Generell Topologi

Richard Williamson

May 6, 2013

6 Thursday 31st January

6.1 Connected topological spaces — equivalent conditions, an example, and two non-examples

Observation 6.1. Both X and Y are open in $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$. Indeed, $i_X^{-1}(X \sqcup Y) = X$, and $X \in \mathcal{O}_X$. Similarly, $i_Y^{-1}(X \sqcup Y) = Y$, and $Y \in \mathcal{O}_Y$.

Moreover, $(X \sqcup Y) \setminus X = Y$. Thus, since Y is open in $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$, we have that X is closed in $X \sqcup Y$. Similarly, $(X \sqcup Y) \setminus Y = X$. Since X is open in $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$, we have that Y is closed in $X \sqcup Y$.

Putting everything together, we have that X and Y are each both open and closed in $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$.

Definition 6.2. A space $(X, \mathcal{O}_X)$ is *connected* if there do not exist $X_0,\ X_1 \subset X$ such that the following all hold.

(1) $X = X_0 \sqcup X_1$.

(2) $X_0 \in \mathcal{O}_X$, and $X_0 \neq \emptyset$.

(3) $X_1 \in \mathcal{O}_X$, and $X_1 \neq \emptyset$.

Proposition 6.3. Let $(X, \mathcal{O}_X)$ be a topological space. Then X is connected if and only if the only subsets of X which are both open and closed in $(X, \mathcal{O}_X)$ are $\emptyset$ and X.

Proof. Suppose that there exists a subset A of X which is both open and closed. Then A and $X \setminus A$ are both open in X, and also $X = A \sqcup (X \setminus A)$. If X is connected, A must therefore be $\emptyset$ or X.

Suppose now that the only subsets of X which are both open and closed are $\emptyset$ and X, and that $X = A \sqcup A'$, with both A and A' open. Then since $X \setminus A = A'$, $X \setminus A$ is open in X, and thus A is closed in X. Thus A is $\emptyset$ or X. Hence X is connected. $\square$

Observation 6.4. Let X be a set, and let A and A' be subsets of X. Then $X = A \sqcup A'$ if and only if the following conditions are satisfied.

(1) $X = A \cup A'$.

(2) $A \cap A' = \emptyset$.

Proposition 6.5. Let $\{0,1\}$ be equipped with the discrete topology $\mathcal{O}^{\text{disc}}_{\{0,1\}}$. A topological space $(X, \mathcal{O}_X)$ is connected if and only if there does not exist a surjective continuous map

$$X \longrightarrow \{0,1\}.$$

Proof. Suppose that there exists a surjective continuous map

$$X \xrightarrow{f} \{0,1\}.$$

We make the following observations.

(1) $f^{-1}(0)$ and $f^{-1}(1)$ are both open in X, since f is continuous and $\{0\}$ and $\{1\}$ both belong to $\mathcal{O}^{\text{disc}}_{\{0,1\}}$.

(2) $f^{-1}(0)$ and $f^{-1}(1)$ are both non-empty in X, since f is surjective.

(3) $f^{-1}(0) \cup f^{-1}(1) = f^{-1}(\{0,1\}) = X$.

(4) $f^{-1}(0) \cap f^{-1}(1) = \{x \in X \mid f(x) = 0 \text{ and } f(x) = 1\} = \emptyset$, since f is a well-defined map.

By (3) and (4) and Observation 6.4, $X = f^{-1}(0) \sqcup f^{-1}(1)$. Thus, by (1) and (2), X is not connected.

Suppose now that X is not connected. Thus we have $X = A \sqcup A'$ for a pair of open subsets A and A' of X. Define

$$X \xrightarrow{f} \{0,1\}$$

by

$$\begin{cases} x \mapsto 0 & \text{if } x \in A, \\ x \mapsto 1 & \text{if } x \in A'. \end{cases}$$

Then $f^{-1}(0) = A$ and $f^{-1}(1) = A'$, and thus f is continuous. □

Examples 6.6.

(1) Let $X = \{a,b\}$ be a set with two elements, and let $\mathcal{O} := \{\emptyset, \{b\}, X\}$. In other words, $(X, \mathcal{O})$ is the Sierpiński interval. Then $(X, \mathcal{O})$ is connected, since the only way to express X as a disjoint union is $X = \{a\} \sqcup \{b\}$, but $\{a\} \notin \mathcal{O}$.

(2) Take $X = \{a,b,c,d,e\}$. Let $\mathcal{O}$ be the topology

$$\{\emptyset, \{a\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{c,d,e\}, \{a,b,c,d\}, \{a,c,d,e\}, X\}$$

on X. Then $(X, \mathcal{O})$ is not connected, since $X = \{a,b\} \sqcup \{c,d,e\}$, and we have that both $\{a,b\}$ and $\{c,d,e\}$ belong to $\mathcal{O}$.

(3) Equip $\mathbb{Q}$ with the subspace topology $\mathcal{O}_{\mathbb{Q}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not connected. Indeed, pick any irrational $x \in \mathbb{R}$, such as $x = \sqrt{2}$. Then

$$\mathbb{Q} = \big(\mathbb{Q} \cap (-\infty, x)\big) \sqcup \big(\mathbb{Q} \cap (x, \infty)\big),$$

and since both $(-\infty, x)$ and (x, ∞) belong to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, we have that both $\mathbb{Q} \cap (-\infty, x)$ and $\mathbb{Q} \cap (x, \infty)$ belong to $\mathcal{O}_{\mathbb{Q}}$.

6.2 Connectedness of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$

Lemma 6.7. Let A be a subset of $\mathbb{R}$ which is bounded below. Let b denote the greatest lower bound of A, which exists by the completeness of $\mathbb{R}$, as expressed in Theorem 1.10. Then $b \in \overline{A}$, and for every $x \in \overline{A}$, we have that $b < x$. Here $\overline{A}$ is the closure of A with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Proof. Let U be a neighbourhood of b in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By definition of $\mathcal{O}_{\mathbb{R}}$, U is a union of open intervals in $\mathbb{R}$. One of these open intervals must contain b. Let us denote it by (x, x'). There exists $b' \in A$ such that $b \leq b' < x'$, since otherwise x' would be a lower bound of A with the property that $x' > b$. We thus have that $b' \in (x, x')$, and since $(x, x') \subset U$, we deduce that $b' \in U$.

We have now shown that $b' \in A \cap U$. We conclude that b is a limit point of A in $\mathbb{R}$, and thus that $b \in \overline{A}$.

Suppose now that $a \in \overline{A}$. If $a < b$, let $\epsilon := b - a$. Since $\epsilon > 0$, we have that $(a-\epsilon, a+\epsilon)$ is a neighbourhood of a in $\mathbb{R}$. Since a is a limit point of A in $\mathbb{R}$, we deduce that there exists $a' \in \mathbb{R}$ with $a' \in A \cap (a-\epsilon, a+\epsilon)$. But then $a' < a + \epsilon$, and since $a + \epsilon = b$, we have that $a' < b$. Together with the fact that $a' \in A$, this contradicts our assumption that b is a lower bound of A.

We therefore have that $a \geq b$, as required.

□

Example 6.8. For a prototypical illustration of Lemma 6.7, let A denote the open interval $(0, 1)$.

Then 0 is the greatest lower bound of A, and $\overline{A} = [0, 1]$.

Proposition 6.9. The topological space $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.

Proof. Suppose that there exists a subset U of $\mathbb{R}$ which is both open and closed in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, and such that $U \neq \mathbb{R}$ and $U \neq \emptyset$. Let $x \in \mathbb{R} \setminus U$. Since $U \neq \emptyset$, either $U \cap [x, \infty) \neq \emptyset$ or $U \cap (-\infty, x] \neq \emptyset$. Suppose that $U \cap [x, \infty) \neq \emptyset$, and let us denote $U \cap [x, \infty)$ by A. Then

$$\begin{aligned}\mathbb{R} \setminus A &= \mathbb{R} \setminus \big(U \cap [x, \infty)\big) \\ &= \big(\mathbb{R} \setminus U\big) \cup \big(\mathbb{R} \setminus [x, \infty)\big) \\ &= (\mathbb{R} \setminus U) \cup (-\infty, x).\end{aligned}$$

Since U is closed in $\mathbb{R}$, $\mathbb{R} \setminus U$ is open in $\mathbb{R}$. Also, $(-\infty, x)$ is open in $\mathbb{R}$. Thus $\mathbb{R} \setminus A$ is open in $\mathbb{R}$, and hence A is closed in $\mathbb{R}$.

In addition, since $x \notin U$, we have that $A = U \cap (x, \infty)$. Since U is open in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, and since (x, ∞) is also open in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, we have that A is open in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

By definition of A, x is a lower bound of A. Thus, by the completeness of $\mathbb{R}$ as expressed in Theorem 1.10, A admits a greatest lower bound. Let us denote it by $b \in \mathbb{R}$. Since A is closed in $\mathbb{R}$, by Lemma 6.7 and Proposition 5.7, we have that $b \in A$, and that for every $a \in A$, $b \leq a$.

But since A is open in $\mathbb{R}$, it is a union of open intervals in $\mathbb{R}$, one of which must contain b. Let us denote it by (a', a''). Then $a' < b$, which since $a' \in A$ contradicts that $b \leq a$ for all $a \in A$.

The proof in the case that $U \cap (-\infty, x] \neq \emptyset$ is entirely analogous. $\square$