Generell Topologi

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7.1 Characterisation of connected subspaces of \((\mathbb{R}, \mathcal{O}_\mathbb{R})\)

**Proposition 7.1.** Let \((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\) be topological spaces, and let

\[ X \xrightarrow{f} Y \]

be a continuous map. Suppose that \((X, \mathcal{O}_X)\) is connected. Let \(f(X)\) be equipped with the subspace topology \(\mathcal{O}_{f(X)}\) with respect to \((Y, \mathcal{O}_Y)\). Then \((f(X), \mathcal{O}_{f(X)})\) is connected.

**Proof.** Suppose that \(f(X) = U_0 \sqcup U_1\), and that \(U_0\) and \(U_1\) are open in \(f(X)\). By definition, \(U_0 = f(X) \cap Y_0\) for an open subset \(Y_0\) of \(Y\), and \(U_1 = f(X) \cap Y_1\) for an open subset \(Y_1\) of \(Y\). We make the following observations.

1. Since \(f\) is continuous, \(f^{-1}(Y_0)\) is open in \(X\). We have that
   \[
   f^{-1}(U_0) = f^{-1}(f(X) \cap Y_0) = f^{-1}(f(X)) \cap f^{-1}(Y_0) = X \cap f^{-1}(Y_0) = f^{-1}(Y_0).
   \]
   Thus \(f^{-1}(U_0)\) is open in \(X\).

2. By an analogous argument, \(f^{-1}(U_1)\) is open in \(X\).

3. We have that \(f^{-1}(U_0) \cap f^{-1}(U_1) = f^{-1}(U_0 \cap U_1)\). Since \(U_0 \cap U_1 = \emptyset\), we deduce that \(f^{-1}(U_0 \cap U_1) = \emptyset\). Thus \(f^{-1}(U_0) \cap f^{-1}(U_1) = \emptyset\).

4. We have that \(f^{-1}(U_0) \cup f^{-1}(U_1) = f^{-1}(U_0 \cup U_1)\). Since \(U_0 \cup U_1 = f(X)\), and since \(f^{-1}(f(X)) = X\), we deduce that \(f^{-1}(U_0) \cup f^{-1}(U_1) = X\).

By (3) and (4), we have that \(X = f^{-1}(U_0) \sqcup f^{-1}(U_1)\). Thus, by (1), (2), and the fact that \((X, \mathcal{O}_X)\) is connected, we must have that either \(f^{-1}(U_0) = X\) or that \(f^{-1}(U_0) = \emptyset\). Hence either \(U_0 = f(X)\) or \(U_0 = \emptyset\).

**Remark 7.2.** We will sometimes refer to the conclusion of Proposition 7.1 as: ‘the continuous image of a connected topological space is connected’.

**Corollary 7.3.** Let \((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\) be topological spaces, and let

\[ X \xrightarrow{f} Y \]

be a homeomorphism. If \((X, \mathcal{O}_X)\) is connected, then \((Y, \mathcal{O}_Y)\) is connected.

**Proof.** By Proposition 3.15 we have that \(f\) is surjective, or in other words we have that \(f(X) = Y\). It follows immediately from Proposition 7.1 that \(Y\) is connected.
Proposition 7.4. Let \((X, \mathcal{O}_X)\) be a connected topological space, and let \(A\) and \(A'\) be subsets of \(X\). Let \(A\) be equipped with the subspace topology \(\mathcal{O}_A\) with respect to \((X, \mathcal{O}_X)\), and let \(A'\) be equipped with the subspace topology with respect to \((X, \mathcal{O}_X)\). Suppose that \((A', \mathcal{O}_{A'})\) is connected, and that \(A' \subset A \subset \overline{A}\). Then \((A, \mathcal{O}_A)\) is connected.

Proof. Suppose that \(A = U_0 \sqcup U_1\), where both \(U_0\) and \(U_1\) belong to \(\mathcal{O}_A\). By definition of \(\mathcal{O}_A\), we have that \(U_0 = A \cap U_0'\) for an open subset \(U_0'\) of \(X\), and that \(U_1 = A \cap U_1'\) for an open subset \(U_1'\) of \(X\).

We have that

\[
(A' \cap U_0) \cup (A' \cap U_1) = A' \cap (U_0 \cup U_1) = A' \cap A = A',
\]

where for the final equality we appeal to the fact that \(A' \subset A\).

Moreover, we have that

\[
(A' \cap U_0) \cap (A' \cap U_1) = A' \cap (U_0 \cap U_1) = A' \cap \emptyset = \emptyset.
\]

Putting the last two observations together, we have that \(A' = (A' \cap U_0) \cup (A' \cap U_1)\). Moreover, we have that

\[
A' \cap U_0 = A' \cap (A \cap U_0') = (A' \cap A) \cap U_0' = A' \cap U_0',
\]

and that

\[
A' \cap U_1 = A' \cap (A \cap U_1') = (A' \cap A) \cap U_1' = A' \cap U_1'.
\]

For the last equality in each case we appeal again to the fact that \(A' \subset A\). By definition of \(\mathcal{O}_{A'}\), we thus have that \(A' \cap U_0 = A' \cap U_0'\) and \(A' \cap U_1 = A' \cap U_1'\) are open in \(A'\). Since \((A', \mathcal{O}_{A'})\) is connected, we deduce that either \(A' \cap U_0 = \emptyset\) or \(A' \cap U_1 = \emptyset\).

Suppose that \(A' \cap U_0 = \emptyset\). Since \(A' \cap U_0 = A' \cap U_0'\), we then have that \(A' \cap U_0' = \emptyset\). Thus \(A' \subset X \setminus U_0'\). Since \(U_0'\) is open in \(X\), we have that \(X \setminus U_0'\) is closed in \(X\). By Proposition 5.9, we deduce that \(\overline{A'} \subset X \setminus U_0'\).

By assumption we have that \(A \subset \overline{A}\). Thus \(U_0 = A \cap U_0' \subset \overline{A} \cap X_0 = \emptyset\), so that \(U_0 = \emptyset\).

An entirely analogous argument gives that if \(A' \cap U_1 = \emptyset\), then \(U_1 = \emptyset\). Putting everything together, we have proven that if \(A = U_0 \sqcup U_1\), where both \(U_0\) and \(U_1\) are open in \((A, \mathcal{O}_A)\), then either \(U_0 = \emptyset\) or \(U_1 = \emptyset\). Thus \((A, \mathcal{O}_A)\) is connected.

\(\square\)
Remark 7.5. We will sometimes refer to Proposition 7.4 as the ‘sandwich proposition’.

Corollary 7.6. Let \((X, \mathcal{O}_X)\) be a connected topological space, and let \(A\) be a subset of \(X\). Let \(A\) be equipped with the subspace topology \(\mathcal{O}_A\) with respect to \((X, \mathcal{O}_X)\), and let \(\overline{A}\) be equipped with the subspace topology \(\mathcal{O}_{\overline{A}}\) with respect to \((X, \mathcal{O}_X)\). Suppose that \((A, \mathcal{O}_A)\) is connected. Then \((\overline{A}, \mathcal{O}_{\overline{A}})\) is connected.

Proof. Follows immediately from Proposition 7.4, taking both \(A\) and \(A'\) to be \(A\).

Lemma 7.7. Let \((X, \mathcal{O})\) be a topological space. If \(X\) is empty or consists of exactly one element, then \((X, \mathcal{O})\) is connected.

Proof. If \(X\) is empty or consists of exactly one element, the only ways to express \(X\) as a disjoint union of subsets are \(X = \emptyset \sqcup X\) and \(X = X \sqcup \emptyset\). For \(X = \emptyset \sqcup X\), condition (2) of Definition 6.2 is not satisfied. For \(X = X \sqcup \emptyset\), condition (3) of Definition 6.2 is not satisfied.

Lemma 7.8. Let \(X\) be a non-empty subset of \(\mathbb{R}\). Then \(X\) is an open interval, a closed interval, or a half open interval if and only if for every \(x, x' \in X\) and \(y \in \mathbb{R}\) with \(x < y < x'\) we have that \(y \in X\).

Proof. Suppose that \(X = [a, b]\), for \(a, b \in \mathbb{R}\) with \(a \leq b\). If \(x, x' \in X\), then by definition of \([a, b]\) we have that \(a \leq x \leq b\) and \(a \leq x' \leq b\). Suppose that \(x < x'\) and that \(y \in \mathbb{R}\) has the property that \(x < y < x'\). Then we have that \(a \leq x < y < x' \leq b\), and in particular \(a \leq y \leq b\). Thus, by definition of \([a, b]\), we have that \(y \in [a, b]\).

If \(X\) is an open interval or a half open interval, an entirely analogous argument proves that if \(x, x' \in X\) and \(y \in \mathbb{R}\) have the property that \(x < y < x'\), then \(y \in X\).

Conversely, suppose that for every \(x, x' \in X\) and \(y \in \mathbb{R}\) with \(x < y < x'\) we have that \(y \in X\). Let \(a = \inf X\) and let \(b = \sup X\). As in Lecture 1, if \(X\) is not bounded below we adopt the convention that \(\inf X = -\infty\), and if \(X\) is not bounded above we adopt the convention that \(\sup X = \infty\).

Suppose that \(y \in X\) has the property that \(a < y < b\). Since \(y > a\) there is an \(x \in X\) with \(x < y\), since otherwise \(y\) would be a lower bound of \(X\), contradicting the fact that \(a\) is by definition the greatest lower bound of \(X\). Since \(y < b\) there is an \(x' \in X\) with \(y < x'\), since otherwise \(y\) would be an upper bound of \(X\), contradicting the fact that \(b\) is by definition the least upper bound of \(X\). We have shown that \(x < y < x'\), with \(x, x' \in X\). By assumption, we deduce that \(y \in X\).

We have now shown that for all \(y \in X\) such that \(a < y < b\), we have that \(y \in X\). To complete the proof, there are four cases to consider.

1. If \(a, b \in X\), we have by definition of \([a, b]\) that \(X = [a, b]\).
2. If \(a \in X\) and \(b \notin X\), we have by definition of \([a, b]\) that \(X = [a, b]\).
3. If \(a \notin X\) and \(b \in X\), we have by definition of \((a, b]\) that \(X = (a, b]\).
4. If \(a \notin X\) and \(b \notin X\), we have by definition of \((a, b]\) that \(X = (a, b]\).
Proposition 7.9. Let $X$ be a subset of $\mathbb{R}$, and let $X$ be equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}, \mathcal{O}_\mathbb{R})$. Then $X$ is connected if and only if $X$ is an open interval, a closed interval, a half open interval, or $\emptyset$.

Proof. If $X$ is not an open interval, a closed interval, a half open interval, or $\emptyset$, then by Lemma 7.8 there are $x, x' \in X$ and $y \in \mathbb{R} \setminus X$ with $x < y < x'$. Let $U_0 := X \cap (-\infty, y)$, and let $U_1 := X \cap (y, \infty)$. By definition of $\mathcal{O}_X$, both $U_0$ and $U_1$ are open in $X$. Moreover, $X = X_0 \sqcup X_1$. Thus $X$ is not connected. This completes one direction of the proof.

Conversely, let us prove that if $X$ is an open interval, a closed interval, a half open interval, or $\emptyset$, then $X$ is connected. By Lemma 7.7, if $X = \emptyset$ or if $X$ consists of exactly one element, then $X$ is connected.

Suppose instead that $X$ is an open interval. By Examples 4.7 (2) we have that $X$ is homeomorphic to $\mathbb{R}$. By Proposition 6.9 and Corollary 7.3, we deduce that $X$ is connected.

Suppose now that $X$ is either a closed interval or a half open interval, with more than one element. Let us denote the open interval $X \setminus \partial_X \mathbb{R}$ by $X'$. We have by earlier in the proof that $X'$ is connected. Moreover we have that $X' \subset X \subset X'$. By Proposition 7.4, we deduce that $X$ is connected.

Corollary 7.10. Let $(X, \mathcal{O}_X)$ be a connected topological space, and let

$$f : X \rightarrow \mathbb{R}$$

be a continuous map. Let $x$ and $x'$ be elements of $X$ with $f(x) \leq f(x')$. Then for every $y \in \mathbb{R}$ such that $f(x) \leq y \leq f(x')$, there is an $x'' \in X$ such that $f(x'') = y$.

Proof. By Proposition 7.1, $f(X)$ is connected. By Proposition 7.9 we deduce that $f(X)$ is an open interval, a closed interval, or a half open interval. By Lemma 7.8 we conclude that every $y \in \mathbb{R}$ such that $f(x) \leq y \leq f(x'')$ belongs to $f(X)$.

Remark 7.11. Taking $(X, \mathcal{O}_X)$ to be $(\mathbb{R}, \mathcal{O}_\mathbb{R})$, Corollary 7.10 is the ‘intermediate value theorem’ that you met in real analysis/calculus!

7.2 Examples of connected topological spaces

Lemma 7.12. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. For every $x$ in $X$, let $\{x\} \times Y$ be equipped with the subspace topology $\mathcal{O}_{\{x\} \times Y}$ with respect to $(X \times Y, \mathcal{O}_{X \times Y})$. Then $(\{x\} \times Y, \mathcal{O}_{\{x\} \times Y})$ is homeomorphic to $(Y, \mathcal{O}_Y)$.

For every $y$ in $Y$, let $X \times \{y\}$ be equipped with the subspace topology $\mathcal{O}_{X \times \{y\}}$ with respect to $(X \times Y, \mathcal{O}_{X \times Y})$. Then $(X \times \{y\}, \mathcal{O}_{X \times \{y\}})$ is homeomorphic to $(X, \mathcal{O}_X)$.

Proof. Exercise.

Proposition 7.13. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then $(X \times Y, \mathcal{O}_{X \times Y})$ is connected if and only if both $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are connected.
Proof. Suppose that \((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\) are connected. Let \(X \times Y\) be equipped with the product topology \(\mathcal{O}_{X \times Y}\), and let \(\{0,1\}\) be equipped with the discrete topology. Let 

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{f} & \{0,1\},
\end{array}
\]

be a continuous map.

Let \(x, x' \in X\), and let \(y, y' \in Y\). Let \(\{x\} \times Y\) with the subspace topology with respect to \((X \times Y, \mathcal{O}_{X \times Y})\), and let \(i_x\) denote the inclusion 

\[
\{x\} \times Y \hookrightarrow X \times Y.
\]

By Proposition \ref{prop:cont_inclusion}, \(i_x\) is continuous. By Proposition \ref{prop:cont_inclusion}, we deduce that the map 

\[
\{x\} \times Y \xrightarrow{f \circ i_x} \{0,1\}
\]

is continuous.

By Lemma \ref{lem:prod_conn}, \(\{x\} \times Y\) is homeomorphic to \(Y\). Thus, since \(Y\) is connected, Corollary \ref{cor:conn_sub} implies that \(\{x\} \times Y\) is connected.

We now have that \(f \circ i_x\) is continuous and \(\{x\} \times Y\) is connected. We deduce by Proposition \ref{prop:cont_comp} that \(f \circ i_x\) cannot be surjective. Since \(\{0,1\}\) has only two elements, we conclude that \(f \circ i_x\) is constant, and in particular that \(f(x, y) = f(x, y')\).

Let \(X \times \{y'\}\) be equipped with the subspace topology with respect to \((X \times Y, \mathcal{O}_{X \times Y})\). Let \(i_{y'}\) denote the inclusion 

\[
X \times \{y'\} \hookrightarrow X \times Y.
\]

By Proposition \ref{prop:cont_inclusion}, \(i_{y'}\) is continuous. Hence, by Proposition \ref{prop:cont_inclusion}, the map 

\[
X \times \{y'\} \xrightarrow{f \circ i_{y'}} \{0,1\}
\]

is continuous.

By Lemma \ref{lem:prod_conn}, \(X \times \{y'\}\) is homeomorphic to \(X\). Since \(X\) is connected, we deduce by Corollary \ref{cor:conn_sub} that \(X \times \{y'\}\) is connected.

We now have that \(f \circ i_{y'}\) is continuous and that \(X \times \{y'\}\) is connected. We deduce by Proposition \ref{prop:cont_comp} that \(f \circ i_{y'}\) cannot be surjective. Since \(\{0,1\}\) has only two elements, we conclude that \(f \circ i_{y'}\) is constant, and in particular that \(f(x, y') = f(x', y')\).

Putting everything together, we have that \(f(x, y) = f(x', y')\). Since \(x, x' \in X\) and \(y, y' \in Y\) were arbitrary, we conclude that \(f\) is constant. In particular, \(f\) is not surjective.

We have proven that there does not exist a continuous surjection 

\[
X \times Y \longrightarrow \{0,1\}.
\]

By Proposition \ref{prop:cont_surj}, we conclude that \((X \times Y, \mathcal{O}_{X \times Y})\) is connected.
Conversely, suppose that \((X \times Y, \mathcal{O}_{X \times Y})\) is connected. By Proposition 3.2, we have that the map

\[ X \times Y \xrightarrow{p_X} X \]

is continuous. We have that \(p_X(X \times Y) = X\). By Proposition 7.1, we conclude that \(X\) is connected.

Similarly, by Proposition 3.2, the map

\[ X \times Y \xrightarrow{p_Y} X \]

is continuous. We have that \(p_Y(X \times Y) = Y\). By Proposition 7.1, we conclude that \(Y\) is connected.

**Examples 7.14.**

(1) By Proposition 6.9, we have that \((\mathbb{R}, \mathcal{O}_\mathbb{R})\) is connected. By Proposition 7.13, we deduce \((\mathbb{R}^2, \mathcal{O}_{\mathbb{R} \times \mathbb{R}})\) is connected. By Proposition 7.13 and induction, we moreover have that \((\mathbb{R}^n, \mathcal{O}_{\mathbb{R} \times \ldots \times \mathbb{R}})\) is connected for any \(n \in \mathbb{N}\).

(2) By Proposition 7.9, \(I\) is connected. By Proposition 7.13, we deduce that \((I^2, \mathcal{O}_{I \times I})\) is connected. By Proposition 7.13 and induction, we moreover have that \((I^n, \mathcal{O}_{I \times \ldots \times I})\) is connected for any \(n \in \mathbb{N}\).

**Proposition 7.15.** Let \((X, \mathcal{O}_X)\) be a connected topological space, and let \(\sim\) be an equivalence relation on \(X\). Then \((X/\sim, \mathcal{O}_{X/\sim})\) is connected.

**Proof.** Let

\[ X \xrightarrow{\pi} X/\sim \]

denote the quotient map. By Proposition 3.7, we have that \(\pi\) is continuous. Moreover \(\pi\) is surjective, namely \(\pi(X) = X/\sim\). By Proposition 7.1, we deduce that \(X/\sim\) is connected.

**Example 7.16.** All the topological spaces of Examples 3.9 (1) – (5) are connected. Indeed, by Examples 7.14 (2) we have that \(I\) and \(I^2\) are connected. Thus, by Proposition 7.15, a quotient of \(I\) or \(I^2\) is connected.

On Exercise Sheet 4 we will prove that \(D^2 \cong I^2\). Since \(I^2\) is connected by Examples 7.14 (2), we deduce by Corollary 7.3 that \(D^2\) is connected. By Proposition 7.15, we conclude that \(S^2\), constructed as a quotient of \(D^2\) as in Examples 3.9 (6), is connected.