Generell Topologi

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13.1 $(I, \mathcal{O}_I)$ is compact

Lemma 13.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{U_j\}_{j\in J}$ and $\{W_k\}_{k\in K}$ be open coverings of X. Suppose that for every $k \in K$ there is a $j_k \in J$ such that $W_k \subset U_{j_k}$. If $\{W_k\}_{k\in K}$ has a finite subcovering then $\{U_j\}_{j\in J}$ has a finite subcovering.

Proof. Let K' be a finite subset of K such that $\{W_{k'}\}_{k'\in K}$ is an open covering of X. We have that

$$\begin{aligned} X &= \bigcup_{k'\in K'} W_{k'} \\ &\subset \bigcup_{k'\in K'} U_{j_{k'}}. \end{aligned}$$

Then $X = \bigcup_{k'\in K'} U_{j_{k'}}$. Thus $\{U_{j_{k'}}\}_{k'\in K'}$ is a finite subcovering of $\{U_j\}_{j\in J}$. □

Proposition 13.2. The unit interval $(I, \mathcal{O}_I)$ is compact.

Proof. By definition of $\mathcal{O}_{\mathbb{R}}$ we have that $\big\{(a,b) \mid a,b \in \mathbb{R}\big\}$ is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Question 2 of Exercise Sheet 2 we deduce that

$$\mathcal{O}' = \big\{I \cap (a,b) \mid a,b \in \mathbb{R}\big\}$$

is a basis for $(I, \mathcal{O}_I)$.

Let $\{U_j\}_{j\in J}$ be an open covering of I. By Question 3 (a) of Exercise Sheet 2 with respect to $\mathcal{O}'$, for every $t \in I$ and every $j \in J$ there is an open interval (a_t, b_t) such that $I \cap (a_t, b_t) \subset U_j$ and $t \in I \cap (a_t, b_t)$. Let $A_t = I \cap (a_t, b_t)$. Then A_t is either an open interval, a half open interval, or a closed interval. For brevity we shall simply refer to A_t as an interval.

We have that

$$\begin{aligned} I &= \bigcup_{t\in I} \{t\} \\ &\subset \bigcup_{t\in I} A_t. \end{aligned}$$

Thus $\{A_t\}_{t\in I}$ is an open covering of I. By Lemma 13.1 it suffices to prove that $\{A_t\}_{t\in I}$ admits a finite subcovering.

Let $\{0,1\}$ be equipped with its discrete topology. Define

$$I \xrightarrow{f} \{0,1\}$$

to be the map given by $s \mapsto 0$ if there is a finite subset J of I such that

$$[0,s] \subset \bigcup_{t'\in J} A_{t'}$$

and by $s \mapsto 1$ otherwise. Let us first prove that f is continuous.

Let $t \in I$. Let $s \in A_t$ and suppose that $f(s) = 0$. Then by definition of f there is a finite subset J of I such that

$$[0, s] \subset \bigcup_{t' \in J} A_{t'}.$$

Let $s' \in A_t$, and let $s'' \in I$ be such that $s'' \in [0, s']$. If $s'' \leq s$ then since

$$[0, s] \subset \bigcup_{t' \in J} A_{t'}$$

we have that $s'' \in A_{t'}$ for some $t' \in J$. If $s \leq s''$ then since A_t is an interval and both s and s' belong to A_t we have by Lemma 7.8 that $s'' \in A_t$. Thus for any $s'' \in [0, s']$ we have that

$$s'' \in \bigcup_{t'' \in J \cup \{t\}} A_{t''}.$$

We deduce that

$$[0, s'] \subset \bigcup_{t'' \in J \cup \{t\}} A_{t''}.$$

Since J is finite we have that $J \cup \{t\}$ is finite. Hence $f(s') = 0$.

We have now proven that for any $t \in I$, if $f(s) = 0$ for some $s \in A_t$ then $f(A_t) = \{0\}$. We draw the following conclusions.

(1) We have that

$$f^{-1}\big(\{0\}\big) = \bigcup_{t \in I \text{ such that } f(s) = 0 \text{ for some } s \in A_t} A_t.$$

We deduce that $f^{-1}\big(\{0\}\big) \in \mathcal{O}_I$ since $A_t \in \mathcal{O}_I$ for all $t \in I$.

(2) We have that

$$f^{-1}\big(\{1\}\big) = \bigcup_{t \in I \text{ such that } f(s) = 1 \text{ for all } s \in A_t} A_t.$$

We deduce that $f^{-1}\big(\{1\}\big) \in \mathcal{O}_I$ since $A_t \in \mathcal{O}_I$ for all $t \in I$.

This completes our proof that f is continuous.

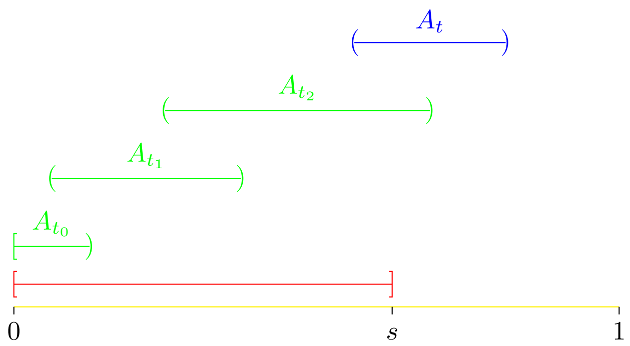
By Proposition 7.9 we have that $(I, \mathcal{O}_I)$ is connected. Since f is continuous we deduce by Proposition 6.5 that f is constant. Moreover $f(0) = 0$ since $[0, 0] = \{0\} \subset A_0$.

We deduce that $f(s) = 0$ for all $s \in I$. In particular we have that $f(1) = 0$. Thus by definition of f there is a finite subset J of I such that $I = \bigcup_{t' \in J} A_{t'}$. We conclude that $\{A_{t'}\}_{t' \in J}$ is a finite subcovering of $\{A_t\}_{t \in I}$.

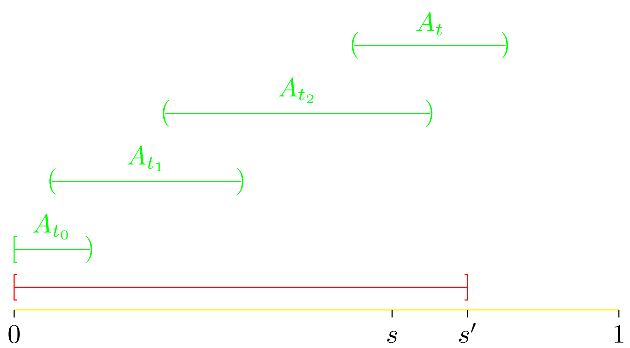
$\square$

Scholium 13.3. The proof of Proposition 13.2 is perhaps the most difficult in the course. The idea is to carry out a kind of inductive argument. We begin by noting that we need only the singleton set $\{A_0\}$ to ensure that $0 \in \bigcup_{t \in I}\{A_t\}$.

Next we suppose that we know that for some $t \in I$ and some $s \in A_t$ we need only a finite number of the sets $A_{t'}$ to ensure that $[0, s] \subset \bigcup_{t' \in I} A_{t'}$. In the picture below we suppose that we need only three sets $A_0 = A_{t_0}$, A_{t_1}, and A_{t_2}.



We then observe that for any $s' \in A_t$ we require only the finite number of sets $A_{t'}$ which we needed to cover $[0, s]$ together with the set A_t to ensure that $[0, s'] \subset \bigcup_{t' \in I} A_{t'}$.



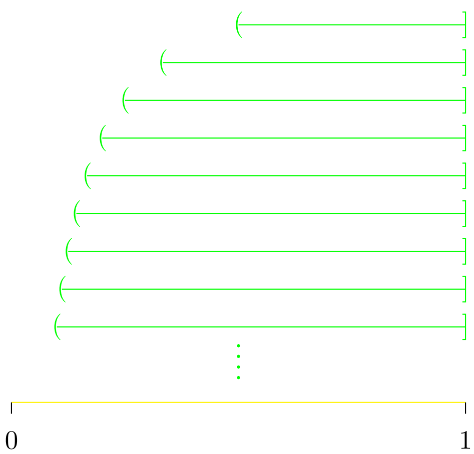
The tricky part is to show that by 'creeping along' in this manner we can arrive at 1 after a finite number of steps. Ultimately this is a consequence of the completeness of $\mathbb{R}$ — indeed it is possible to give a proof in which one appeals to the completeness of $\mathbb{R}$ directly.

We instead gave a proof which builds upon the hard work we already carried out to prove that $(I, \mathcal{O}_I)$ is connected. The role of the completeness of $\mathbb{R}$ lay in the proof of Lemma 7.8.

Remark 13.4. To help us to appreciate why $(I, \mathcal{O}_I)$ is compact, let us compare it to $(0,1]$. We equip $(0,1]$ with its subspace topology $\mathcal{O}_{(0,1]}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. The open covering

$$\left\{ \left(\frac{1}{n}, 1\right] \right\}_{n \in \mathbb{N}}$$

of $(0,1]$ has no finite subcovering.



Thus $(0,1]$ is not compact.

To obtain an open covering of $[0,1]$ we have to add an open set containing 0. Let us take this open set to be $[0,t)$ for some $0 < t < 1$. Thus our open covering of $[0,1]$ is

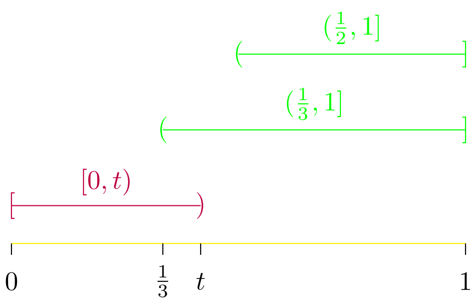
$$\left\{ \left(\frac{1}{n}, 1\right] \right\}_{n \in \mathbb{N}} \cup \{[0,t)\}$$

where $\{[0,t)\}$ is the singleton set containing $[0,t)$.

This open covering has a finite subcovering! For instance

$$\left\{ \left(\frac{1}{n}, 1\right] \right\}_{n \in \mathbb{N} \text{ such that } n \leq m} \cup \{[0,t)\}$$

where $m \in \mathbb{N}$ is such that $\frac{1}{m} < t$.



Corollary 13.5. Let the closed interval $[a,b]$ be equipped with its subspace topology $\mathcal{O}_{[a,b]}$ with respect to $(\mathbb{R},\mathcal{O}_{\mathbb{R}})$. Then $\big([a,b],\mathcal{O}_{[a,b]}\big)$ is compact.

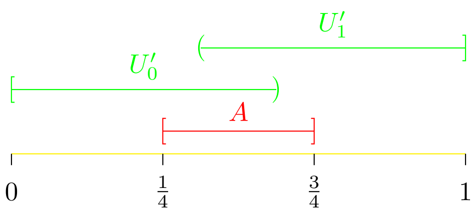
Proof. Follows immediately by Corollary 12.11 from Proposition 13.2, since by Examples 4.7 (4) we have that $\big([a,b],\mathcal{O}_{[a,b]}\big)$ is homeomorphic to $(I,\mathcal{O}_I)$. $\square$

Remark 13.6. Corollary 13.5 is one of the cornerstones of mathematics. Analysis relies indispensably upon it, and it is at the heart of many constructions in topology.

13.2 Compact vs Hausdorff vs closed

Proposition 13.7. Let $(X,\mathcal{O}_X)$ be a compact topological space, and let A be a closed subset of X. Then $(A,\mathcal{O}_A)$ is compact. Here $\mathcal{O}_A$ is the subspace topology on A with respect to $(X,\mathcal{O}_X)$.

Proof. Let $\{U_j\}_{j\in J}$ be an open covering of $(A,\mathcal{O}_A)$. By definition of $\mathcal{O}_A$ we have that $U_j = A\cap U'_j$ for some $U'_j\in\mathcal{O}_X$. Suppose first that $\{U'_j\}_{j\in J}$ is an open covering of $(X,\mathcal{O}_X)$. An example is pictured below.

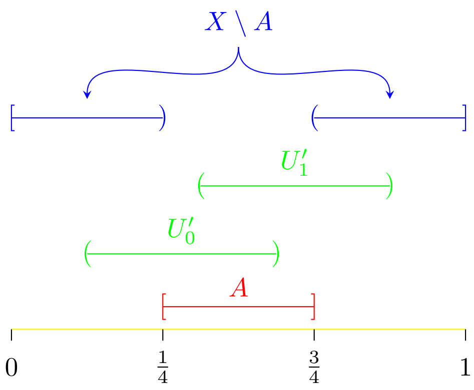


Since $(X,\mathcal{O}_X)$ is compact there is a finite subset J' of J such that $\{U'_{j'}\}_{j'\in J'}$ is an open covering of $(X,\mathcal{O}_X)$. Then:

$$\begin{aligned} A &= A\cap X \\ &= A\cap\Big(\bigcup_{j'\in J'} U'_{j'}\Big) \\ &= \bigcup_{j'\in J'} A\cap U'_{j'} \\ &= \bigcup_{j'\in J'} U_{j'}. \end{aligned}$$

Thus $\{U_{j'}\}_{j'\in J'}$ is a finite subcovering of $\{U_j\}_{j\in J}$.

Suppose now that $X\setminus\bigcup_{j\in J}U'_j\neq\emptyset$. Since A is closed in $(X,\mathcal{O}_X)$ we have that $X\setminus A$ is open in $(X,\mathcal{O}_X)$. Hence $\{U'_j\}_{j\in J}\cup\{X\setminus A\}$ is an open covering of $(X,\mathcal{O}_X)$. Here $\{X\setminus A\}$ denotes the set with the single element $X\setminus A$. An example is pictured below.



Since $(X, \mathcal{O}_X)$ is compact this open covering admits a finite subcovering. Moreover $X \setminus A$ must belong to this finite subcovering by our assumption that $X \setminus \bigcup_{j \in J} U'_j \neq \emptyset$. Thus there is a finite subset J' of J such that

$$\{U'_{j'}\}_{j' \in J'} \cup \{X \setminus A\}$$

is an open covering of X. We now observe that:

$$\begin{aligned}
A &= A \cap X \\
&= A \cap \left(\left(\bigcup_{j' \in J'} U'_{j'} \right) \cup \{X \setminus A\} \right) \\
&= \left(A \cap \bigcup_{j' \in J'} U'_{j'} \right) \cup \left\{ A \cap (X \setminus A) \right\} \\
&= \left(\bigcup_{j' \in J'} A \cap U'_{j'} \right) \cup \emptyset \\
&= \bigcup_{j' \in J'} U_{j'}.
\end{aligned}$$

Thus $\{U_{j'}\}_{j' \in J'}$ is a finite subcovering of $\{U_j\}_{j \in J}$. □

Terminology 13.8. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X equipped with its subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Then A is a *compact subset* of X if $(A, \mathcal{O}_A)$ is compact.

Lemma 13.9. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let A be a compact subset of X. Suppose that $x \in X \setminus A$. There is a pair of open subsets U and U' of X such that:

(1) $A \subset U$,

(2) U' is a neighbourhood of x,

(3) $U \cap U' = \emptyset$.

Proof. Let $a \in A$. Since $(X, \mathcal{O}_X)$ is Hausdorff there is a neighbourhood U_a of a in $(X, \mathcal{O}_X)$ and a neighbourhood U'_a of x in $(X, \mathcal{O}_X)$ such that $U_a \cap U'_a = \emptyset$.

Let A be equipped with its subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Then $\{A \cap U_a\}_{a \in A}$ defines an open covering of $(A, \mathcal{O}_A)$, since

$$\begin{aligned} A &= \bigcup_{a \in A} \{a\} \\ &\subset \bigcup_{a \in A} A \cap U_a \end{aligned}$$

and thus $A = \bigcup_{a \in A} A \cap U_a$.

Since $(A, \mathcal{O}_A)$ is compact there is a finite subset J of A such that $\{A \cap U_a\}_{a \in J}$ is an open covering of A. Let $U = \bigcup_{a \in J} U_a$. Let $U' = \bigcap_{a \in J} U'_a$. Then:

(1) We have that $A \subset U$, since

$$\begin{aligned} A &= \bigcup_{a \in A} A \cap U_a \\ &= \bigcup_{a \in J} A \cap U_a \\ &\subset \bigcup_{a \in J} U_a \\ &= U. \end{aligned}$$

Moreover we have that U_a is open in $(X, \mathcal{O}_X)$ for all $a \in A$, and in particular for all $a \in J$. Thus we have that U is open in $(X, \mathcal{O}_X)$.

(2) Since $x \in U'_a$ for all $a \in A$ we have that

$$x \in \bigcap_{a \in A} U'_a \subset \bigcap_{a \in J} U'_a.$$

Moreover we have that U'_a is open in $(X, \mathcal{O}_X)$ for all $a \in A$, and in particular for all $a \in J$. Since J is finite we thus have that $\bigcap_{a \in J} U'_a$ is open in $(X, \mathcal{O}_X)$.

(3) Since $U_a \cap U_{a'} = \emptyset$ for all $a \in A$ we have that

$$\begin{aligned} U \cap U' &= \Big(\bigcup_{a \in J} U_a \Big) \cap \Big(\bigcap_{a' \in J} U'_{a'} \Big) \\ &= \bigcup_{a \in J} \Big(U_a \cap \bigcap_{a' \in J} U'_{a'} \Big) \\ &\subset \bigcup_{a \in J} (U_a \cap U_a) \\ &= \bigcup_{a \in J} \emptyset \\ &= \emptyset. \end{aligned}$$

$\square$

Remark 13.10. The proof of Lemma 13.9 is a very typical example of an appeal to compactness in practise. The key step is (2). Our conclusion that $\bigcap_{a \in J} U'_a$ is open in $(X, \mathcal{O}_X)$ relies on the fact that J is finite — as we know, an arbitrary intersection of open sets in a topological space need not be open.

Proposition 13.11. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let A be a compact subset of X. Then A is closed in $(X, \mathcal{O}_X)$.

Proof. Let $x \in X \setminus A$. By Lemma 13.9 there is a pair of open subsets U and U' of X such that:

(1) $A \subset U$,

(2) U' is a neighbourhood of x,

(3) $U \cap U' = \emptyset$.

In particular $U' \cap A \subset U' \cap U = \emptyset$. Thus x is not a limit point of A in $(X, \mathcal{O}_X)$.

We deduce that $\overline{A} = A$. By Proposition 5.7 we conclude that A is closed in X. $\square$

Remark 13.12. Proposition 13.11 does not necessarily hold if $(X, \mathcal{O}_X)$ is not Hausdorff. Here are two examples.

(1) Let $X = \{0, 1\}$ be equipped with the topology $\mathcal{O} = \{\emptyset, \{1\}, X\}$. In other words, $(X, \mathcal{O})$ is the Sierpiński interval.

By Examples 11.4 (1) a finite topological space is T1 if and only if its topology is the discrete topology. Since $\mathcal{O}$ is not the discrete topology on X, the Sierpiński interval is not T1 and hence not Hausdorff.

Since X is finite every subset of X is compact. In particular $\{1\}$ is a compact subset of $(X, \mathcal{O})$. But as we observed in Examples 5.6 (1) the set $\{1\}$ is not closed in $(X, \mathcal{O})$.

(2) Let $\left((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R} \sqcup \mathbb{R})/\sim}\right)$ be the real line with two origins of Example 12.1. As in Example 12.1 to avoid confusion we adopt the notation $]a, b[$ for the open interval from a to b. Similarly we denote by $]a, b]$ the half open interval from a to b.

Let

$$\mathbb{R} \sqcup \mathbb{R} \xrightarrow{\pi} (\mathbb{R} \sqcup \mathbb{R})/\sim$$

denote the quotient map.

Let

$$I \xrightarrow{i} \mathbb{R} \sqcup \mathbb{R}$$

denote the map $t \mapsto (t, 0)$. Here as usual we think of $\mathbb{R} \sqcup \mathbb{R}$ as

$$\big(\mathbb{R} \times \{0\}\big) \cup \big(\mathbb{R} \times \{1\}\big).$$

We have that i is the composition of the following two maps.

(1) The inclusion map

$$I \hookrightarrow \mathbb{R},$$

which is continuous by Proposition 2.15

(2) The map

$$\mathbb{R} \longrightarrow \mathbb{R} \sqcup \mathbb{R}$$

given by $x \mapsto (x, 0)$, which is continuous by Observation 5.21.

We deduce by Proposition 2.16 that i is continuous. Thus by Proposition 13.2 and Proposition 12.10 we have that $\pi\big(i(I)\big)$ is a compact subset of

$$\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big).$$

We claim that $\pi\big(i(I)\big)$ is not a closed subset of $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$.

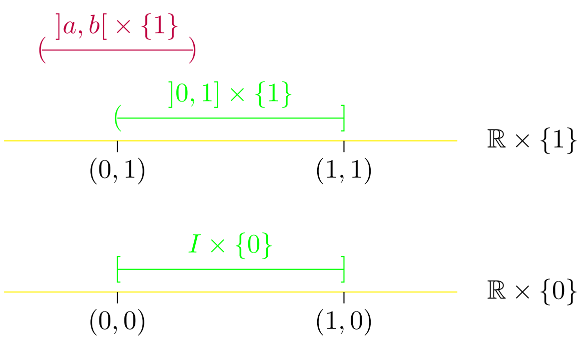
Let us prove that $\pi\big((0,1)\big)$ is a limit point of $\pi\big(i(I)\big)$ in $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$. Let U be a neighbourhood of $\pi\big((0,1)\big)$ in $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$. Then $\pi^{-1}(U)$ is a neighbourhood of $(0,1)$ in $\mathbb{R} \sqcup \mathbb{R}$.

As in Example 12.1 we have that

$$\big\{\,]a,b[\times \{0\} \mid a, b \in \mathbb{R}\big\} \cup \big\{\,]a,b[\times \{1\} \mid a, b \in \mathbb{R}\big\}$$

is a basis for $(\mathbb{R}\sqcup\mathbb{R}, \mathcal{O}_{\mathbb{R}\sqcup\mathbb{R}})$. By Question 3 (a) of Exercise Sheet 2 there are $a, b \in \mathbb{R}$ such that $0 \in\,]a,b[$ and $]a,b[\times \{1\} \subset \pi^{-1}(U)$.

We have that $\pi^{-1}\Big(\pi\big(i(I)\big)\Big) = \big(I \times \{0\}\big) \times \big(\,]0,1] \times \{1\}\big)$.



Then

$$]a,b[\,\cap\,]0,1] = \,]0,b[\, \neq \emptyset.$$

Hence

$$\big(\,]a,b[\,\times\{1\}\big)\cap\big(\,]0,1]\times\{1\}\big)\neq\emptyset.$$

Thus

$$\pi^{-1}\Big(U\cap\pi\big(i(I)\big)\Big) = \pi^{-1}(U)\cap\pi^{-1}\Big(\pi\big(i(I)\big)\Big)\neq\emptyset.$$

We deduce that

$$U\cap\pi\big(i(I)\big)\neq\emptyset.$$

This completes our proof that $\pi\big((0,1)\big)$ is a limit point of $\pi\big(i(I)\big)$ in

$$\big((\mathbb{R}\sqcup\mathbb{R})/\sim,\mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big).$$

But $\pi\big((0,1)\big)\notin\pi\big(i(I)\big)$. Thus $\pi\big(i(I)\big)$ is not closed in $\big((\mathbb{R}\sqcup\mathbb{R})/\sim,\mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$.

Proposition 13.13. Let $(X,\mathcal{O}_X)$ be a compact topological space. Let $(Y,\mathcal{O}_Y)$ be a Hausdorff topological space. A map

$$X \xrightarrow{f} Y$$

is a homeomorphism if and only if f is continuous and bijective.

Proof. If f is a homeomorphism then by Proposition 3.15 we have that f is continuous and bijective.

Suppose instead that f is continuous and bijective. That f is bijective implies that $x\mapsto f^{-1}(x)$ gives a well defined map

$$Y \xrightarrow{g} X.$$

We have that g is inverse to f. To prove that f is a homeomorphism we shall prove that g is continuous.

By Question 1 (a) of Exercise Sheet 3 we have that g is continuous if and only if $g^{-1}(A)$ is a closed subset of Y for any closed subset A of X. By definition of g we have that $g^{-1}(A) = f(A)$. Thus it suffices to prove that if A is a closed subset of X then $f(A)$ is a closed subset of Y.

Suppose that A ia a closed subset of X. Then since $(X,\mathcal{O}_X)$ is compact we have by Proposition 13.7 that A is a compact subset of X. Thus by Proposition 12.10 we have that $f(A)$ is a compact subset of Y. Since $(Y,\mathcal{O}_Y)$ is Hausdorff we deduce by Proposition 13.11 that $f(A)$ is closed in $(Y,\mathcal{O}_Y)$ as required.

□

Proposition 13.14. Let $(X,\mathcal{O}_X)$ be a compact topological space and let $\sim$ be an equivalence relation on X. Then $(X/\sim,\mathcal{O}_{X/\sim})$ is compact.

Proof. The quotient map

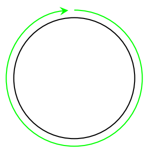
$$X \xrightarrow{\pi} X/\sim$$

is continuous and surjective. We deduce that $(X/\sim, \mathcal{O}_{X/\sim})$ is compact by Proposition 12.10. $\square$

Example 13.15. As in Examples 3.9 (1) let $\sim$ be the equivalence relation on I generated by $0 \sim 1$. Let

$$I \xrightarrow{\phi} S^1$$

be the continuous map constructed in Question 9 of Exercise Sheet 3.



We have that $\phi(0) = \phi(1)$. Thus we obtain a map

$$I/\sim \xrightarrow{f} S^1$$

given by $[t] \mapsto \phi(t)$. By Question 11 (a) of Exercise Sheet 4 we have that f is continuous. Moreover f is a bijection.

(1) By Examples 11.7 we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. Thus by Proposition 11.11 we have that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff. By Proposition 11.10 we deduce that $(S^1, \mathcal{O}_{S^1})$ is Hausdorff.

(2) By Proposition 13.2 we have that $(I, \mathcal{O}_I)$ is compact. Thus by Proposition 13.14 we have that $(I/\sim, \mathcal{O}_{I/\sim})$ is compact.

We conclude by Proposition 13.13 that f is a homeomorphism. This gives a rigorous affirmative answer to Question 3.10.