Generell Topologi

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5.1 Limits points, closure, boundary — continued

Definition 5.1. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. The *closure* of A in X is the set of limit points of A in X.

Remark 5.2. This choice of terminology will be explained by Proposition 5.7.

Notation 5.3. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. We denote the closure of A in X by $\overline{A}$.

Definition 5.4. Let $(X, \mathcal{O}_X)$ be a topological space. A subset A of X is *dense* in X if $X = \overline{A}$.

Observation 5.5. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. Every $a \in A$ is a limit point of A, so $A \subset \overline{A}$.

Examples 5.6.

(1) Let $X = \{a, b\}$, and let $\mathcal{O} := \big\{\emptyset, \{b\}, X\big\}$. In other words, $(X, \mathcal{O})$ is the Sierpiński interval. Let $A := \{b\}$. We have that a is a limit point of A. Indeed, X is the only neighbourhood of a in X, and it contains b. Thus $\overline{A} = X$, and we have that A is dense in X.

(2) Let $X = \{a, b, c, d, e\}$, and let $\mathcal{O}$ denote the topology on X given by

$$\big\{\emptyset, \{a\}, \{b\}, \{c, d\}, \{a, b\}, \{a, c, d\}, \{b, e\}, \{b, c, d\}, \{b, c, d, e\}, \{a, b, c, d\}, \{a, b, e\}, X\big\}.$$

Let $A := \{d\}$. Then c is a limit point of A, since the neighbourhoods of $\{c\}$ in X are $\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{b, c, d, e\}$, $\{a, b, c, d\}$, and X, all of which contain d.

But b is not a limit point of A, since $\{b\}$ is a neighbourhood of b, and $\{b\} \cap A = \emptyset$. Similarly, a is not a limit point of A.

Also, $\{e\}$ is not a limit point of A, since the neighbourhood $\{b, e\}$ of e does not contain d. Thus $\overline{A} = \{c, d\}$.

Let $A' := \{b, d\}$. Then c is a limit point of A', since every neighbourhood of c in X contains d, as we already observed.

In addition, e is a limit point of A, since the neighbourhoods of e in X are $\{b, e\}$, $\{b, c, d, e\}$, $\{a, b, e\}$, and X, all of which contain either b or d, or both.

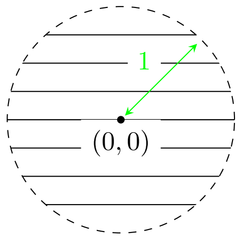
But a is not a limit point of A', since $\{a\} \cap A' = \emptyset$. Thus $\overline{A'} = \{b, c, d, e\}$.

(3) Let $A := [0, 1)$. Then 1 is a limit point of A in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. See Exercise Sheet 4.

(4) Let $A := \mathbb{Q}$, the set of rational numbers. Then every $x \in \mathbb{R}$ is a limit point of A in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Indeed, for every open interval (a, b) such that $a, b \in \mathbb{R}$ and $x \in (a, b)$, there is a rational number q with $a < q < x$. Thus $\overline{A} = \mathbb{R}$, and we have that $\mathbb{Q}$ is dense in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

(5) Let $A := \mathbb{Z}$, the set of integers. Then no real number which is not an integer is a limit point of A in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Indeed, let $x \in \mathbb{R}$, and suppose that $x \notin \mathbb{Z}$. Then $(\lfloor x \rfloor, \lceil x \rceil)$ is a neighbourhood of x not containing any integer. Thus $\overline{A} = \mathbb{Z}$. Here $\lfloor x \rfloor$ is the floor of x, namely the largest integer z such that $z \leq x$, and $\lceil x \rceil$ is the roof of x, namely the smallest integer z such that $z \geq x$.

(6) Let $A := \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| < 1\}$, the *open disc* around 0 in $\mathbb{R}^2$ of radius 1.



Then $(x, y) \in \mathbb{R}^2$ is a limit point of A in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$ if and only if $(x, y) \in D^2$. Let us prove this. If $(x, y) \notin D^2$, then $\|(x, y)\| > 1$. Let $\epsilon \in \mathbb{R}$ be such that

$$0 < \epsilon \leq |x| - \frac{|x|}{\|(x, y)\|},$$

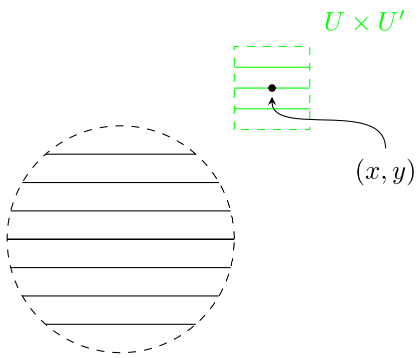
and let $\epsilon' \in \mathbb{R}$ be such that

$$0 < \epsilon' \leq |y| - \frac{|y|}{\|(x, y)\|}.$$

GLet $U := (x - \epsilon, x + \epsilon)$, and let $U' := (y - \epsilon', y + \epsilon')$. By definition of $\mathcal{O}_{\mathbb{R}\times\mathbb{R}}$, $U \times U' \in \mathcal{O}_{\mathbb{R}\times\mathbb{R}}$. Moreover, for every $(u, u') \in U \times U'$, we have that

$$\begin{aligned}
\|(u, u')\| &= \|(|u|, |u'|)\| \\
&> \|(|x| - \epsilon, |y| - \epsilon')\| \\
&\geq \| \frac{1}{\|(|x|, |y|)\|}(x, y)\| \\
&= 1.
\end{aligned}$$

Thus $U \times U'$ is a neighbourhood of (x, y) in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$ with the property that $A \cap (U \times U') = \emptyset$. We deduce that (x, y) is not a limit point of A.

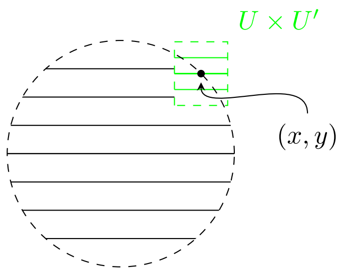


Suppose now that $(x, y) \in S^1$. Let W be a neighbourhood of (x, y) in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. By definition of $\mathcal{O}_{\mathbb{R}\times\mathbb{R}}$, there is an open interval U in $\mathbb{R}$ and an open interval U' in $\mathbb{R}$ such that $x \in U$, $y \in U'$, and $U \times U' \subset W$.

Let us denote the open interval $\{|u| \mid u \in U\}$ in $\mathbb{R}$ by (a, b) for $a, b \in \mathbb{R}$, and let us denote the open interval $\{|u'| \mid u' \in U'\}$ in $\mathbb{R}$ by (a', b') for $a', b' \in \mathbb{R}$. Let $x' \in U$ be such that $a < |x'| < |x|$, and let $y' \in U$ be such that $a' < |y'| < |y|$. Then we have that

$$\begin{aligned}\|(x', y')\| &= \|(|x'|, |y'|)\| \\ &< \|(|x|, |y|)\| \\ &= 1.\end{aligned}$$

Thus $(x', y') \in A \cap (U \times U')$, and hence $(x', y') \in A \cap W$. Thus (x, y) is a limit point of A.



Putting everything together, we conclude that $\overline{A} = D^2$.

Proposition 5.7. Let $(X, \mathcal{O}_X)$ be a topological space, and let V be a subset of X. Then V is closed in $(X, \mathcal{O}_X)$ if and only if $V = \overline{V}$.

Proof. Suppose that V is closed. By definition, $X \setminus V$ is then open. Thus, for any $x \in X$ such that $x \notin V$, we have that $X \setminus V$ is a neighbourhood of x. Moreover, by definition, $X \setminus V$ does not contain any element of V. Thus x is not a limit point of V in X. We conclude that $V = \overline{V}$.

Suppose now that $V = \overline{V}$. Then for every $x \notin V$ there is a neighbourhood of x which does not contain any element of V. Let us denote this neighbourhood by U_x. We make three observations.

(1) $X \setminus V \subset \bigcup_{x \in X \setminus V} U_x$, since $x \in U_x$.

(2) $\bigcup_{x \in X \setminus V} U_x \subset X \setminus V$, since

$$V \cap \Big(\bigcup_{x \in X \setminus V} U_x \Big) = \bigcup_{x \in X \setminus V} (U_x \cap V) = \bigcup_{x \in X \setminus V} \emptyset = \emptyset.$$

(3) $\bigcup_{x \in X \setminus V} U_x \in \mathcal{O}_X$, since $U_x \in \mathcal{O}_X$ for all $x \in X \setminus V$.

Putting (1) and (2) together, we have that $\bigcup_{x \in X \setminus V} U_x = X \setminus V$. Hence, by (3), $X \setminus V \in \mathcal{O}_X$. Thus V is closed. □

Remark 5.8. In other words, a subset V of a topological space $(X, \mathcal{O}_X)$ is closed if and only if every limit point of V belongs to V.

Proposition 5.9. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. Suppose that V is a closed subset of X with $A \subset V$. Then $\overline{A} \subset V$.

Proof. See Exercise Sheet 4. □

Remark 5.10. In other words, $\overline{A}$ is the smallest closed subset of X containing A.

Corollary 5.11. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. Then

$$\overline{A} = \bigcap_{V} V,$$

where the intersection is taken over all closed subsets V of X with the property that $A \subset V$.

Proof. Follows immediately from Proposition 5.9. □

Definition 5.12. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. The *boundary* of A in X is the set $x \in X$ such that every neighbourhood of x in X contains at least one element of A and at least one element of $X \setminus A$.

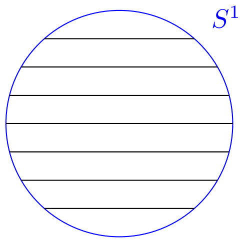
Notation 5.13. We denote the boundary of A in X by $\partial_X A$.

Observation 5.14. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. Every limit point of A which does not belong to A belongs to $\partial_X A$.

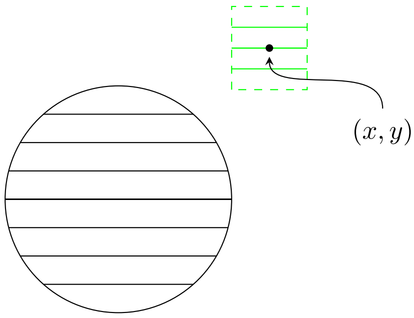
Terminology 5.15. The boundary of A in X is also known as the *frontier* of A in X.

Examples 5.16.

(1) Let $X := (\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$, and let $A := D^2$. Then $\partial_X A = S^1$.



Let us prove this. By exactly the argument of the first part of the proof in Examples 5.6 (6), every $(x, y) \in \mathbb{R}^2 \setminus D^2$ is not a limit point of D^2. Thus $\partial_A X \subset D^2$.



Suppose that $(x, y) \in D^2$, but that $(x, y) \notin S^1$. Then $\|(x, y)\| < 1$. Let $\epsilon \in \mathbb{R}$ be such that

$$0 < \epsilon \leq \frac{|x|}{\|(x, y)\|} - |x|,$$

and let $\epsilon' \in \mathbb{R}$ be such that

$$0 < \epsilon' \leq \frac{|y|}{\|(x, y)\|} - |y|.$$

Let $U := (x - \epsilon, x + \epsilon)$, and let $U' := (y - \epsilon', y + \epsilon')$. By definition of $\mathcal{O}_{\mathbb{R}\times\mathbb{R}}$, $U \times U' \in \mathcal{O}_{\mathbb{R}\times\mathbb{R}}$. Moreover, for every $(u, u') \in U \times U'$, we have that

$$\begin{aligned}\|(u, u')\| &= \|(|u|, |u'|)\| \\ &< \|(|x| + \epsilon, |y| + \epsilon')\| \\ &\leq \|\frac{1}{\|(x, y)\|}(|x|, |y|)\| \\ &= 1.\end{aligned}$$

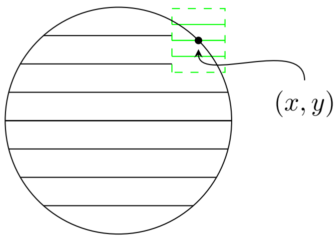
Thus $U \times U'$ is a neighbourhood of (x, y) in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$ with the property that $(\mathbb{R}^2 \setminus D^2) \cap (U \times U') = \emptyset$. We deduce that $(x, y) \notin \partial_A X$.

We now have that $\partial_X A \subset S^1$. Suppose that $(x, y) \in S^1$, and let W be a neighbourhood of (x, y) in $\mathbb{R}^2$. By definition of $\mathcal{O}_{\mathbb{R}\times\mathbb{R}}$, there is an open interval U in $\mathbb{R}$ and an open interval U' in $\mathbb{R}$ such that $x \in U$, $y \in U'$, and $U \times U' \subset W$.

Let us denote the open interval $\{|u| \mid u \in U\}$ in $\mathbb{R}$ by (a, b) for $a, b \in \mathbb{R}$, and let us denote the open interval $\{|u'| \mid u' \in U'\}$ in $\mathbb{R}$ by (a', b') for $a', b' \in \mathbb{R}$. Let $x' \in U$ be such that $|x| < |x'| < b$, and let $y' \in U$ be such that $|y| < |y'| < b'$. Then we have that

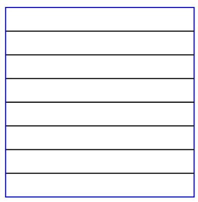
$$\begin{aligned} \|(x', y')\| &= \|(|x'|, |y'|)\| \\ &> \|(|x|, |y|)\| \\ &= 1. \end{aligned}$$

Thus $(x', y') \in (\mathbb{R}^2 \setminus D^2) \cap (U \times U')$, and hence $(x', y') \in (\mathbb{R}^2 \setminus D^2) \cap W$. In addition, (x, y) belongs to both D^2 and W. We deduce that $(x, y) \in \partial_X A$, and conclude that $S^1 \subset \partial_X A$.



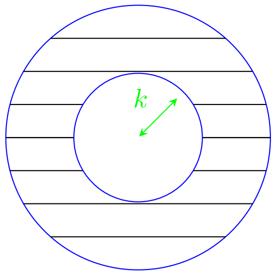
Putting everything together, we have that $\partial_A X = S^1$. Alternatively, this may be deduced from Example (2) below, via a homeomorphism between D^2 and I^2.

(2) Let $X := (\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$, and let $A := I^2$. Then $\partial_A X$ is as indicated in blue below.



We have at least three ways to prove this. Firstly, as a corollary of Example (1), via a homeomorphism between I^2 and D^2. Secondly directly, by an argument similar to that in Example (1). Thirdly as a corollary of Example (4) below, using a general result on the boundary of a product of topological spaces which we will prove in Exercise Sheet 4.

(3) Let $X := (\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$, and let $A := A_k$, an annulus, for some $k \in \mathbb{R}$ with $0 < k < 1$. Then $\partial_X A$ is as indicated in blue below. This may be proven by an argument similar to that in Example (1).



(4) Let $X := (\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then $\partial_X (0,1) = \partial_X (0,1] = \partial_X [0,1) = \partial_X [0,1] = \{0,1\}$. See Exercise Sheet 4.

(5) Let $X := \{a,b,c,d,e\}$, and let $\mathcal{O}$ denote the topology

$$\big\{\emptyset, \{a\}, \{b\}, \{c,d\}, \{a,b\}, \{a,c,d\}, \{b,e\}, \{b,c,d\}, \{b,c,d,e\}, \{a,b,c,d\}, \{a,b,e\}, X\big\}$$

on X, as in Examples 5.6 (2). Let $A := \{b,d\}$.

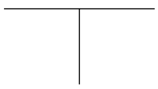
We saw in Examples 5.6 (2) that the limit points of A which do not belong to A are $\{c\}$ and $\{e\}$. Also $d \in \partial_X A$. Indeed, the neighbourhoods of d in X are $\{c,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, $\{b,c,d,e\}$, $\{a,b,c,d\}$, and X. Each of these neighbourhoods contains c, which does not belong to A.

But b does not belong to $\partial_X A$, since $\{b\}$ is a neighbourhood of b in X, and $\{b\}$ does not contain an element of $X \setminus A$. Thus $\partial_X A = \{c,d,e\}$.

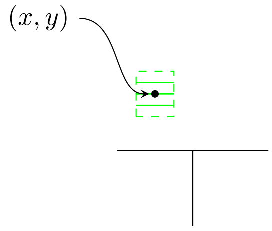
(6) Let A denote the letter T, viewed as the subset

$$\{(0,y) \mid 0 \leq y \leq 1\} \cup \{(x,1) \mid -1 \leq x \leq 1\}$$

of $\mathbb{R}^2$.



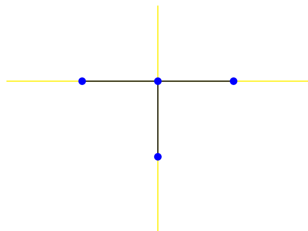
Let $X := (\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. Then $\partial_X \mathsf{T} = \mathsf{T}$. Indeed, for every $(x,y) \notin \mathsf{T}$, there exists a neighbourhood $U \times U' \subset \mathbb{R}^2$ of (x,y) such that $(U \times U') \cap \mathsf{T} = \emptyset$.



Instead, let X denote the subset

$$\{(0,y) \mid y \in \mathbb{R}\} \cup \{(x,1) \mid x \in \mathbb{R}\}$$

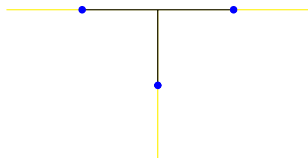
of $\mathbb{R}^2$, equipped with the subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. Then $\partial_X \mathsf{T}$ consists of the four elements of T indicated in blue in the following picture, in which X is drawn in yellow.



Now let X denote the subset

$$\{(0,y) \mid y \leq 1\} \cup \{(x,1) \mid x \in \mathbb{R}\}$$

of $\mathbb{R}^2$, equipped with the subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. Then $\partial_X \mathsf{T}$ consists of the three elements of T indicated in blue in the following picture, in which again X is drawn in yellow.



As Examples 5.16 (6) illustrates, a set A may have a different boundary depending upon which topological space it is regarded as a subset of.

5.2 Coproduct topology

Recollection 5.17. Let X and Y be sets. The *disjoint union* of X and Y is the set $(X \times \{0\}) \cup (Y \times \{1\})$.

Let

$$X \xrightarrow{i_X} X \sqcup Y$$

denote the map given by $x \mapsto (x, 0)$, and let

$$Y \xrightarrow{i_Y} X \sqcup Y$$

denote the map given by $y \mapsto (y, 1)$.

Terminology 5.18. A disjoint union is also known as a *coproduct*.

Proposition 5.19. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\mathcal{O}_{X \sqcup Y}$ be the set of subsets U of $X \sqcup Y$ such the following conditions are satisfied.

(1) $i_X^{-1}(U) \in \mathcal{O}_X$.

(2) $i_Y^{-1}(U) \in \mathcal{O}_Y$.

Then $\mathcal{O}_{X \sqcup Y}$ defines a topology on $X \sqcup Y$.

Proof. Exercise. □

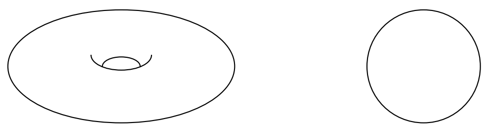
Terminology 5.20. We refer to $\mathcal{O}_{X \sqcup Y}$ as the *coproduct topology* on $X \sqcup Y$.

Observation 5.21. It is immediate from the definition of $\mathcal{O}_{X \sqcup Y}$ that i_X and i_Y are continuous.

Examples 5.22.

(1) $T^2 \sqcup T^2$.

(2) $T^2 \sqcup S^1$.



The disjoint union of two sets is very different from the union. Indeed, $T^2 \cup T^2 = T^2$. Two doughnuts are very different from one doughnut!