Generell Topologi

Richard Williamson

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7.1 Characterisation of connected subspaces of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$

Proposition 7.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let

$$X \xrightarrow{f} Y$$

be a continuous map. Suppose that $(X, \mathcal{O}_X)$ is connected. Let $f(X)$ be equipped with the subspace topology $\mathcal{O}_{f(X)}$ with respect to $(Y, \mathcal{O}_Y)$. Then $\big(f(X), \mathcal{O}_{f(X)}\big)$ is connected.

Proof. Suppose that $f(X) = U_0 \sqcup U_1$, and that U_0 and U_1 are open in $f(X)$. By definition, $U_0 = f(X) \cap Y_0$ for an open subset Y_0 of Y, and $U_1 = f(X) \cap Y_1$ for an open subset Y_1 of Y. We make the following observations.

(1) Since f is continuous, $f^{-1}(Y_0)$ is open in X. We have that

$$\begin{aligned} f^{-1}(U_0) &= f^{-1}\big(f(X) \cap Y_0\big) \\ &= f^{-1}\big(f(X)\big) \cap f^{-1}(Y_0) \\ &= X \cap f^{-1}(Y_0) \\ &= f^{-1}(Y_0). \end{aligned}$$

Thus $f^{-1}(U_0)$ is open in X.

(2) By an analogous argument, $f^{-1}(U_1)$ is open in X.

(3) We have that $f^{-1}(U_0) \cap f^{-1}(U_1) = f^{-1}(U_0 \cap U_1)$. Since $U_0 \cap U_1 = \emptyset$, we deduce that $f^{-1}(U_0 \cap U_1) = \emptyset$. Thus $f^{-1}(U_0) \cap f^{-1}(U_1) = \emptyset$.

(4) We have that $f^{-1}(U_0) \cup f^{-1}(U_1) = f^{-1}(U_0 \cup U_1)$. Since $U_0 \cup U_1 = f(X)$, and since $f^{-1}\big(f(X)\big) = X$, we deduce that $f^{-1}(U_0) \cup f^{-1}(U_1) = X$.

By (3) and (4), we have that $X = f^{-1}(U_0) \sqcup f^{-1}(U_1)$. Thus, by (1), (2), and the fact that $(X, \mathcal{O}_X)$ is connected, we must have that either $f^{-1}(U_0) = X$ or that $f^{-1}(U_0) = \emptyset$. Hence either $U_0 = f(X)$ or $U_0 = \emptyset$. □

Remark 7.2. We will sometimes refer to the conclusion of Proposition 7.1 as: 'the continuous image of a connected topological space is connected'.

Corollary 7.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let

$$X \xrightarrow{f} Y$$

be a homeomorphism. If $(X, \mathcal{O}_X)$ is connected, then $(Y, \mathcal{O}_Y)$ is connected.

Proof. By Proposition 3.15 we have that f is surjective, or in other words we have that $f(X) = Y$. It follows immediately from Proposition 7.1 that Y is connected. □

Proposition 7.4. Let $(X, \mathcal{O}_X)$ be a connected topological space, and let A and A' be subsets of X. Let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$, and let A' be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Suppose that $(A', \mathcal{O}_{A'})$ is connected, and that $A' \subset A \subset \overline{A'}$. Then $(A, \mathcal{O}_A)$ is connected.

Proof. Suppose that $A = U_0 \sqcup U_1$, where both U_0 and U_1 belong to $\mathcal{O}_A$. By definition of $\mathcal{O}_A$, we have that $U_0 = A \cap U_0'$ for an open subset U_0' of X, and that $U_1 = A \cap U_1'$ for an open subset U_1' of X.

We have that

$$\begin{aligned}(A' \cap U_0) \cup (A' \cap U_1) &= A' \cap (U_0 \cup U_1) \\ &= A' \cap A \\ &= A',\end{aligned}$$

where for the final equality we appeal to the fact that $A' \subset A$.

Moreover, we have that

$$\begin{aligned}(A' \cap U_0) \cap (A' \cap U_1) &= A' \cap (U_0 \cap U_1) \\ &= A' \cap \emptyset \\ &= \emptyset.\end{aligned}$$

Putting the last two observations together, we have that $A' = (A' \cap U_0) \sqcup (A' \cap U_1)$. Moreover, we have that

$$\begin{aligned}A' \cap U_0 &= A' \cap (A \cap U_0') \\ &= (A' \cap A) \cap U_0' \\ &= A' \cap U_0',\end{aligned}$$

and that

$$\begin{aligned}A' \cap U_1 &= A' \cap (A \cap U_1') \\ &= (A' \cap A) \cap U_1' \\ &= A' \cap U_1'.\end{aligned}$$

For the last equality in each case we appeal again to the fact that $A' \subset A$. By definition of $\mathcal{O}_{A'}$, we thus have that $A' \cap U_0 = A' \cap U_0'$ and $A' \cap U_1 = A' \cap U_1'$ are open in A'. Since $(A', \mathcal{O}_{A'})$ is connected, we deduce that either $A' \cap U_0 = \emptyset$ or $A' \cap U_1 = \emptyset$.

Suppose that $A' \cap U_0 = \emptyset$. Since $A' \cap U_0 = A' \cap U_0'$, we then have that $A' \cap U_0' = \emptyset$. Thus $A' \subset X \setminus U_0'$. Since U_0' is open in X, we have that $X \setminus U_0'$ is closed in X. By Proposition 5.9, we deduce that $\overline{A'} \subset X \setminus U_0'$.

By assumption we have that $A \subset \overline{A'}$. Thus $U_0 = A \cap U_0' \subset \overline{A'} \cap X_0 = \emptyset$, so that $U_0 = \emptyset$.

An entirely analogous argument gives that if $A' \cap U_1 = \emptyset$, then $U_1 = \emptyset$. Putting everything together, we have proven that if $A = U_0 \sqcup U_1$, where both U_0 and U_1 are open in $(A, \mathcal{O}_A)$, then either $U_0 = \emptyset$ or $U_1 = \emptyset$. Thus $(A, \mathcal{O}_A)$ is connected.

□

Remark 7.5. We will sometimes refer to Proposition 7.4 as the 'sandwich proposition'.

Corollary 7.6. *Let $(X, \mathcal{O}_X)$ be a connected topological space, and let A be a subset of X. Let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$, and let $\overline{A}$ be equipped with the subspace topology $\mathcal{O}_{\overline{A}}$ with respect to $(X, \mathcal{O}_X)$. Suppose that $(A, \mathcal{O}_A)$ is connected. Then $(\overline{A}, \mathcal{O}_{\overline{A}})$ is connected.*

Proof. Follows immediately from Proposition 7.4, taking both A and A' to be A. $\square$

Lemma 7.7. *Let $(X, \mathcal{O})$ be a topological space. If X is empty or consists of exactly one element, then $(X, \mathcal{O})$ is connected.*

Proof. If X is empty or consists of exactly one element, the only ways to express X as a disjoint union of subsets are $X = \emptyset \sqcup X$ and $X = X \sqcup \emptyset$. For $X = \emptyset \sqcup X$, condition (2) of Definition 6.2 is not satisfied. For $X = X \sqcup \emptyset$, condition (3) of Definition 6.2 is not satisfied. $\square$

Lemma 7.8. *Let X be a non-empty subset of $\mathbb{R}$. Then X is an open interval, a closed interval, or a half open interval if and only if for every $x, x' \in X$ and $y \in \mathbb{R}$ with $x < y < x'$ we have that $y \in X$.*

Proof. Suppose that $X = [a, b]$, for $a, b \in \mathbb{R}$ with $a \leq b$. If $x, x' \in X$, then by definition of $[a, b]$ we have that $a \leq x \leq b$ and $a \leq x' \leq b$. Suppose that $x < x'$ and that $y \in \mathbb{R}$ has the property that $x < y < x'$. Then we have that $a \leq x < y < x'' \leq b$, and in particular $a \leq y \leq b$. Thus, by definition of $[a, b]$, we have that $y \in [a, b]$.

If X is an open interval or a half open interval, an entirely analogous argument proves that if $x, x' \in X$ and $y \in \mathbb{R}$ have the property that $x < y < x'$, then $y \in X$.

Conversely, suppose that for every $x, x' \in X$ and $y \in \mathbb{R}$ with $x < y < x'$ we have that $y \in X$. Let $a = \inf X$ and let $b = \sup X$. As in Lecture 1, if X is not bounded below we adopt the convention that $\inf X = -\infty$, and if X is not bounded above we adopt the convention that $\sup X = \infty$.

Suppose that $y \in X$ has the property that $a < y < b$. Since $y > a$ there is an $x \in X$ with $x < y$, since otherwise y would be a lower bound of X, contradicting the fact that a is by definition the greatest lower bound of X. Since $y < b$ there is an $x' \in X$ with $y < x'$, since otherwise y would be an upper bound of X, contradicting the fact that b is by definition the least upper bound of X. We have shown that $x < y < x'$, with $x, x' \in X$. By assumption, we deduce that $y \in X$.

We have now shown that for all $y \in X$ such that $a < y < b$, we have that $y \in X$. To complete the proof, there are four cases to consider.

(1) If $a, b \in X$, we have by definition of $[a, b]$ that $X = [a, b]$.

(2) If $a \in X$ and $b \notin X$, we have by definition of $[a, b)$ that $X = [a, b)$.

(3) If $a \notin X$ and $b \in X$, we have by definition of $(a, b]$ that $X = (a, b]$.

(4) If $a \notin X$ and $b \notin X$, we have by definition of (a, b) that $X = (a, b)$.

□

Proposition 7.9. Let X be a subset of $\mathbb{R}$, and let X be equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then X is connected if and only if X is an open interval, a closed interval, a half open interval, or $\emptyset$.

Proof. If X is not an open interval, a closed interval, a half open interval, or $\emptyset$, then by Lemma 7.8 there are $x, x' \in X$ and $y \in \mathbb{R} \setminus X$ with $x < y < x'$. Let $U_0 := X \cap (-\infty, y)$, and let $U_1 := X \cap (y, \infty)$. By definition of $\mathcal{O}_X$, both U_0 and U_1 are open in X. Moreover, $X = X_0 \sqcup X_1$. Thus X is not connected. This completes one direction of the proof.

Conversely, let us prove that if X is an open interval, a closed interval, a half open interval, or $\emptyset$, then X is connected. By Lemma 7.7, if $X = \emptyset$ or if X consists of exactly one element, then X is connected.

Suppose instead that X is an open interval. By Examples 4.7 (2) we have that X is homeomorphic to $\mathbb{R}$. By Proposition 6.9 and Corollary 7.3, we deduce that X is connected.

Suppose now that X is either a closed interval or a half open interval, with more than one element. Let us denote the open interval $X \setminus \partial_X \mathbb{R}$ by X'. We have by earlier in the proof that X' is connected. Moreover we have that $X' \subset X \subset \overline{X'}$. By Proposition 7.4, we deduce that X is connected. □

Corollary 7.10. Let $(X, \mathcal{O}_X)$ be a connected topological space, and let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Let x and x' be elements of X with $f(x) \leq f(x')$. Then for every $y \in \mathbb{R}$ such that $f(x) \leq y \leq f(x')$, there is an $x'' \in X$ such that $f(x'') = y$.

Proof. By Proposition 7.1, $f(X)$ is connected. By Proposition 7.9 we deduce that $f(X)$ is an open interval, a closed interval, or a half open interval. By Lemma 7.8 we conclude that every $y \in \mathbb{R}$ such that $f(x) \leq y \leq f(x'')$ belongs to $f(X)$. □

Remark 7.11. Taking $(X, \mathcal{O}_X)$ to be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, Corollary 7.10 is the 'intermediate value theorem' that you met in real analysis/calculus!

7.2 Examples of connected topological spaces

Lemma 7.12. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. For every x in X, let $\{x\} \times Y$ be equipped with the subspace topology $\mathcal{O}_{\{x\} \times Y}$ with respect to $(X \times Y, \mathcal{O}_{X \times Y})$. Then $(\{x\} \times Y, \mathcal{O}_{\{x\} \times Y})$ is homeomorphic to $(Y, \mathcal{O}_Y)$.

For every y in Y, let $X \times \{y\}$ be equipped with the subspace topology $\mathcal{O}_{X \times \{y\}}$ with respect to $(X \times Y, \mathcal{O}_{X \times Y})$. Then $(X \times \{y\}, \mathcal{O}_{X \times \{y\}})$ is homeomorphic to $(X, \mathcal{O}_X)$.

Proof. Exercise. □

Proposition 7.13. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then $(X \times Y, \mathcal{O}_{X \times Y})$ is connected if and only if both $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are connected.

Proof. Suppose that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are connected. Let $X \times Y$ be equipped with the product topology $\mathcal{O}_{X \times Y}$, and let $\{0, 1\}$ be equipped with the discrete topology. Let

$$X \times Y \xrightarrow{f} \{0, 1\},$$

be a continuous map.

Let $x, x' \in X$, and let $y, y' \in Y$. Let $\{x\} \times Y$ with the subspace topology with respect to $(X \times Y, \mathcal{O}_{X \times Y})$, and let i_x denote the inclusion

$$\{x\} \times Y \hookrightarrow X \times Y.$$

By Proposition 2.15, i_x is continuous. By Proposition 2.16, we deduce that the map

$$\{x\} \times Y \xrightarrow{f \circ i_x} \{0, 1\}$$

is continuous.

By Lemma 7.12, $\{x\} \times Y$ is homeomorphic to Y. Thus, since Y is connected, Corollary 7.3 implies that $\{x\} \times Y$ is connected.

We now have that $f \circ i_x$ is continuous and $\{x\} \times Y$ is connected. We deduce by Proposition 6.5 that $f \circ i_x$ cannot be surjective. Since $\{0, 1\}$ has only two elements, we conclude that $f \circ i_x$ is constant, and in particular that $f(x, y) = f(x, y')$.

Let $X \times \{y'\}$ be equipped with the subspace topology with respect to $(X \times Y, \mathcal{O}_{X \times Y})$. Let $i_{y'}$ denote the inclusion

$$X \times \{y'\} \hookrightarrow X \times Y.$$

By Proposition 2.15, $i_{y'}$ is continuous. Hence, by Proposition 2.16, the map

$$X \times \{y'\} \xrightarrow{f \circ i_y} \{0, 1\}$$

is continuous.

By Lemma 7.12, $X \times \{y'\}$ is homeomorphic to X. Since X is connected, we deduce by Corollary 7.3 that $X \times \{y'\}$ is connected.

We now have that $f \circ i_{y'}$ is continuous and that $X \times \{y'\}$ is connected. We deduce by Proposition 6.5 that $f \circ i_{y'}$ cannot be surjective. Since $\{0, 1\}$ has only two elements, we conclude that $f \circ i_{y'}$ is constant, and in particular that $f(x, y') = f(x', y')$.

Putting everything together, we have that $f(x, y) = f(x', y')$. Since $x, x' \in X$ and $y, y' \in Y$ were arbitrary, we conclude that f is constant. In particular, f is not surjective.

We have proven that there does not exist a continuous surjection

$$X \times Y \longrightarrow \{0, 1\}.$$

By Proposition 6.5, we conclude that $(X \times Y, \mathcal{O}_{X \times Y})$ is connected.

Conversely, suppose that $(X \times Y, \mathcal{O}_{X \times Y})$ is connected. By Proposition 3.2, we have that the map

$$X \times Y \xrightarrow{p_X} X$$

is continuous. We have that $p_X(X \times Y) = X$. By Proposition 7.1, we conclude that X is connected.

Similarly, by Proposition 3.2, the map

$$X \times Y \xrightarrow{p_Y} X$$

is continuous. We have that $p_Y(X \times Y) = Y$. By Proposition 7.1, we conclude that Y is connected.

$\square$

Examples 7.14.

(1) By Proposition 6.9, we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected. By Proposition 7.13, we deduce $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R} \times \mathbb{R}})$ is connected. By Proposition 7.13 and induction, we moreover have that $(\mathbb{R}^n, \mathcal{O}_{\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}})$ is connected for any $n \in \mathbb{N}$.

(2) By Proposition 7.9, I is connected. By Proposition 7.13, we deduce that $(I^2, \mathcal{O}_{I \times I})$ is connected. By Proposition 7.13 and induction, we moreover have that

$$(I^n, \mathcal{O}_{\underbrace{I \times \cdots \times I}})$$

is connected for any $n \in \mathbb{N}$.

Proposition 7.15. Let $(X, \mathcal{O}_X)$ be a connected topological space, and let $\sim$ be an equivalence relation on X. Then $(X/\sim, \mathcal{O}_{X/\sim})$ is connected.

Proof. Let

$$X \xrightarrow{\pi} X/\sim$$

denote the quotient map. By Proposition 3.7, we have that π is continuous. Moreover π is surjective, namely $\pi(X) = X/\sim$. By Proposition 7.1, we deduce that $X/\sim$ is connected. $\square$

Example 7.16. All the topological spaces of Examples 3.9 (1) – (5) are connected. Indeed, by Examples 7.14 (2) we have that I and I^2 are connected. Thus, by Proposition 7.15, a quotient of I or I^2 is connected.

On Exercise Sheet 4 we will prove that $D^2 \cong I^2$. Since I^2 is connected by Examples 7.14 (2), we deduce by Corollary 7.3 that D^2 is connected. By Proposition 7.15, we conclude that S^2, constructed as a quotient of D^2 as in Examples 3.9 (6), is connected.