Generell Topologi

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9.1 Using connectedness to distinguish between topological spaces —- II, continued

Examples 9.1.

(1) Let us regard the letter K as a subset of $\mathbb{R}^2$, equipped with its subspace topology $\mathcal{O}_{\mathsf{K}}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ be as in Examples 8.16 (1). Let us prove that $(\mathsf{K}, \mathcal{O}_{\mathsf{K}})$ is not homeomorphic to $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$. Let x be the point of K indicated below.

Then $\mathsf{K} \setminus \{x\}$ equipped with the subspace topology with respect to $(\mathsf{K}, \mathcal{O}_{\mathsf{K}})$ has four connected components.

However, the topological space obtained by removing a point from $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ and equipping the resulting set with the subspace topology with respect to $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ has at most three connected components.

We deduce by Proposition 8.14 that $\mathsf{K} \setminus \{x\}$ is not homeomorphic to $\mathsf{T} \setminus \{y\}$ for any $y \in \mathsf{T}$. We conclude that $(\mathsf{K}, \mathcal{O}_{\mathsf{K}})$ is not homeomorphic to $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ by Proposition 8.1.

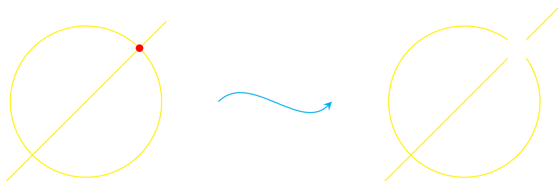
(2) Let us regard the letter $\mathsf{Ø}$ as a subset of $\mathbb{R}^2$, equipped with its subspace topology $\mathcal{O}_{\mathsf{Ø}}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let I and $\mathcal{O}_{\mathsf{I}}$ be as in Examples 8.16 (1). We cannot distinguish $(\mathsf{Ø}, \mathcal{O}_{\mathsf{Ø}})$ from $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ by remvoing one point from $\mathsf{Ø}$.

Let us see why. Removing one point from $\mathsf{Ø}$ and equipping the resulting set with the subspace topology with respect to $(\mathsf{Ø}, \mathcal{O}_{\mathsf{Ø}})$ we obtain a topological space with either one or two connected components.

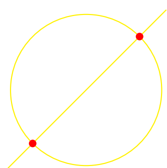
For instance, we obtain one connected component by removing a point as shown below.

We obtain two connected components by removing a point as shown below, for example.



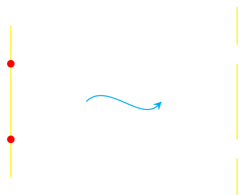
Since we may also obtain a topological space with either one or two connected components by removing a point from $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$, we cannot conclude that $(\mathsf{Ø}, \mathcal{O}_{\mathsf{Ø}})$ is not homeomorphic to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$.

However, let x and y be the two points of $\mathsf{Ø}$ indicated below.



Then $\varnothing \setminus \{x, y\}$ equipped with the subspace topology with respect to $(\varnothing, \mathcal{O}_{\varnothing})$ has five connected components.

Removing two points from I and equipping the resulting set with the subspace topology with respect to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ gives a topological space with at most three connected components.



We deduce by Proposition 8.14 that $\varnothing \setminus \{x, y\}$ is not homeomorphic to $\mathsf{I} \setminus \{x', y'\}$ for any $x', y' \in \mathsf{I}$. We conclude that $(\varnothing, \mathcal{O}_{\varnothing})$ is not homeomorphic to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ by Proposition 8.1.

Lemma 9.2. For any $n > 1$ and any $x \in \mathbb{R}^n$ we have that $\mathbb{R}^n \setminus \{x\}$ equipped with the subspace topology $\mathcal{O}_{\mathbb{R}^n \setminus \{x\}}$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is connected.

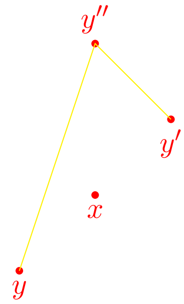
Proof. Let $y, y' \in \mathbb{R}^n \setminus \{x\}$. Since $n > 1$, there exists a line L through y and a line L' through y' such that $L \subset \mathbb{R}^n \setminus \{x\}$ $L' \subset \mathbb{R}^n \setminus \{x\}$, and $L \cap L' \neq \emptyset$. For instance, let y'' denote the point $x + (0, \ldots, 0, 1)$ of $\mathbb{R}^n$. We can take L to be

$$\{y + ty'' \mid t \in [0, 1]\}$$

equipped with the subspace topology $\mathcal{O}_L$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$, and take L' to be

$$\{y' + ty'' \mid t \in [0, 1]\}$$

equipped with the subspace topology with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.



Since $(L, \mathcal{O}_L)$ is homeomorphic to $(I, \mathcal{O}_I)$, and since $(I, \mathcal{O}_I)$ is connected by Proposition 7.9, we deduce by Corollary 7.3 that $(L, \mathcal{O}_L)$ is connected. By exactly the same argument, we also have that $(L', \mathcal{O}_{L'})$ is connected. We deduce by Proposition 8.5 that $L \cup L'$ is connected.

Thus y and y' belong to the same connected component of $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n \setminus \{x\}})$. Since y and y' were arbitrary, we deduce that this connected component is $\mathbb{R}^n \setminus \{x\}$ itself. We conclude by Corollary 8.6 that $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n \setminus \{x\}})$ is connected. □

Proposition 9.3. The topological space $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not homeomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ for any $n > 1$.

Proof. Let $n > 1$, and suppose that

$$\mathbb{R} \xrightarrow{f} \mathbb{R}^n$$

defines a homeomorphism between $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ and $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. Let $x \in \mathbb{R}$. By Lemma 9.2 we have that $(\mathbb{R}^n \setminus \{f(x)\}, \mathcal{O}_{\mathbb{R}^n \setminus \{f(x)\}})$ is connected.

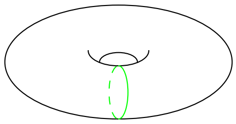
But $(\mathbb{R} \setminus \{x\}, \mathcal{O}_{\mathbb{R} \setminus \{x\}})$ is not connected.

By Proposition 8.1 and Corollary 7.3 we conclude that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not homeomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.

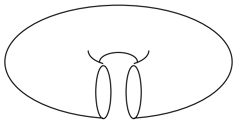
□

Question 9.4. In all our examples of distinguishing topological spaces by means of connectedness at least one of the topological spaces has been 'one dimensional', built out of lines and circles. Can we apply our technique to distinguish between higher dimensional topological spaces?

Remark 9.5. For example, let us try to formulate an argument to distinguish T^2 from S^2. Let X denote the circle on T^2 depicted below.



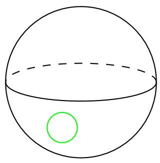
Then $T^2 \setminus X$ is as depicted below.



Equipped with the subspace topology with respect to $(T^2, \mathcal{O}_{T^2})$, it is homeomorphic to a cylinder.

In particular, $T^2 \setminus X$ is connected.

Let us now consider a subset Y of S^2 which, equipped with the subspace topology with respect to $(S^2, \mathcal{O}_{S^2})$, is homeomorphic to a circle.



Then $S^2 \setminus Y$ intuitively appears to have two connected components: the interior of Y and $S^2 \setminus Y$.

If our intuition is correct, by Proposition 8.14 we deduce that $T^2 \setminus X$ is not homeomorphic to $S^2 \setminus Y$ for any subset Y of S^2 which, equipped with the subspace topology with respect to $(S^2, \mathcal{O}_{S^2})$, is homeomorphic to $(S^1, \mathcal{O}_{S^1})$. We conclude that $(T^2, \mathcal{O}_{T^2})$ is not homeomorphic to $(S^2, \mathcal{O}_{S^2})$ by Proposition 8.1.

We have to be very careful! Homeomorphism is a very flexible notion, and Y could be very wild, much more complicated than the circle on S^2 drawn above.

We need to be sure that the requirement that we have a homeomorphism, as opposed to only a continuous surjection, excludes examples which are as wild as the Peano curve that we will meet on a later Exercise Sheet.

In other words, in order to carry out the argument of Remark 9.5 we have to rigorously prove that $S^2 \setminus Y$ has two connected components for any possible Y. This is subtle!

Answer 9.6. Nevertheless, it is true! This is known as the *Jordan curve theorem*, which we will prove towards the end of the course. Thus the argument of Remark 9.5 does after further work prove that $(T^2, \mathcal{O}_{T^2})$ is not homeomorphic to $(S^2, \mathcal{O}_{S^2})$.

Towards the end of the course we will also be able to prove by our technique, using a generalisation of the Jordan curve theorem to higher dimensions, that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ for any $m, n > 0$.

More sophisticated tools, which you will meet if you take Algebraic Topology I in the future, give a simple — after some foundational work! — proof of the Jordan curve theorem and its generalisation to higher dimensions.

Example 9.7. Whilst we do not yet have the tools to explore very wild phenomena such as the Peano curve that we will meet on a later Exercise Sheet, let us give an example of the kind of wildness that topology allows.

We will construct a pair of spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ such that there exists a continuous bijection from X to Y and a continuous bijection from Y to X, but such that X is not homeomorphic to Y.

Let us define

$$X = (0,1) \cup \{2\} \cup (3,4) \cup \{5\} \cup (6,7) \cup \{8\} \cdots .$$

In other words

$$X = \bigcup_{n \in \mathbb{Z},\ n \geq 0} (3n, 3n+1) \cup \{3n+2\}.$$

Here as usual $(3n, 3n+1)$ denotes the open interval from $3n$ to $3n+1$ in $\mathbb{R}$. We equip X with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let us define

$$Y = (0,1] \cup (3,4) \cup \{5\} \cup (6,7) \cup \{8\} \cup \cdots .$$

In other words,

$$Y = (0,1] \cup \Big(\bigcup_{n \in \mathbb{Z},\ n \geq 1} (3n, 3n+1) \cup \{3n+2\} \Big).$$

We equip Y with the subspace topology $\mathcal{O}_Y$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

We have that $(0,1]$ is a connected component of Y. By Corollary 8.2, $(0,1]$ is not homeomorphic to an open or closed interval. We deduce by Observation 8.15 that X is not homeomorphic to Y.

Let

$$X \xrightarrow{f} Y$$

be given by

$$f(x) = \begin{cases} x & \text{if } x \neq 2, \\ 1 & x = 2. \end{cases}$$

Let

$$Y \xrightarrow{\;\; g \;\;} X$$

be given by

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \in (0,1], \\ \frac{x-2}{2} & x \in (3,4), \\ x - 3 & \text{otherwise.} \end{cases}$$

Then both f and g are continuous, by Question 3 (f) of Exercise Sheet 3 and Question 1 of Exercise Sheet 5. Morever, f and g are bijective.

Do not be confused — the bijections f and g are not inverse to one another! If they were, we would have that X is homeomorphic to Y.

9.2 Locally connected topological spaces

Proposition 9.8. Let $(X, \mathcal{O})$ be a topological space. Given $x \in X$, let A_x denote the connected component of x in X. Then A_x is a closed subset of X.

Proof. By Corollary 8.6, A_x is a connected subset of X. We deduce by Corollary 7.6 that $\overline{A_x}$ is a connected subset of X. Hence by definition of A_x we have that $\overline{A_x} \subset A_x$. Since $A_x \subset \overline{A_x}$, we deduce that $\overline{A_x} = A_x$. We conclude by Proposition 5.7 that A_x is closed. $\square$

Remark 9.9. By Examples 8.10 (4) a connected component need not be open.

Definition 9.10. A topological space $(X, \mathcal{O})$ is *locally connected* if for every $x \in X$ and every neighbourhood U of x in $(X, \mathcal{O}_X)$ there is a neighbourhood U' of x in $(X, \mathcal{O}_X)$ such that U' is a connected subset of X and $U' \subset U$.

Proposition 9.11. A topological space $(X, \mathcal{O}_X)$ is locally connected if and only if it admits a basis consisting of connected subsets.

Proof. This is an immediate consequence of Question 3 (a) and Question 3 (b) on Exercise Sheet 2. $\square$

Lemma 9.12. Let $(X, \mathcal{O}_X)$ be a topological space, let $x \in X$, and let U be a neighbourhood of x in $(X, \mathcal{O}_X)$. Equip U with its subspace topology $\mathcal{O}_U$ with respect to $(X, \mathcal{O}_X)$. Let A be a connected subset of X with $A \subset U$. Then A is a connected subset of $(U, \mathcal{O}_U)$.

Proof. See Exercise Sheet 5. $\square$

Proposition 9.13. A topological space $(X, \mathcal{O})$ is locally connected if and only if for every open subset U of X the connected components of $(U, \mathcal{O}_U)$ are open subsets of X, where $\mathcal{O}_U$ denotes the subspace topology on U with respect to $(X, \mathcal{O}_X)$.

Proof. Suppose that $(X, \mathcal{O})$ is locally connected. Let U be an open subset of X, equipped with the subspace topology $\mathcal{O}_U$ with respect to $(X, \mathcal{O}_X)$. Let $x \in U$, and for any $y \in U$, let A_y denote the connected component of y in $(U, \mathcal{O}_U)$. We have that $A_y = A_x$ for all $y \in A_x$.

By Proposition 9.11, $(X, \mathcal{O})$ admits a basis $\{U_j\}_{j \in J}$ such that U_j is a connected subset of $(X, \mathcal{O}_X)$ for every $j \in J$. Thus by Question 3 (a) of Exercise Sheet 2, there is a $j \in J$ such that $y \in U_j$ and $U_j \subset U$. Since U_j is a connected subset of $(X, \mathcal{O}_X)$, we deduce by Lemma 9.12 that U_j is a connected subset of $(U, \mathcal{O}_U)$. Hence $U_j \subset A_y = A_x$. By Question 3 (b) of Exercise Sheet 2, we conclude that A_x is an open subset of X.

Conversely, suppose that for every open subset U of X the connected components of $(U, \mathcal{O}_U)$ are open subsets of X, where $\mathcal{O}_U$ denotes the subspace topology on U with respect to $(X, \mathcal{O}_X)$. For $x \in U$, let A_x^U denote the connected component of x in U.

Then $\{A_x^U\}_{U \in \mathcal{O},\ x \in U}$ defines a basis for $(X, \mathcal{O})$. Indeed, for any $U \in \mathcal{O}$ we have by Proposition 8.9 that $U = \bigcup_{x \in U} A_x^U$.

□

Examples 9.14.

(1) $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally connected. Indeed by definition of $\mathcal{O}_{\mathbb{R}}$ we have that

$$\{(a, b) \mid a, b \in \mathbb{R}\}$$

is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Proposition 7.9, (a, b) is a connected subset of $\mathbb{R}$ for every $a, b \in \mathbb{R}$.

(2) Products and quotients of locally connected topological spaces are locally connected. We will prove this on the Exercise Sheet 5. We deduce that all of the topological spaces of Examples 1.38 and Examples 3.9 are locally connected.

(3) The subset $X = (0, 1) \sqcup (2, 3)$ of $\mathbb{R}$ equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally connected, since $(0, 1)$ and $(2, 3)$ are connected by Proposition 7.9. However X is evidently not connected.

(4) By Examples 8.10 (4), $\mathbb{Q}$ equipped with its subspace topology $\mathcal{O}_{\mathbb{Q}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not locally connected, since its connected components are the singleton sets $\{q\}_{q \in \mathbb{Q}}$, which are not open in $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$.