Generell Topologi

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3 Tuesday 22nd January

3.1 Projection maps are continuous — pictures versus rigour

Notation 3.1. Let X and Y be sets. We denote by

$$X \times Y \xrightarrow{p_1} X$$

the map given by $(x, y) \mapsto x$. We denote by

$$X \times Y \xrightarrow{p_2} Y$$

the map given by $(x, y) \mapsto y$.

Proposition 3.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $X \times Y$ be equipped with the product topology $\mathcal{O}_{X \times Y}$. Then

$$X \times Y \xrightarrow{p_1} X$$

and

$$X \times Y \xrightarrow{p_2} Y$$

define continuous maps.

Proof. Suppose that $U \subset X$ belongs to $\mathcal{O}_X$. Then $p_1^{-1}(U) = U \times Y$, which belongs to $\mathcal{O}_{X \times Y}$.

Suppose that $U' \subset Y$ belongs to $\mathcal{O}_Y$. Then $p_2^{-1}(U') = X \times U'$, which belongs to $\mathcal{O}_{X \times Y}$. □

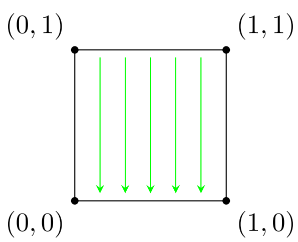
Remark 3.3. It is often helpful to our intuition to picture p_1 and p_2. Let us consider

$$I \times I \xrightarrow{p_1} I$$

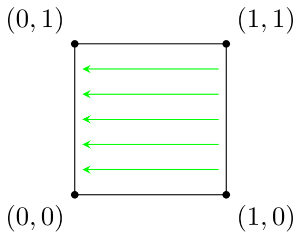
and

$$I \times I \xrightarrow{p_2} I.$$

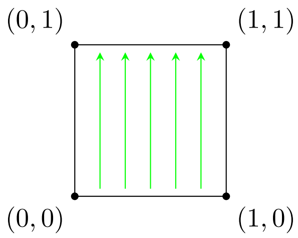
Up to a bijection between I and $I \times \{0\} = \{(x, 0) \mid x \in [0, 1]\}$, we may think of p_1 as the map $(x, y) \mapsto (x, 0)$.



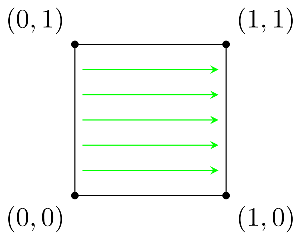
Up to a bijection between I and $\{0\} \times I = \{(0, y) \mid y \in [0, 1]\}$, we may think of p_2 as the map $(x, y) \mapsto (0, y)$.



It is important to note, though, that there will typically be many good ways that we may picture p_1 and p_2. In this example, we may for instance equally think of p_1 as the map given by $(x, y) \mapsto (x, 1)$



and/or think of p_2 as the map given by $(x, y) \mapsto (1, y)$.



The moral to draw from this is that pictures help our intuition, often profoundly. In topology we often see a proof before we can write it down!

But we must never forget that it is with rigorous definitions and proofs — which are independent of any particular picture — that we must ultimately be able to capture our intuition.

3.2 Quotient topologies

Notation 3.4. Let X be a set, and let $\sim$ be an equivalence relation on X. We denote by $X/\sim$ the set

$$\{[x] \mid x \in X\}$$

of equivalences classes of X with respect to $\sim$. We denote by

$$X \xrightarrow{\pi} X/\sim$$

the map given by $x \mapsto [x]$.

Proposition 3.5. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\sim$ be an equivalence relation on X. Then

$$\mathcal{O}_{X/\sim} := \{U \in X/\sim \mid \pi^{-1}(U) \in \mathcal{O}_X\}$$

defines a topology on $X/\sim$.

Proof. Exercise. □

Terminology 3.6. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\sim$ be an equivalence relation on X. We refer to $\mathcal{O}_{X/\sim}$ as the *quotient topology* upon $X/\sim$.

Observation 3.7. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\sim$ be an equivalence relation on X. Let $X/\sim$ be equipped with the quotient topology. Then

$$X \xrightarrow{\pi} X/\sim$$

is continuous. Indeed, $\mathcal{O}_{X/\sim}$ is defined exactly so as to ensure this.

Notation 3.8. In Examples 3.9 we will adopt the following notation. Let X be a set, and let $\approx$ be a transitive relation on X. We denote by $\sim$ the equivalence relation on X defined by

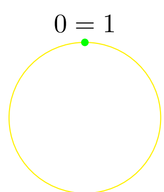
$$x \sim x' \Leftrightarrow \begin{cases} x \approx x' \text{ or } x' \approx x \text{ or } x = x' & \text{if } x, x' \in X', \\ x = x' & \text{otherwise.} \end{cases}$$

We refer to $\sim$ as the equivalence relation on X which is *generated* by $\approx$.

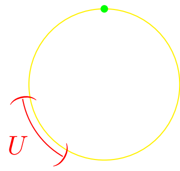
Examples 3.9.

(1) Define $\approx$ on I by $0 \approx 1$.

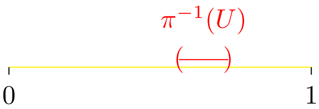
Then $I/\sim$ is obtained by glueing 0 to 1.



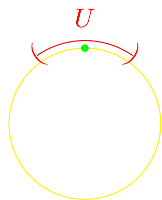
Let us explore subsets U of $I/\sim$ which belong to $\mathcal{O}_{I/\sim}$. Let $U \subset I/\sim$ be as depicted below, and suppose that $[0] = [1] \notin U$.



Then $\pi^{-1}(U)$ is an open interval in I, and thus $U \in \mathcal{O}_{I/\sim}$.

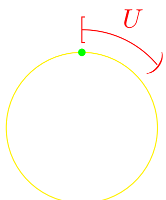


Suppose now that $[0] = [1] \in U$.

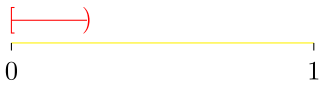


Then $\pi^{-1}(U)$ is a disjoint union of subsets of I which belong to $\mathcal{O}_I$, and thus $U \in \mathcal{O}_{I/\sim}$.

Do not be misled by this into thinking that subsets of $I/\sim$ belonging to are exactly images under π of subsets of I belonging to $\mathcal{O}_I$. Indeed, suppose that U is as depicted below.



This is the image under π of a half open interval as depicted below, which belongs to $\mathcal{O}_I$.



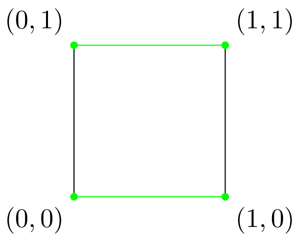
Then $\pi^{-1}(U)$ is the disjoint union of the half open interval depicted above with the singleton set $\{1\}$. Thus $\pi^{-1}(U) \notin \mathcal{O}_I$.

We see that $I/\sim$ looks like the circle S^1, which we equipped with a topology $\mathcal{O}_{S^1}$ in a different way in Examples 1.38 (6). Moreover, the subsets of $I/\sim$ which belong to $\mathcal{O}_{I/\sim}$ seem very similar to the subsets of S^1 which belong to $\mathcal{O}_{S^1}$.

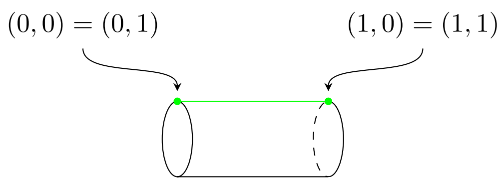
Question 3.10. Are $(I/\sim, \mathcal{O}_{I/\sim})$ and $(S^1, \mathcal{O}_{S^1})$ the same topological space, in an appropriate sense?

Answer 3.11. Yes! The appropriate notion of sameness for topological spaces will be defined at the end of this lecture. In a later lecture we will prove that $(I/\sim, \mathcal{O}_{I/\sim})$ and $(S^1, \mathcal{O}_{S^1})$ are the same in this sense.

(2) Define $\approx$ on I^2 by $(x, 1) \approx (x, 0)$ for all $x \in [0, 1]$.



Then $I/\sim$ is obtained by glueing the upper horizontal edge of I^2 to the lower horizontal edge.

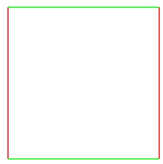


In a later lecture we will see a way to prove that $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ is the same, in the appropriate sense, to the cylinder $(S^1 \times I, \mathcal{O}_{S^1 \times I})$ which was defined in a different way in Examples 1.38 (9).

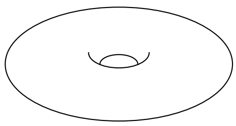
(3) Define $\approx$ on I^2 by

$$\begin{cases} (x,1) \approx (x,0) & \text{for all } x \in [0,1], \\ (1,y) \approx (0,y) & \text{for all } y \in [0,1]. \end{cases}$$

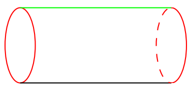
Then $I^2/\sim$ is obtained by glueing together the two horizontal edges of I^2 and glueing together the two vertical edges of I^2.



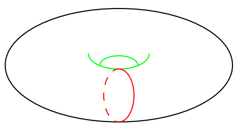
We may picture $I^2/\sim$ as follows.



Indeed we may for example first glue the horizontal edges together as in (2), obtaining a cylinder.

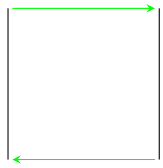


We then glue the two red circles together.

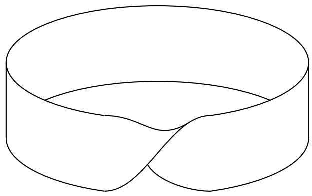


We refer to $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ as the *torus*, and denote it by T^2.

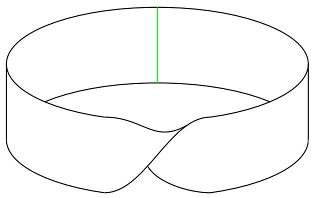
(4) Define $\approx$ on I^2 by $(x, 1) \approx (1 - x, 0)$ for all $x \in [0, 1]$. Then $I^2/\sim$ is obtained by glueing together the two horizontal edges of I^2 with a twist, indicated by the arrows in the picture below.



We may picture $I^2/\sim$ as follows.



In this picture, the glued horizontal edges of I^2 can be thought of as a line in $I^2/\sim$.

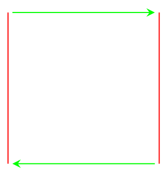


We refer to $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ as the *Möbius band*, and denote it by M^2.

(5) Define $\approx$ on I^2 by

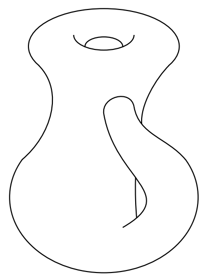
$$\begin{cases} (x, 1) \approx (1 - x, 0) & \text{for all } x \in [0, 1], \\ (1, y) \approx (0, y) & \text{for all } y \in [0, 1]. \end{cases}$$

Then $I^2/\sim$ is obtained by glueing together the two vertical edges of I^2 and glueing together the two horizontal edges of I^2 with a twist.

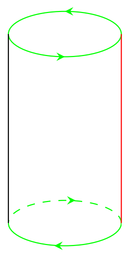


We refer to $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ as the *Klein bottle*, and denote it by K^2.

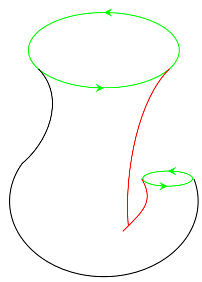
We cannot truly picture K^2 in $\mathbb{R}^3$. Nevertheless we can gain an intuitive feeling for K^2 through a picture as follows.



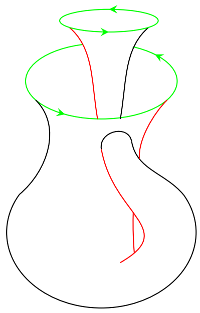
Indeed we may for example first glue the vertical edges to obtain a cylinder.



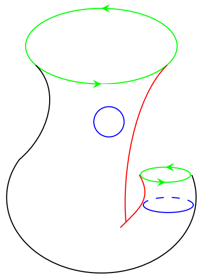
We then bend this cylinder so that the directions of the arrows on the circles at its ends match up.



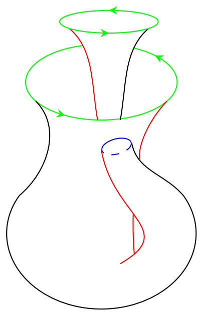
Next we push the cylinder through itself.



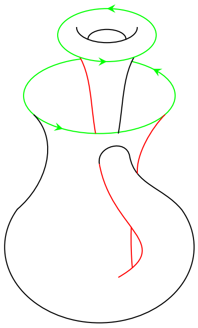
It is this step that is not possible in a true picture of K^2. It can be thought of as the glueing of two circles: a cross-section of the cylinder, and a circle on the side of the cylinder.



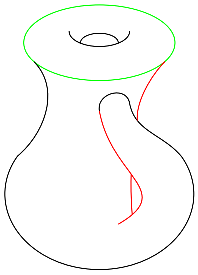
This is not specified by $\sim$. The circle obtained after glueing these two circles is indicated below.



Next we fold back one of the ends of the cylinder, giving a 'mushroom with a hollow stalk'.



Finally we glue the ends of the cylinder together, as specified by $\sim$.



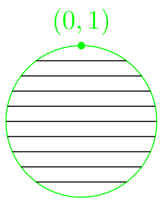
A rite of passage when learning about topology for the first time is to be confronted with the following limerick — I'm sure that I remember Colin Rourke enunciating it during the lecture in which I first met the Klein bottle!

> A mathematician named Klein
> Thought the Möbius band was divine.
> Said he: "If you glue
> The edges of two,
> You'll get a weird bottle like mine!"

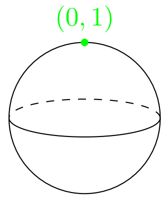
We will investigate the meaning of this in Exercise Sheet 4!

Colin Rourke also had a glass model of the topological space depicted above — I'm sorry that I could not match up to this!

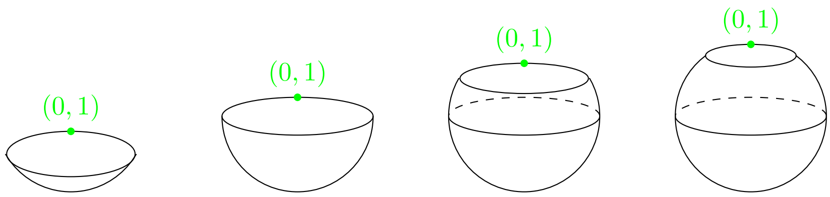
(6) Define $\approx$ on D^2 by $(x, y) \approx (0, 1)$ for all $(x, y) \in S^1$.



Then $D^2/\sim$ can be depicted as a hollow ball, as follows.



We think of $D^2/\sim$ as obtained by 'contracting the boundary of D^2 to the point $(0,1)$'. For instance, think of the boundary circle of D^2 as a loop of fishing line, and suppose that we have a reel at the point $(0,1)$. Then $D^2/\sim$ is obtained by reeling in tight all of our fishing line.



We refer to $(D^2/\sim, \mathcal{O}_{D^2/\sim})$ as the *2-sphere*, and denote it by S^2. It can be proven that $(S^2, \mathcal{O}_{S^2})$ is the same — in the appropriate sense, which we are about to introduce — as

$$\{x \in \mathbb{R}^3 \mid \|x\| = 1\}$$

equipped with the subspace topology with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}})$.

We could choose any single point on S^1 instead of $(0,1)$ in the definition of $\approx$.

3.3 Homeomorphisms

Notation 3.12. Let X be a set. We denote by id_X the identity map

$$X \longrightarrow X,$$

namely the map given by $x \mapsto x$.

Recollection 3.13. The following definitions of a bijective map

$$X \xrightarrow{f} Y$$

are equivalent.

(1) There is a map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$.

(2) The map f is both injective and surjective.

We leave (1) $\Rightarrow$ (2) as an exercise. For (2) $\Rightarrow$ (1), observe that if f is both injective and surjective, then $x \mapsto f^{-1}(x)$ gives a well-defined map

$$Y \longrightarrow X.$$

with the required properties.

Definition 3.14. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is a *homeomorphism* if:

(1) f is continuous,

(2) there is a continuous map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$.

Proposition 3.15. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is a homeomorphism if and only if:

(1) f is bijective,

(2) for every $U \subset X$, we have that $f(U) \in \mathcal{O}_Y$ if and only if $U \in \mathcal{O}_X$.

Proof. Exercise.

□