathcal{O}_X$.

Remark 2.12. A map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous with respect to the standard topology on both copies of $\mathbb{R}$ if and only if it is continuous in the $\epsilon - \delta$ sense that you know from real analysis/calculus. See the Exercise Sheet.

Examples 2.13.

(1) Let $X := \{a, b\}$, and let $\mathcal{O}$ denote the topology on X given by $\big\{\emptyset, \{b\}, X\big\}$, so that $(X, \mathcal{O})$ is the Sierpiński interval. Let $X' := \{a', b', c'\}$, and let $\mathcal{O}'$ denote the topology on X' given by

$$\big\{\emptyset, \{a'\}, \{c'\}, \{a', c'\}, \{b', c'\}, X'\big\}.$$

Let

$$X \xrightarrow{f} Y$$

be given by $a \mapsto b'$ and $b \mapsto c'$. Then f is continous.

Proof. We verify that $f^{-1}(U) \in \mathcal{O}_X$ for every $U \in \mathcal{O}_Y$, as follows.

(1) $f^{-1}(\emptyset) = \emptyset$

(2) $f^{-1}\big(\{a'\}\big) = \emptyset$

(3) $f^{-1}\big(\{c'\}\big) = \{b'\}$

(4) $f^{-1}\big(\{a', c'\}\big) = \{b\}$

(5) $f^{-1}(Y) = X$.

Let

$$Y \xrightarrow{g} X$$

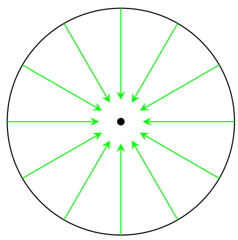
be given by $a \mapsto c'$ and $b \mapsto b'$. Then g is not continuous, since for example $g^{-1}\big(\{c'\}\big) = \{a\}$, which does not belong to $\mathcal{O}_X$. $\square$

(2) Let

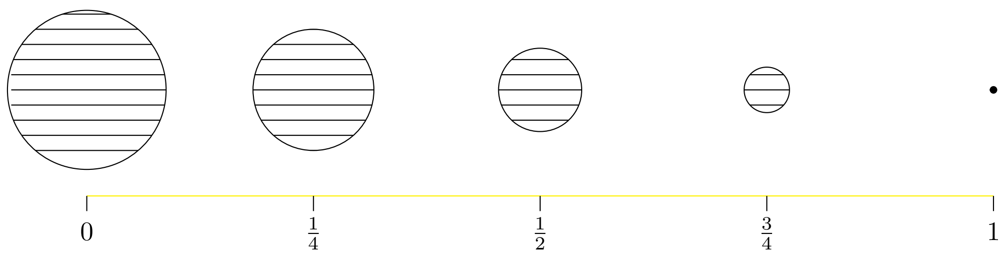
$$D^2 \times I \xrightarrow{f} D^2$$

be given by $(x, y, t) \mapsto \big((1-t)x, (1-t)y\big)$. We will prove on the Exercise Sheet that f is continuous.

We may think of f as a 'shrinking of D^2 onto its centre', as t moves from 0 to 1.



We can picture the image of $D^2 \times \{t\}$ under f as follows as t moves from 0 to 1.

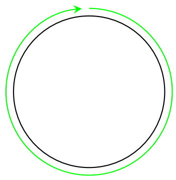


(3) Fix $k \in \mathbb{R}$. Let

$$I \xrightarrow{f} S^1$$

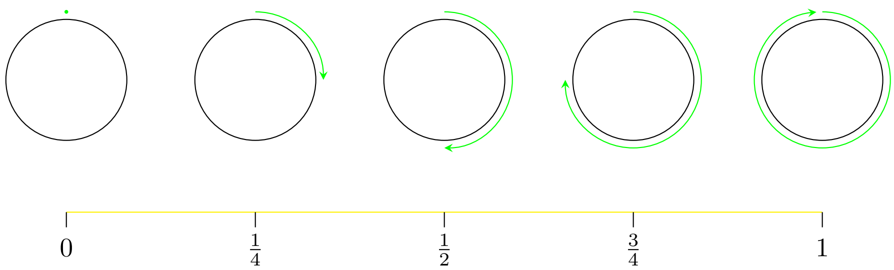
be given by $t \mapsto \phi(kt)$, where ϕ is the continuous map of Question 8 of Exercise Sheet 3. Let us picture f for a few values of k.

(1) Let $k = 1$. In words, f begins at the point $(0, 1)$, and travels exactly once around S^1.

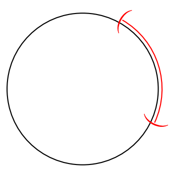


Don't be misled by the picture — the path really travels around the circle, not slightly outside it.

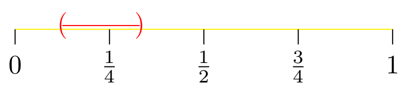
We may picture $f\big([0, t]\big)$ as t moves from 0 to 1 as follows.



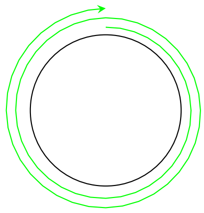
Recall from Examples 1.38 (6) that a typical open subset U of S^1 is as depicted below.



Then $f^{-1}(U)$ is as depicted below. In particular, $f^{-1}(U)$ is open in I. Thus intuitively we can believe that f is continuous!

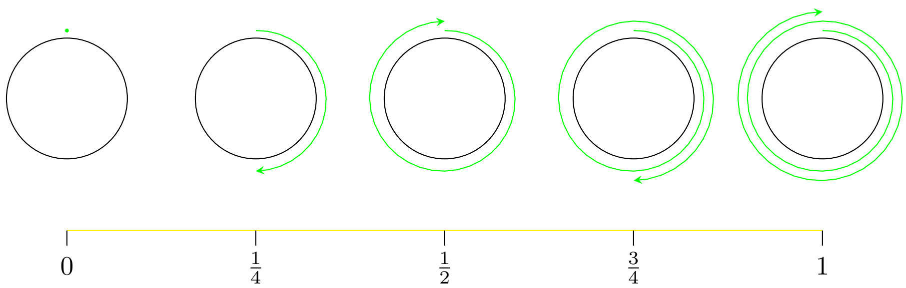


(2) Let $k = 2$. In words, f begins at the point $(0, 1)$, and travels exactly twice around S^1.

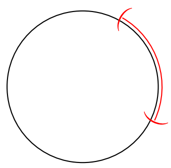


Again, don't be misled by the picture — the path really travels twice around the circle, thus passing through every point on the circle twice, not in a spiral outside the circle as drawn.

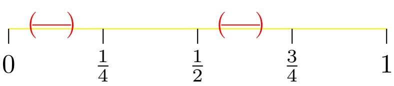
We may picture $f\big([0, t]\big)$ as t moves from 0 to 1 as follows.



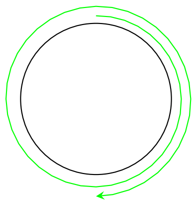
Let $U \subset S^1$ be the open subset depicted below.



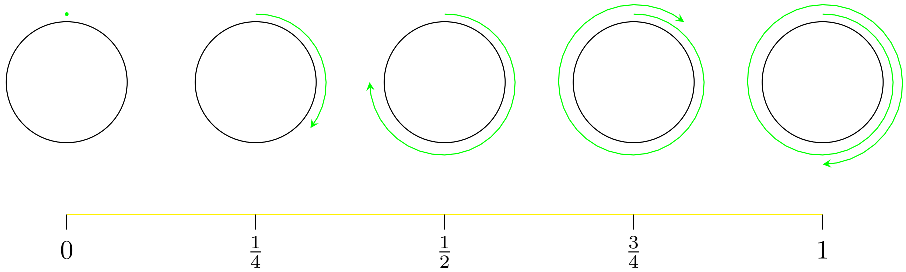
Then $f^{-1}(U)$ a disjoint union of open intervals as depicted below, so is open in I.



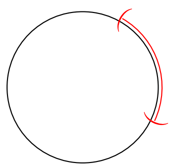
(3 Let $k = \frac{3}{2}$. In words, f begins at the point $(0, 1)$, and travels exactly one and a half times around S^1.



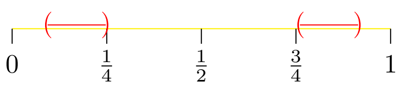
We may picture $f\big([0, t]\big)$ as t moves from 0 to 1 as follows.



Let $U \subset S^1$ be the open subset depicted below.



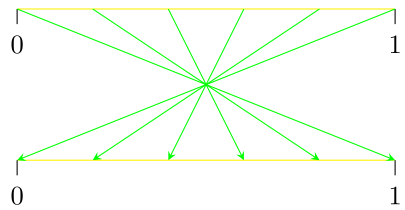
Then $f^{-1}(U)$ is a disjoint union of open subsets of I as depicted below, so is open in I.



(4) Let

$$I \xrightarrow{f} I$$

be given by $t \mapsto 1 - t$. We will prove on the Exercise Sheet that f is continuous. We may depict f as follows.



Let $U \subset I$ be the open subset depicted below.

Then $f^{-1}(U)$ is as depicted below. In particular, $f^{-1}(U)$ is open in I.

(5) Let

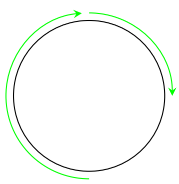
$$I \xrightarrow{f} S^1$$

be the map given by

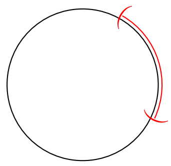
$$t \mapsto \begin{cases} \phi\left(\frac{1}{2}t\right) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \phi(t) & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

As in (3), ϕ is the map of Question 8 of Exercise Sheet 3.

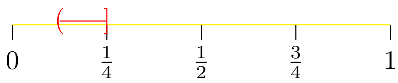
We may depict f as follows.



Then f is not continuous. Indeed, consider an open subset U of S^1 as depicted below.



Then $f^{-1}(U)$ is a half open interval as depicted below.

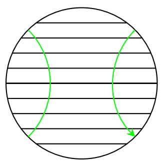


In particular, $f^{-1}(U)$ is not an open subset of I.

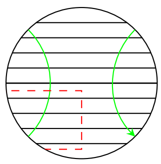
(5) Consider a map

$$I \xrightarrow{f} D^2$$

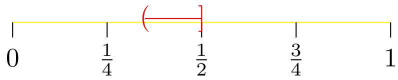
as depicted below. A precise definition of this map is not important here — the path should be interpreted as beginning on the top left of the disc, moving to the bottom left, jumping to the top right, and then moving to the bottom right.



Let $U \subset D^2$ be an open subset of D^2 depicted as a dashed rectangle below.



Then $f^{-1}(U)$ is a half open interval in I as depicted below. In particular, $f^{-1}(U)$ is not open in I.



Terminology 2.14. Let X be a set, and let A be a subset of X. The *inclusion map* with respect to A and X is the map

$$A \longrightarrow X$$

given by $a \mapsto a$. We will often denote it by

$$A \hookrightarrow X.$$

Proposition 2.15. Let $(X, \mathcal{O}_X)$ be a topological space. Let $A \subset X$ be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Then the inclusion map

$$A \overset{i}{\hookrightarrow} X$$

is continuous.

Proof. Let U be a subset of X belonging to $\mathcal{O}_X$. Then $i^{-1}(U) = A \cap U$. By definition of $\mathcal{O}_A$, we have that $A \cap U$ belongs to $\mathcal{O}_A$. Hence $i^{-1}(U)$ belongs to $\mathcal{O}_A$. □

Proposition 2.16. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

and

$$Y \xrightarrow{g} Z$$

be continuous maps. Then the map

$$X \xrightarrow{g \circ f} Z$$

is continuous.

Proof. Let U be a subset of Z belonging to $\mathcal{O}_Z$. Then

$$\begin{aligned} (g \circ f)^{-1}(U) &= \{x \in X \mid g\big(f(x)\big) \in U\} \\ &= \{x \in X \mid f(x) \in g^{-1}(U)\} \\ &= f^{-1}\big(g^{-1}(U)\big). \end{aligned}$$

Since g is continuous, we have that $g^{-1}(U) \in \mathcal{O}_Y$. Hence, since f is continuous, we have that $f^{-1}\big(g^{-1}(U)\big) \in \mathcal{O}_X$. Thus $(g \circ f)^{-1}(U) \in \mathcal{O}_X$. □

Terminology 2.17. Let X and Y be sets. A map

$$X \xrightarrow{f} Y$$

is *constant* if $f(x) = f(x')$ for all $x, x' \in X$.

Proposition 2.18. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a constant map. Then f is continuous.

Proof. Since f is constant, $f(x) = y$ for some $y \in Y$ and all $x \in X$. Let $U \in \mathcal{O}_Y$. If $y \notin U$, then $f^{-1}(U) = \emptyset$, which belongs to $\mathcal{O}_X$. If $y \in U$, then $f^{-1}(U) = X$, which also belongs to $\mathcal{O}_X$. $\square$