Generell Topologi

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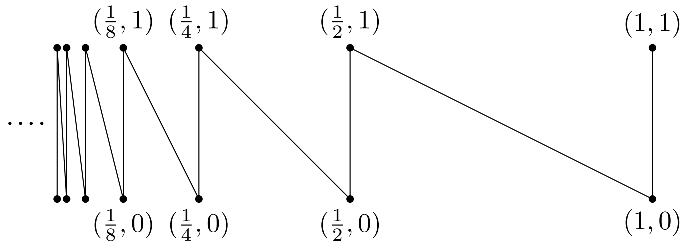
10 Thursday 14th February

10.1 An example of a topological space which is connected but not locally connected

Example 10.1. Let $X \subset \mathbb{R}^2$ be the set depicted below.



To be explicit X is the union of the sets

$$\bigcup_{n \geq 0} \{ (\frac{1}{2^n}, y) \mid y \in [0, 1] \}$$

and

$$\bigcup_{n \geq 0} \{ (x, -2^{n+1}x + 2) \mid x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}] \}.$$

Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. We have that $(X, \mathcal{O}_X)$ is connected. This may be proven as follows.

(1) For every $n \geq 0$, let A_n be the line segment

$$\{ (\frac{1}{2^n}, y) \mid y \in [0, 1] \}.$$

Let $\mathcal{O}_{A_n}$ denote the subspace topology on A_n with respect to X, or equivalently with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

The projection map

$$A_n \longrightarrow [0, 1]$$

given by $(x, y) \mapsto y$ defines a homeomorphism between $(A_n, \mathcal{O}_{A_n})$ and $([0, 1], \mathcal{O}_{[0,1]})$, where $\mathcal{O}_{[0,1]}$ denotes the subspace topology on $[0, 1]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

By Proposition 7.9 we have that $([0, 1], \mathcal{O}_{[0,1]})$ is connected. By Corollary 7.3 we deduce that $(A_n, \mathcal{O}_{A_n})$ is connected.

(2) For every $n \geq 0$, let A'_n be the line segment

$$\left\{\left(x, -2^{n+1}x + 2\right) \mid y \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]\right\}.$$

Let $\mathcal{O}_{A'_n}$ denote the subspace topology on A'_n with respect to X, or equivalently with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

The projection map

$$A'_n \longrightarrow \left[\tfrac{1}{2^{n+1}}, \tfrac{1}{2^n}\right]$$

given by $(x, y) \mapsto x$ defines a homeomorphism between $(A_n, \mathcal{O}_{A_n})$ and

$$\left(\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \mathcal{O}_{\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]}\right),$$

where $\mathcal{O}_{\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]}$ denotes the subspace topology on $\left[\frac{1}{2^{n+1}}, \frac{1}{2^m}\right]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

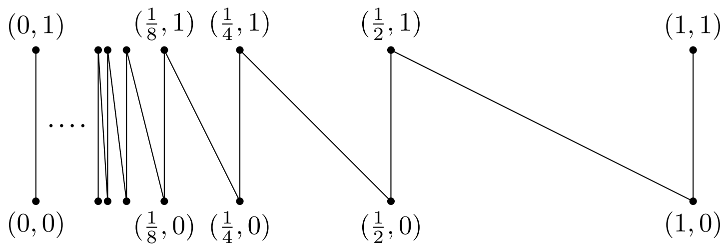
By Proposition 7.9 we have that $\left(\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \mathcal{O}_{\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]}\right)$ is connected. By Corollary 7.3 we deduce that $(A'_n, \mathcal{O}_{A'_n})$ is connected.

(3) We conclude that $(X, \mathcal{O}_X)$ is connected by Proposition 8.5.

The closure $\overline{X}$ of X in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is

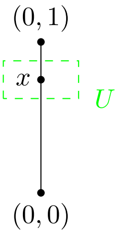
$$X \bigcup \Big(\{(0, y) \mid y \in [0, 1]\}\Big).$$

See Exercise Sheet 4.

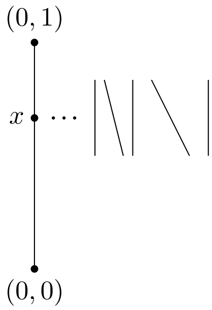


Let $\mathcal{O}_{\overline{X}}$ denote the subspace topology on $\overline{X}$ in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Since X is connected we have by Corollary 7.6 that $(\overline{X}, \mathcal{O}_{\overline{X}})$ is connected.

However $(\overline{X}, \mathcal{O}_{\overline{X}})$ is not locally connected. Let us prove this. Let $x \in \big\{(0, y) \mid y \in [0, 1]\big\}$. There is an open rectangle $U \subset \mathbb{R}^2$ such that $x \in U$ with the property that if $(x', y') \in U$ then $0 < y' < 1$.



Then $U \cap \overline{X}$ is a disjoint union of infinitely many open intervals.



In particular $U \cap \overline{X}$ is not a connected subset of $(\overline{X}, \mathcal{O}_{\overline{X}})$.

Remark 10.2. The topological space $(\overline{X}, \mathcal{O}_{\overline{X}})$ is a variant of the *topologist's sine curve*. If you wish to look up this kind of example in another reference, this is the phrase that you will need!

10.2 Path connected topological spaces

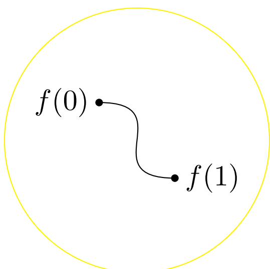
Definition 10.3. Let $(X, \mathcal{O}_X)$ be a topological space. A *path* in X is a continuous map

$$I \longrightarrow X.$$

Terminology 10.4. Let $(X, \mathcal{O})$ be a topological space, and let

$$I \xrightarrow{f} X$$

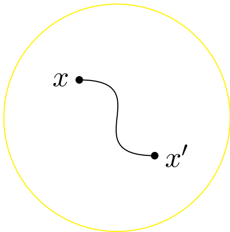
be a path in X. The picture shows a path in D^2.



Terminology 10.5. Let $(X, \mathcal{O}_X)$ be a topological space. Let (x, x') be a pair of elements of X. A path *from x to x'* in X is a path

$$I \xrightarrow{f} X$$

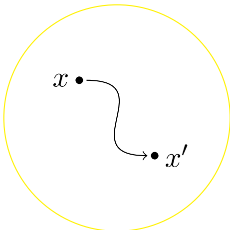
in X such that $f(0) = x$ and $f(1) = x'$.



Notation 10.6. When drawing a path

$$I \xrightarrow{f} X$$

from x to x' in a topological space $(X, \mathcal{O}_X)$ we often use an arrow to indicate that $x = f(0)$ and $x' = f(1)$.



Example 10.7. Look back at Examples 2.13 (3) for several examples of paths in S^1.

Proposition 10.8. Let $(X, \mathcal{O})$ be a topological space. Let $x, x' \in X$, and let

$$I \xrightarrow{f} X$$

be a path from x to x' in X.

Let

$$I \xrightarrow{v} I$$

be the map given by $t \mapsto 1 - t$.

The map

$$I \xrightarrow{f \circ v} X$$

defines a path from x' to x in X.

Proof. By Question 3 (f) of Exercise Sheet 3 we have that v is continuous. By Proposition 2.16 we deduce that $f \circ v$ is continuous.

We have that

$$\begin{aligned}(f \circ v)(0) &= f\big(v(0)\big) \\ &= f(1) \\ &= x'\end{aligned}$$

and that

$$\begin{aligned}(f \circ v)(0) &= f\big(v(1)\big) \\ &= f(0) \\ &= x.\end{aligned}$$

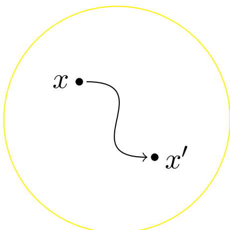
□

Remark 10.9. We met the map v in Examples 2.13 (4).

Remark 10.10. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$I \xrightarrow{f} X$$

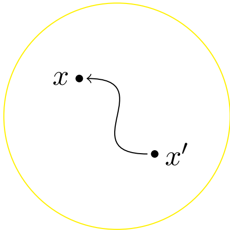
be a path in X.



We think of the path

$$I \xrightarrow{f \circ v} X$$

as obtained by travelling along f in reverse.



Proposition 10.11. Let $(X, \mathcal{O})$ be a topological space. Let $x, x', x'' \in X$. Let

$$I \xrightarrow{f} X$$

be a path from x to x' in X, and let

$$I \xrightarrow{f'} X$$

be a path from x' to x'' in X.

Let

$$I \xrightarrow{g} X$$

be the map given by

$$t \mapsto \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ f'(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then g defines a path from x to x'' in X.

Proof. Note that g is well-defined, since

$$\begin{aligned} f(2 \cdot \frac{1}{2}) &= f(1) \\ &= f'(0) \\ &= f'(2 \cdot \frac{1}{2} - 1). \end{aligned}$$

Moreover by Question 7 (b) of Exercise Sheet 3 we have that g is continuous.

We have that

$$\begin{aligned} g(0) &= f(2 \cdot 0) \\ &= f(0) \\ &= x \end{aligned}$$

and that

$$\begin{aligned} g(1) &= f'(2 \cdot 1 - 1) \\ &= f'(1) \\ &= x'. \end{aligned}$$

$\square$

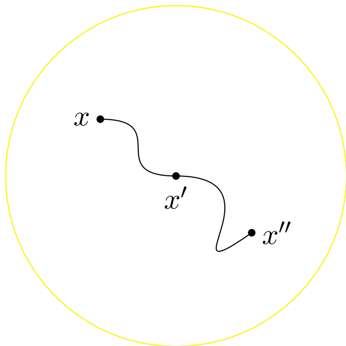
Remark 10.12. Let $(X, \mathcal{O})$ be a topological space. Let $x, x', x'' \in X$. Let

$$I \xrightarrow{f} X$$

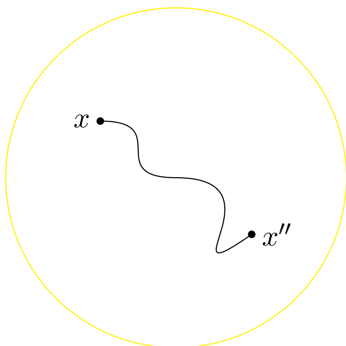
be a path from x to x' in X, and let

$$I \xrightarrow{f'} X$$

be a path from x' to x'' in X.



The corresponding path g from x to x'' of Proposition 10.11 can be thought of as first travelling at double speed from x to x' along f, and then travelling at double speed from x' to x'' along f'.



Proposition 10.13. Let $(X, \mathcal{O})$ be a topological space. Let $x \in X$. The constant map

$$I \xrightarrow{f} X$$

given by $t \mapsto x$ for all $t \in I$ defines a path from x to x in X.

Proof. Easy exercise! $\square$

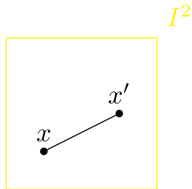
Definition 10.14. A topological space $(X, \mathcal{O})$ is *path connected* if for every pair (x, x') of elements of X there is a path from x to x'.

Examples 10.15.

(1) The topological spaces $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ and $(I^n, \mathcal{O}_{I^n})$ are path connected for every n. Indeed for any x and x' in $\mathbb{R}^n$ or I^n the straight line

$$I \xrightarrow{f} \mathbb{R}^n$$

given by $t \mapsto x + t(x' - x)$ defines a path from x to x'.



(2) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homeomorphic topological spaces. Then X is path connected if and only if Y is path connected. Moreover quotients and products of path connected spaces are path connected. Thus all of the topological spaces of Examples 3.9 (1) – (5) are path connected.

On Exercise Sheet 5 you will be asked to prove these assertions.

(3) Let $X = \{a, b\}$ be equipped with the topology $\mathcal{O} = \big\{\emptyset, \{b\}, X\big\}$. In other words $(X, \mathcal{O})$ is the Sierpiński interval. We have that $(X, \mathcal{O})$ is path connected.

Let us prove this. By virtue of Proposition 10.13 and Proposition 10.8 it suffices to prove that there is a path in X from a to b.

The map

$$I \xrightarrow{f} X$$

given by

$$t \mapsto \begin{cases} a & \text{if } t = 0, \\ b & \text{if } 0 < t \leq 1 \end{cases}$$

is continuous. Indeed $f^{-1}(b) = (0, 1]$, which is an open subset of I.

Moreover $f(0) = a$ and $f(1) = b$. Thus f defines a path from a to b in X.

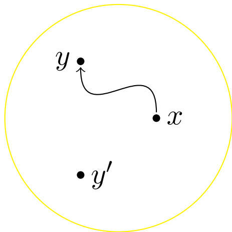
Proposition 10.16. Let $(X, \mathcal{O})$ be a topological space. Let $x \in X$. Then $(X, \mathcal{O})$ is path connected if and only if for every $x' \in X$ there is a path from x to x' in X.

Proof. Suppose that for every $x' \in X$ there is a path from x to x' in X. Let $y, y' \in X$. We must prove that there is a path from y to y' in X.

By assumption there is a path

$$I \xrightarrow{f_{x,y}} X$$

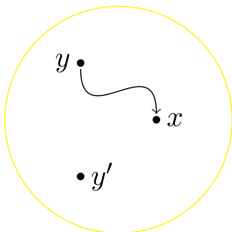
from x to y in X.



By Proposition 10.8 we deduce that there is a path

$$I \xrightarrow{f_{y,x}} X$$

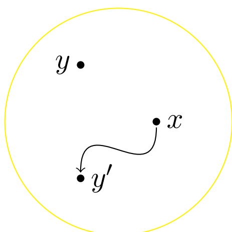
from y to x in X.



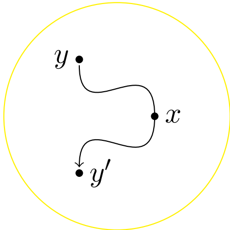
By assumption there is also a path

$$I \xrightarrow{f_{x,y'}} X$$

from x to y' in X,



By Proposition 10.11 applied to the paths $f_{y,x}$ and $f_{x,y'}$ in X we conclude that there is a path from y to y' in X, as required.



Conversely if $(X, \mathcal{O})$ is path connected then by definition there is a path from x to x' for every $x' \in X$.

$\square$

Proposition 10.17. Let $(X, \mathcal{O})$ be a path connected topological space. Then $(X, \mathcal{O})$ is connected.

Proof. Let $x \in X$. Since $(X, \mathcal{O})$ is path connected we have that for every $x' \in X$ there is a path

$$I \xrightarrow{f_{x,x'}} X$$

from x to x' in X.

Since $x' \in f_{x,x'}(I)$ we have that

$$X = \bigcup_{(x,x') \in X \times X} f_{x,x'}(I).$$

By Proposition 7.9 we have that $(I, \mathcal{O}_I)$ is connected. We deduce by Proposition 7.1 that $f_{x,x'}(I)$ is connected for all $x, x' \in X$.

We conclude by Proposition 8.5 that $(X, \mathcal{O})$ is connected. $\square$

Remark 10.18. A connected topological space is not necessarily path connected. For instance the topological space $(\overline{X}, \mathcal{O}_{\overline{X}})$ of Example 10.1 is connected but not path connected. You will be asked to prove this on Exercise Sheet 5.

10.3 Locally path connected topological spaces

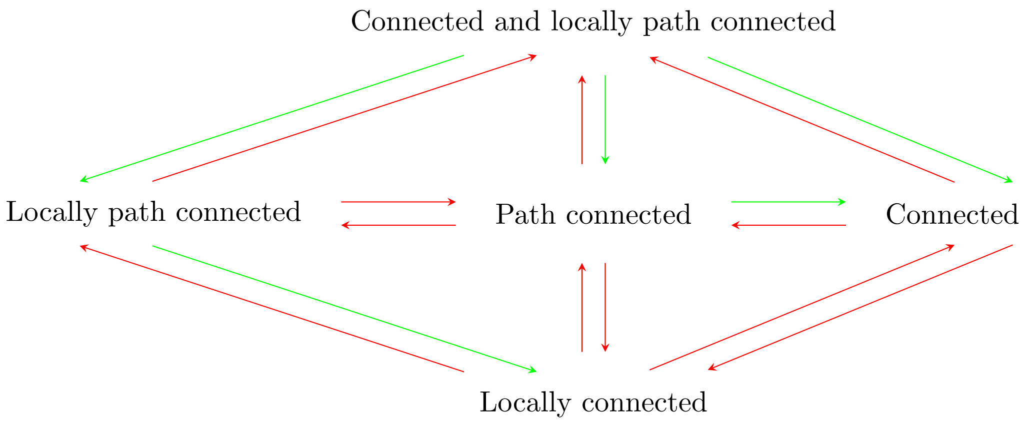
Definition 10.19. A topological space $(X, \mathcal{O}_X)$ is *locally path connected* if for every $x \in X$ and every neighbourhood U of x in $(X, \mathcal{O}_X)$ there is a neighbourhood U' of x in $(X, \mathcal{O}_X)$ such that U' is path connected and $U' \subset U$.

Remark 10.20. We will explore locally path connected topological spaces on Exercise Sheet 5. We will prove that a connected and locally path connected topological space is path connected.

By Proposition 10.17 we have that a locally path connected topological space is locally connected.

By contrast, we will see on Exercise Sheet 5 that a locally connected topological space need not be locally path connected. Moreover we will see that a path connected topological space need not be locally path connected and need not be locally connected.

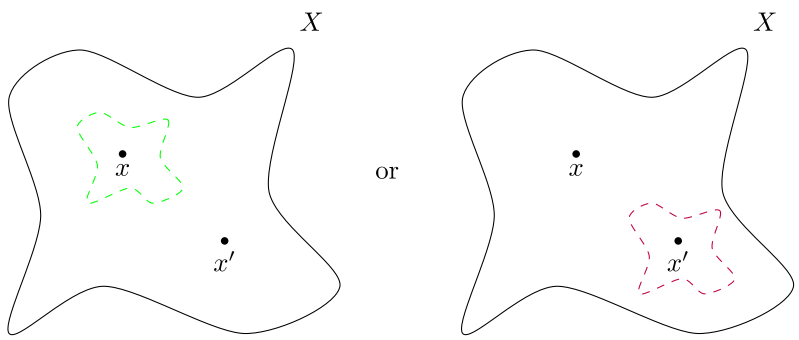
Synopsis 10.21. Let us summarise the relationship between connected, path connected, locally connected, and locally path connected topological spaces. A green arrow indicates an implication. A red arrow indicates that there is no implication.



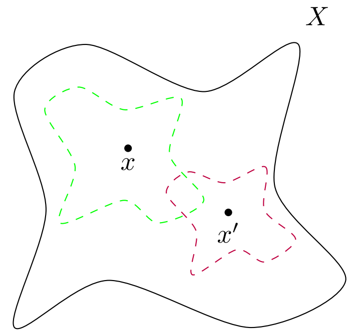
10.4 Separation axioms

Definition 10.22. Let $(X, \mathcal{O})$ be a topological space. The following are axioms which $(X, \mathcal{O})$ may satisfy.

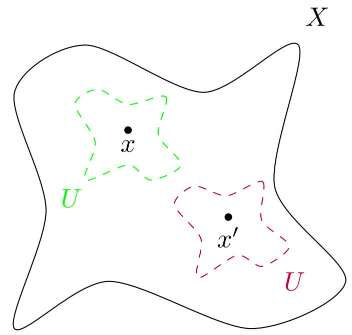
(T0) For every $x, x' \in X$ such that $x \neq x'$ there is a neighbourhood of either x or x' which does not contain both x and x'.



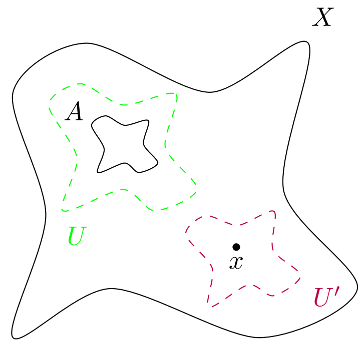
(T1) For every ordered pair (x, x') of elements of X such that $x \neq x'$ there is a neighbourhood of x which does not contain x'.



(T2) For every $x, x' \in X$ such that $x \neq x'$ there is a neighbourhood U of x and a neighbourhood U' of x' such that $U \cap U' = \emptyset$.



(T3) For every non-empty closed subset A of X and every $x \in X \setminus A$ there is an open subset U of X such that $A \subset U$ and a neighbourhood U' of x such that $U \cap U' = \emptyset$.

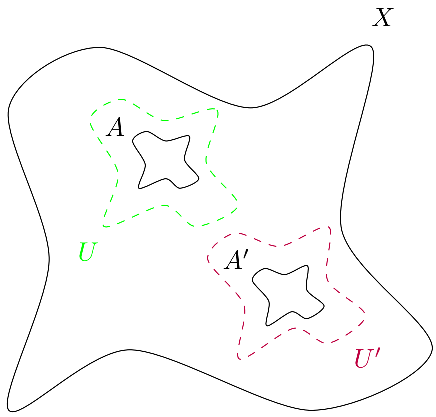


(T3$\frac{1}{2}$) For every non-empty closed subset A of X and every $x \in X \setminus A$ there is a continuous map

$$X \longrightarrow I$$

such that $f(x) = 1$ and $f(A) = \{0\}$.

(T4) For every pair A and A' of closed subsets of X such that $A \cap A' = \emptyset$ there is an open subset U of X with $A \subset U$ and an open subset U' of X with $A' \subset U'$ such that $U \cap U' = \emptyset$.



(T6) For every pair of non-empty closed subsets A and A' of X such that $A \cap A' = \emptyset$ there is a continuous map

$$X \longrightarrow I$$

such that $f^{-1}(0) = A$ and $f^{-1}(1) = A'$.

Remark 10.23. There is an axiom along the above lines which is denoted (T5), but we will not need it. Be careful if you look up these axioms in another source, as there are differing conventions.