Generell Topologi

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8.1 Using connectedness to distinguish between topological spaces — I

Proposition 8.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that

$$X \xrightarrow{f} Y$$

defines a homeomorphism between $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$.

Let A be a subset of X, equipped with the subspace topology $\mathcal{O}_A$ with respect to X. Let $f(A)$ be equipped with the subspace topology $\mathcal{O}_{f(A)}$ with respect to Y. Let $X \setminus A$ be equipped with the subspace topology $\mathcal{O}_{X \setminus A}$ with respect to X, and let $Y \setminus f(A)$ be equipped with the subspace topology with respect to Y.

Then $(A, \mathcal{O}_A)$ is homeomorphic to $\big(f(A), \mathcal{O}_{f(A)}\big)$, and $(X \setminus A, \mathcal{O}_{X \setminus A})$ is homeomorphic to $\big(Y \setminus f(A), \mathcal{O}_{Y \setminus f(A)}\big)$.

Proof. See Exercise Sheet 4. □

Corollary 8.2. Let $[a, b]$, (a, b), $[a, b)$, and $(a, b]$, for $a, b \in \mathbb{R}$, be equipped with their respective subspace topologies with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then:

(1) $[a, b]$ is not homeomorphic to (a, b), $[a, b)$, or $(a, b]$.

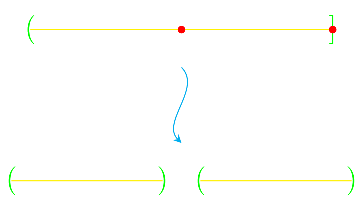
(2) (a, b) is not homeormorphic to $(a, b]$ or $[a, b)$.

(3) $[a, b)$ is homeomorphic to $(a, b]$.

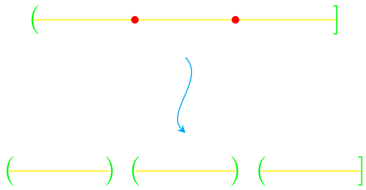
Proof. We have that $[a, b] \setminus \{a, b\} = (a, b)$. By Proposition 7.9, (a, b) equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.

However, by removing any two points from (a, b), $[a, b)$, or $(a, b]$, and equipping the resulting set with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, we will obtain a topological space which is not connected. By Proposition 8.1 and Corollary 7.3, we deduce that $[a, b]$ is not homeomorphic to any of (a, b), $[a, b)$, or $(a, b]$.

An example of removing two points from $(a, b]$ is depicted below. In this case, we obtain a disjoint union of two open intervals.

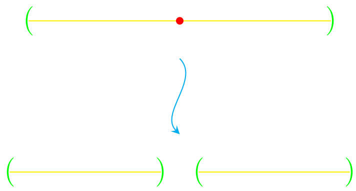


A second example of removing two points from $(a, b]$ is depicted below. In this case, we obtain a disjoint union of two open intervals and a half open interval.



We have that $[a,b) \setminus \{a\} = (a,b)$. Again, by Proposition 7.9, (a,b) equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.

However, removing any one point from (a,b) and equipping the resulting set with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, we will obtain a topological space which is not connected. By Proposition 8.1 and Corollary 7.3, we deduce that $[a,b)$ is not homeomorphic to (a,b).



We also have that $(a,b] \setminus \{b\} = (a,b)$. By the same argument as above, we deduce that $(a,b]$ is not homeomorphic to (a,b).

We will prove that $[a,b) \cong (a,b]$ on Exercise Sheet 4. □

8.2 Connected components

Terminology 8.3. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Then A is a *connected subset* of X if $(A, \mathcal{O}_A)$ is a connected topological space.

Definition 8.4. Let $(X, \mathcal{O}_X)$ be a topological space, and let $x \in X$. The *connected component* of x in $(X, \mathcal{O}_X)$ is the union of all connected subsets A of X such that $x \in A$.

Proposition 8.5. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\{A_j\}_{j \in J}$ be a set of connected subsets of X such that $\bigcup_{j \in J} A_j = X$. Suppose that $\bigcap_{j \in J} A_j \neq \emptyset$. Then $(X, \mathcal{O}_X)$ is connected.

Proof. Let $\{0,1\}$ be equipped with the discrete topology, and let

$$X \xrightarrow{f} \{0,1\}$$

be a continuous map. Given $j \in J$ let

$$A_j \xhookrightarrow{i_j} X$$

denote the inclusion map. By Proposition 2.15 we have that i_j is continuous. We deduce by Proposition 2.16 that

$$A_j \xrightarrow{f \circ i_j} \{0,1\}$$

is continuous.

By assumption, A_j is a connected subset of X. We deduce by Proposition 6.5 that $f \circ i_j$ is constant. This holds for all $j \in J$. Since $\bigcap_{j \in J} A_j \neq \emptyset$, we deduce that f is constant. □

Corollary 8.6. Let $(X, \mathcal{O}_X)$ be a topological space, and let $x \in X$. The connected component of x in X is a connected subset of X.

Proof. Follows immediately from Proposition 8.5. □

Remark 8.7. Thus the connected component of x in a topological space $(X, \mathcal{O}_X)$ is the largest connected subset of X which contains x.

Terminology 8.8. Let X be a set. A *partition* of X is a set $\{X_j\}_{j \in J}$ of subsets of X such that $X = \bigsqcup_{j \in J} X_j$.

Proposition 8.9. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\{A_x\}_{x \in X}$ denote the set of connected components of X. Then $\{A_x\}_{x \in X}$ defines a partition of X.

Proof. Since $x \in A_x$, we have that $\bigcup_{x \in X} A_x = X$. Suppose that $x, x' \in X$ and that $A_x \cap A_{x'} \neq \emptyset$. We must prove that $A_x = A_{x'}$. By Proposition 8.5 we then have that $A_x \cup A_{x'}$ is connected. By definition of A_x, we deduce that $A_x \cup A_{x'} \subset A_x$. Since we also have that $A_x \subset A_x \cup A_{x'}$, we conclude that $A_x \cup A_{x'} = A_x$ as required. □

Examples 8.10.

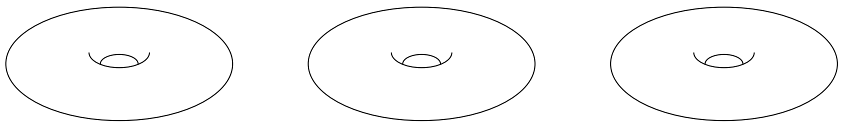
(1) A connected topological space $(X, \mathcal{O})$ has exactly one connected component, namely X itself.

(2) At the other extreme, let X be a set and let $\mathcal{O}^{disc}$ denote the discrete topology on X. The connected components of $(X, \mathcal{O}^{disc})$ are the singleton sets $\{x\}$ for $x \in X$.

Let us prove this. Suppose that $A \subset X$ has more than one element. Let $\mathcal{O}_A$ denote the subspace topology on A with respect to $(X, \mathcal{O}^{disc})$. For any $a \in A$, we have $A = \{a\} \sqcup X \setminus \{a\}$, and both $\{a\}$ and $A \setminus \{a\} = A \cap (X \setminus \{a\})$ are open in $(A, \mathcal{O}_A)$. Thus A is not a connected subset of $(X, \mathcal{O}_X)$.

(3) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be connected topological spaces. Then $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$ has two connected components, namely X and Y. For instance, by Example 7.16 we have that $(T^2, \mathcal{O}_{T^2})$ is connected, and hence that $(T^2 \sqcup T^2, \mathcal{O}_{T^2 \sqcup T^2})$ consists of two connected components.

By induction, we may similarly cook up examples of topological spaces with n connected components for any finite n.



(4) Let the set of rational numbers $\mathbb{Q}$ be equipped with its subspace topology $\mathcal{O}_{\mathbb{Q}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. The connected components of $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ are exactly the singleton sets $\{q\}$ for $q \in \mathbb{Q}$.

Let us prove this. For any $A \subset \mathbb{Q}$ which contains at least two distinct rational numbers q and q', there is an irrational number r with $q < r < q'$. Then

$$A = \Big(A \cap (-\infty, r)\Big) \sqcup \Big(A \cap (r, \infty)\Big).$$

Since both $A \cap (-\infty, r)$ and $A \cap (r, \infty)$ are open in $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$, we deduce that A is not a connected subset of $\mathbb{Q}$.

Terminology 8.11. A topological space $(X, \mathcal{O})$ is *totally disconnected* if the connected components of $(X, \mathcal{O})$ are the singleton sets $\{x\}$ for $x \in X$.

Remark 8.12. By (2) of Examples 8.10, a set equipped with its discrete topology is totally disconnected. However, as (4) of Examples 8.10 demonstrates there are totally disconnected topological spaces $(X, \mathcal{O})$ for which $\mathcal{O}$ is not the discrete topology.

Lemma 8.13. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let

$$X \xrightarrow{f} Y$$

be a continuous map. Let $x \in X$. Let A_x denote the connected component of x in X, and let $B_{f(x)}$ denote the connected component of $f(x)$ in Y. Then $f(A_x) \subset B_{f(x)}$.

Proof. By Corollary 8.6, A_x is a connected subset of $(X, \mathcal{O}_X)$. By Proposition 7.1 we deduce that $f(A)$ is a connected subset of Y. Since $x \in A$, we have that $f(x) \in f(A)$. By definition of $B_{f(x)}$, we conclude that $f(A) \subset B_{f(x)}$. □

Proposition 8.14. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. If $X \cong Y$ then there exists a bijection between the set of connected components of X and the set of connected components of Y.

Proof. Given $x \in X$, let A_x denote the connected component of x in $(X, \mathcal{O}_X)$. Given $y \in Y$, let B_y denote the connected component of y in $(Y, \mathcal{O}_Y)$. Let Γ_X denote the set of connected components of X, and let Γ_Y denote the set of connected components of Y.

Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. By definition of f as a homeomorphism, there is a continuous map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$. For any $x \in X$ we have by Lemma 8.13 that $g(B_{f(x)}) \subset A_{g \circ f(x)} = A_x$. Hence

$$\begin{aligned} B_{f(x)} &= f\big(g(B_{f(x)})\big) \\ &\subset f(A_x). \end{aligned}$$

Moreover, by Lemma 8.13 we have that $f(A_x) \subset B(f_x)$. We deduce that $f(A_x) = B_{f(x)}$. Thus

$$A_x \mapsto f(A_x)$$

defines a map

$$\Gamma_X \xrightarrow{\eta} \Gamma_Y.$$

By an entirely analogous argument we have that $g(B_y) = A_{g(y)}$ for any $y \in Y$. Thus

$$B_y \mapsto g(B_y)$$

defines a map

$$\Gamma_Y \xrightarrow{\zeta} \Gamma_X.$$

We have that $\zeta \circ \eta = id_{\Gamma_X}$ since for any $x \in X$ we have that

$$\begin{aligned} g\big(f(A_x)\big) &= g(B_{f(x)}) \\ &= A_{g \circ f(x)} \\ &= A_x. \end{aligned}$$

Moreover we have that $\eta \circ \zeta = id_{\Gamma_Y}$ since for any $y \in Y$ we have that

$$\begin{aligned} f\big(g(B_y)\big) &= f(A_{g(y)}) \\ &= B_{f \circ g(y)} \\ &= B_y. \end{aligned}$$

□

Observation 8.15. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and suppose that $X \cong Y$. Then every connected component of X, equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$, is homeomorphic to a connected component of Y, equipped with the subspace topology with respect to $(Y, \mathcal{O}_Y)$.

This follows from observations made during the proof of Proposition 8.14. See the Exercise Sheet.

8.3 Using connectedness to distinguish between topological spaces — II

Examples 8.16.

(1) Let us regard the letter T as a subset of $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. For example, we can let

$$\mathsf{T} = \{(x,1) \in \mathbb{R}^2 \mid x \in [-1,1]\} \cup \{(0,y) \in \mathbb{R}^2 \mid y \in [0,1]\}.$$

We equip T with the subspace topology $\mathcal{O}_{\mathsf{T}}$.

Let us also regard the letter I as a subset $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. For example, we can let

$$\mathsf{I} = \{(0,y) \in \mathbb{R}^2 \mid y \in [0,1]\}.$$

We equip I with the subspace topology $\mathcal{O}_{\mathsf{I}}$.

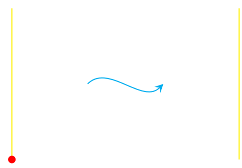
By Examples 4.10 (5) we have that $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ is homeomorphic to the unit interval $(I, \mathcal{O}_I)$.

Let us prove that $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ is not homeomorphic to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$. Let x be the point $(0,1)$ of T.

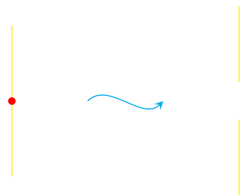
Then $\mathsf{T} \setminus \{x\}$ equipped with the subspace topology with respect to $(\mathsf{T}, \mathcal{O}_\mathsf{T})$ has three connected components.

However, the topological space obtained by removing a single point of I and equipping the resulting set with the subspace topology with respect to $(\mathsf{I}, \mathcal{O}_\mathsf{I})$ has either one connected component or two connected components.

We obtain one connected component if we remove one of the two end points of I. Removing the lower end point $(0,0)$ of I is depicted below.



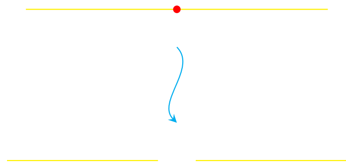
We obtain two connected components if we remove any point of I which is not an end point.



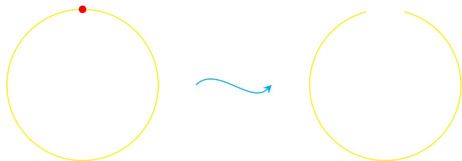
In particular, it is not possible to remove a single point from $(\mathsf{I}, \mathcal{O}_\mathsf{I})$ and obtain a topological space with three connected components.

We deduce by Proposition 8.14 that $\mathsf{T} \setminus \{x\}$ is not homeomorphic to $\mathsf{I} \setminus \{y\}$ for any $y \in \mathsf{I}$. We conclude that T is not homeomorphic to I by Proposition 8.1

(2) The circle S^1 is not homeomorphic to I. Indeed, equipping $I \setminus \{t\}$ for $0 < t < 1$ with the subspace topology with respect to $(I, \mathcal{O}_I)$ gives a topological space with two connected components.



Removing any point from S^1 and equipping the resulting set with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$ gives a topological space with exactly one connected component.



In particular, it is not possible to remove a single point from $(I, \mathcal{O}_I)$ and obtain a topological space with two connected components.

We deduce by Proposition 8.14 that $\mathsf{I} \setminus \{t\}$ is not homeomorphic to $S^1 \setminus \{x\}$ for any $x \in S^1$. We conclude that S^1 is not homeomorphic to I by Proposition 8.1.

(3) Let us regard the letters $\mathsf{Å}$ and A as subsets of $\mathbb{R}^2$, equipped with their respective subspace topologies with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. By Proposition 8.14 we have that $\mathsf{Å}$ is not homeomorphic to A, since $\mathsf{Å}$ has two connected components, whilst A has one.