Generell Topologi

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4 Thursday 24th January

4.1 Homeomorphisms — continued

Definition 4.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is *open* if $f(U) \in \mathcal{O}_Y$ for every $U \in \mathcal{O}_X$.

Remark 4.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. In our new terminology, Proposition 3.15 gives us that a map

$$X \xrightarrow{f} Y$$

is a homeomorphism if and only if it is bijective, continuous, and open.

Observation 4.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. If a map

$$X \xrightarrow{f} Y$$

is a homeomorphism, then

$$Y \xrightarrow{f^{-1}} X$$

is a homeomorphism. We can take the required map

$$X \xrightarrow{g} Y$$

of condition (2) of Definition 3.14 such that $g \circ f^{-1} = id_Y$ and $f^{-1} \circ g = id_X$ to be f.

Proposition 4.4. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

and

$$Y \xrightarrow{f'} Z$$

be homeomorphisms. Then

$$X \xrightarrow{f' \circ f} Z$$

is a homeomorphism.

Proof. Since f is a homeomorphism, there is a map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$. Since f' is a homeomorphism, there is a map

$$Y \xrightarrow{g'} X$$

such that $g' \circ f' = id_Y$ and $f' \circ g' = id_Z$. By Proposition 2.16, we have that $f' \circ f$ and $g \circ g'$ are continuous. Moreover

$$\begin{aligned}(g \circ g') \circ (f' \circ f) &= g \circ (g' \circ f') \circ f \\ &= g \circ id_Y \circ f \\ &= g \circ f \\ &= id_X\end{aligned}$$

and

$$\begin{aligned}(f' \circ f) \circ (g \circ g') &= f' \circ (f \circ g) \circ g' \\ &= f' \circ id_Y \circ g' \\ &= g' \circ f' \\ &= id_X.\end{aligned}$$

□

Definition 4.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then $(X, \mathcal{O}_X)$ is *homeomorphic* to $(Y, \mathcal{O}_Y)$ if there exists a homeomorphism

$$X \longrightarrow Y.$$

Notation 4.6. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. If $(X, \mathcal{O}_X)$ is homeomorphic to $(Y, \mathcal{O}_Y)$, we write $X \cong Y$.

Examples 4.7.

(1) Let $X = \{a, b, c\}$. Define

$$X \xrightarrow{f} X$$

by $a \mapsto b$, $b \mapsto c$, and $c \mapsto a$. We have that f is a bijection. Let

$$\mathcal{O} := \big\{\emptyset, \{a\}, \{b, c\}, X\big\}$$

and let

$$\mathcal{O}' := \big\{\emptyset, \{a,c\}, \{b\}, X\big\}.$$

We have that

$$\begin{aligned}
f^{-1}\big(\emptyset\big) &= \emptyset \in \mathcal{O} \\
f^{-1}\big(\{a,c\}\big) &= \{b,c\} \in \mathcal{O} \\
f^{-1}\big(\{b\}\big) &= \{a\} \in \mathcal{O} \\
f^{-1}\big(X\big) &= X \in \mathcal{O}.
\end{aligned}$$

Thus f defines a continuous map from $(X, \mathcal{O})$ to $(X, \mathcal{O}')$.

Moreover, we have that

$$\begin{aligned}
f\big(\emptyset\big) &= \emptyset \in \mathcal{O}' \\
f\big(\{a\}\big) &= \{b\} \in \mathcal{O}' \\
f\big(\{b,c\}\big) &= \{a,c\} \in \mathcal{O}' \\
f\big(X\big) &= Y \in \mathcal{O}'.
\end{aligned}$$

Thus f defines an open map from $(X, \mathcal{O})$ to $(X, \mathcal{O}')$. Putting everything together, we have that f defines a homeomorphism between $(X, \mathcal{O})$ and $(X, \mathcal{O}')$.

Let

$$\mathcal{O}'' := \big\{\emptyset, \{a,b\}, \{c\}, X\big\}.$$

Then f does not define a continuous map from $(X, \mathcal{O})$ to $(X, \mathcal{O}'')$, since $f^{-1}\big(\{a\}\big) = \{b\} \notin \mathcal{O}$. Thus f is not a homeomorphism.

Nevertheless, $(X, \mathcal{O})$ and $(X, \mathcal{O}'')$ are homeomorphic. Indeed, let

$$X \xrightarrow{\;g\;} Y$$

be given by $a \mapsto c$, $b \mapsto b$, and $c \mapsto a$. Then

$$\begin{aligned}
g^{-1}\big(\emptyset\big) &= \emptyset \in \mathcal{O} \\
g^{-1}\big(\{a,b\}\big) &= \{b,c\} \in \mathcal{O} \\
g^{-1}\big(\{c\}\big) &= \{a\} \in \mathcal{O} \\
g^{-1}\big(Y\big) &= X \in \mathcal{O}.
\end{aligned}$$

Thus g defines a continuous map from $(X, \mathcal{O})$ to $(X, \mathcal{O}'')$.

Moreover, we have that

$$\begin{aligned}
g\big(\emptyset\big) &= \emptyset \in \mathcal{O}'' \\
g\big(\{a\}\big) &= \{c\} \in \mathcal{O} \\
g\big(\{b,c\}\big) &= \{a,b\} \in \mathcal{O}'' \\
g\big(X\big) &= Y \in \mathcal{O}''.
\end{aligned}$$

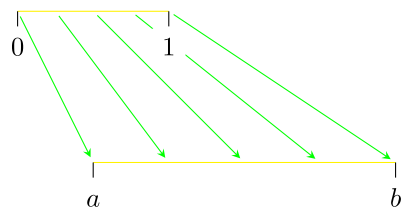
Let $\mathcal{O}''' := \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$. Then f defines a continuous bijection from $(X, \mathcal{O}''')$ to $(X, \mathcal{O}')$, but f is not a homeomorphism. Indeed, $f(\{b\}) = \{c\} \notin \mathcal{O}'$.

More generally, two homeomorphic spaces whose underlying sets are finite must have the same number of open sets, so $(X, \mathcal{O}''')$ is not homeomorphic to $(X, \mathcal{O}')$.

(2) For any $a, b \in \mathbb{R}$ with $a < b$, the open interval (a,b) equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is homeomorphic to the open interval $(0,1)$ equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Indeed, let

$$(0,1) \xrightarrow{f} (a,b)$$

denote the map given by $t \mapsto a(1-t) + bt$. By Question 3 (f) of Exercise Sheet 3, we have that f is continuous. We can think of f as a 'stretching/shrinking and translation' of $(0,1)$.



A continuous inverse

$$(a,b) \xrightarrow{g} (0,1)$$

to f is defined by $t \mapsto \frac{t-a}{b-a}$. Again, that g is continuous is established by Question 3 (f) of Exercise Sheet 3. Thus f is a homeomorphism.

(3) By Proposition 4.4, we deduce from (2) that for any $a, a', b, b' \in \mathbb{R}$ with $a < b$ and $a' < b'$, the open interval (a,b) equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is homeomorphic to (a', b') equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Intuitively, we can 'stretch/shrink' and 'translate' any open interval into any other open interval.

(4) Similarly, for any $a, b \in \mathbb{R}$ with $a < b$, the closed interval $[a,b]$ equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is homeomorphic to $(I, \mathcal{O}_I)$. Indeed, the map

$$I \xrightarrow{f} [a,b]$$

given by $t \mapsto a(1-t)+bt$ again defines a homeomorphism (we just have a different source and target), with a continuous inverse

$$[a,b] \xrightarrow{g} I$$

given by $t \mapsto \frac{t-a}{b-a}$.

It is crucial here that we assume that $a < b$, and do not allow that $a = b$. Indeed the point, which we introduced in Examples 1.7 (2), is not homeomorphic to $(I, \mathcal{O}_I)$, since there is no bijection between a set with one element and I. Note that our argument above breaks down if $a = b$, since then g is not a well-defined map.

(5) By Proposition 4.4, we deduce from (4) that for any $a, a', b, b' \in \mathbb{R}$ with $a < b$ and $a' < b'$, the closed interval $[a, b]$ equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is homeomorphic to $[a', b']$ equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Again, intuitively we can 'stretch/shrink' and 'translate' any closed interval into any other closed interval. The same arguments adapt to prove that any two half open intervals are homeomorphic.

(6) Let the open interval $(-1, 1)$ be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. The map

$$(-1,1) \xrightarrow{f} \mathbb{R}$$

defined by $t \mapsto \frac{t}{1-|t|}$ is continuous by Questions 3 (a) and (f) of Exercise Sheet 3 and Proposition 2.16 — check that you understand how to apply these results to deduce this! A continuous inverse

$$\mathbb{R} \xrightarrow{g} (-1,1)$$

is defined by $x \mapsto \frac{x}{1+|x|}$. Thus f is a homeomorphism.

By Proposition 4.4, we deduce from this and (3) that the open interval (a, b) equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Remark 4.8. We do not yet have any tools for proving that two topological spaces are *not* homeomorphic. For any particular map between two topological spaces, we can hope to verify whether or not it defines a homeomorphism. But to show that two topological spaces are not homeomorphic, we have to be able to prove that we cannot find *any* homeomorphism between them.

To be able to do this, we first need to develop some machinery. After this, we will in a later lecture be able to prove that for any $a, b \in \mathbb{R}$ with $a < b$, the open interval (a, b) is not homeomorphic to the closed interval $[a, b]$.

Proposition 4.9. Let $(X, \mathcal{O}_X)$, $(X', \mathcal{O}_{X'})$, $(Y, \mathcal{O}_Y)$, and $(Y', \mathcal{O}_{Y'})$ be topological spaces, and let

$$X \xrightarrow{f} Y$$

and

$$X' \xrightarrow{f'} Y'$$

be homeomorphisms. Then the map

$$X \times X' \xrightarrow{f \times f'} Y \times Y'$$

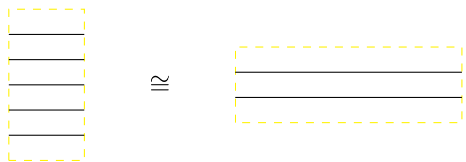
given by $(x, x') \mapsto \big(f(x), f'(x')\big)$ is a homeomorphism.

Proof. Exercise. □

Examples 4.10.

(1) Let the open intervals (a, b), (c, d), (a', b'), and (c', d') be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $(a, b) \times (c, d)$ and $(a', b') \times (c', d')$be equipped with the product topologies.

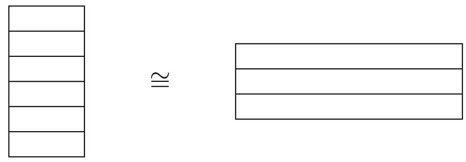
By Proposition 4.9, we deduce from Examples 4.7 (3) that $(a, b) \times (c, d)$ is homeomorphic to $(a', b') \times (c', d')$.



Intuitively, we can squash, stretch, and translate any open rectangle into any other.

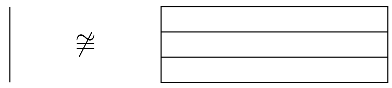
(2) Similarly, suppose that we have closed intervals $[a, b]$, $[c, d]$, $[a', b']$, and $[c', d']$ equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that $a < b$, $c < d$, $a' < b'$, and $c' < d'$.

Let $[a, b] \times [c, d]$ and $[a', b'] \times [c', d']$ be equipped with the product topologies. By Proposition 4.9 we deduce from Examples 4.7 (5) that $[a, b] \times [c, d]$ is homeomorphic to $[a', b'] \times [c', d']$.

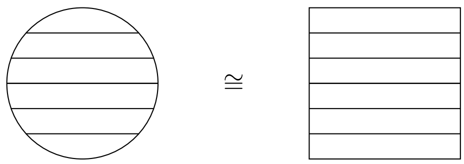


Intuitively, we can squash, stretch, and translate any open rectangle into any other. We can similarly deduce that rectangles $[a, b] \times (c, d)$ and $[a', b'] \times (c', d')$ are homeomorphic, and so on.

As in Examples 4.7 (4), note that these arguments to do not prove that a line $\{x\} \times [c, d]$ is homeomorphic to a rectangle $[a, b] \times [c, d]$. Indeed, we will in a later lecture be able to prove that these two topological spaces are not homeomorphic.



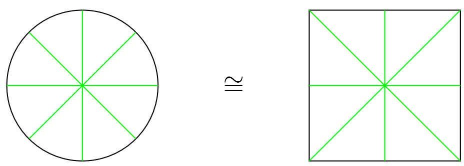
(3) We have that $D^2 \cong I^2$.



We can construct a homeomorphism

$$D^2 \xrightarrow{f} I^2$$

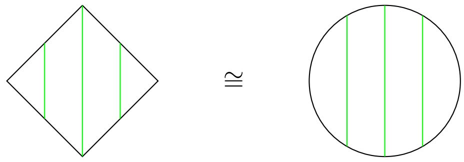
by stretching each line through the origin in D^2 to a line through the origin in I^2.



Alternatively we can for instance construct a homeomorphism

$$I^2 \xrightarrow{g} D^2$$

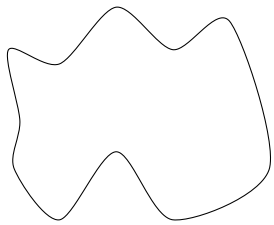
by stretching vertical lines I^2 to vertical lines in D^2.



See Exercise Sheet 4.

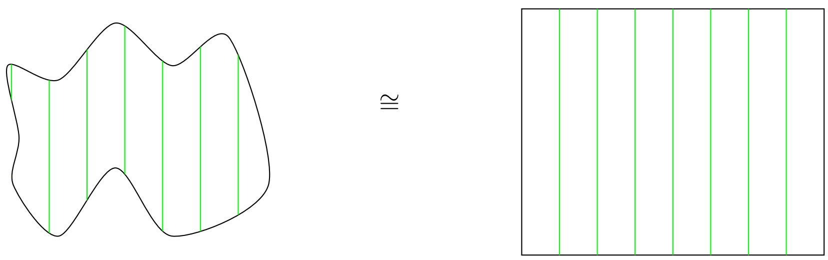
Don't be confused here: g is not inverse to f, just a different homeomorphism!

(4) Let X be a 'blob' in $\mathbb{R}^2$.



By similar ideas to those of (3) one can prove that X equipped with the subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is homeomorphic to I^2.

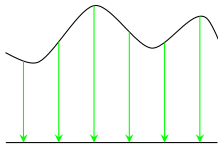
Roughly speaking one cuts X into strips with the property that there is a point in each strip to which every other point in the strip can be joined by a straight line. This property is known as *star convexity* — the strip is said to be *star shaped*.



As in (3) one proves that each strip is homeomorphic to D^2. Glueing two copies of D^2 which intersect in an arc is again homeomorphic to D^2. By induction one deduces that X is homeomorphic to D^2, and hence to I^2.

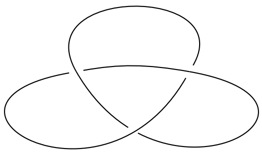
See Exercise Sheet 4.

(5) A ‘squiggle’ in $\mathbb{R}^2$ is homeomorphic to I.



See Exercise Sheet 4.

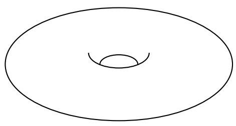
(6) We define a *knot* to be a subset of $\mathbb{R}^3$ which, equipped with the subspace topology with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$, is homeomorphic to S^1. For now we will not work rigorously with knots, but an example known as the ‘trefoil knot’ is pictured below.



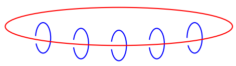
Intuitively, both the trefoil knot and S^1 may be obtained from a piece of string by glueing together the ends — we may bend, twist, and stretch the string as much as we wish before we glue the ends together.

We will look at the theory of knots later in the course.

(7) We have that $S^1 \times S^1 \cong T^2$, where $S^1 \times S^1$ is equipped with the product topology with respect to $\mathcal{O}_{S^1}$.



We will prove this in a later lecture. Intuitively, the idea is that T^2 can be obtained as a ‘circle of circles’.



Remark 4.11. Let us summarise these examples. Intuitively, two topological spaces are homeomorphic if we can bend, stretch, twist, compress, and otherwise ‘manipulate in a continuous manner’ each of these topological spaces so as to obtain the other!

4.2 Neighbourhoods and limit points

Definition 4.12. Let $(X, \mathcal{O}_X)$ be a topological space, and let $x \in X$. A *neighbourhood* of x is a subset U of X such that $x \in U$ and $U \in \mathcal{O}_X$.

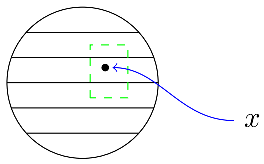
Examples 4.13.

(1) Let $X = \{a, b, c, d\}$, and let

$$\mathcal{O} := \big\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}, X\big\}.$$

The neighbourhoods of a are $\{a\}$, $\{a, b\}$, $\{a, c, d\}$, $\{a, b, d\}$, and X. The neighbourhoods of b are $\{b\}$, $\{a, b\}$, $\{a, b, d\}$, $\{b, c, d\}$, and X. The neighbourhoods of c are $\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, and X. The neighbourhoods of d are $\{c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{b, c, d\}$, and X.

(2) Let $x \in D^2$. A typical example of a neighbourhood of x is an open rectangle in D^2 containing x.



Definition 4.14. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. A *limit point* of A in X is an element $x \in X$ such that every neighbourhood of x in $(X, \mathcal{O}_X)$ contains at least one point of A.