Generell Topologi

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14 Thursday 28th February

14.1 A product of compact topological spaces is compact

Proposition 14.1. Let $(X, \mathcal{O}_X)$ be a topological space and let $(X', \mathcal{O}_{X'})$ be a compact topological space. Let $x \in X$ and let W be a subset of $X \times X'$ satisfying the following conditions.

(1) $W \in \mathcal{O}_{X \times X'}$.

(2) $\{x\} \times X' \subset W$.

Then there is a neighbourhood U of x in $(X, \mathcal{O}_X)$ such that $U \times X' \subset W$.

Proof. Let $x' \in X'$. By (2) we have that $(x, x') \in W$. By (1) and the definition of $\mathcal{O}_{X \times X'}$ we deduce that there is a neighbourhood $U_{x'}$ of x in $(X, \mathcal{O}_X)$ and a neighbourhood $U'_{x'}$ of x' in $(X', \mathcal{O}_{X'})$ such that $U_{x'} \times U'_{x'} \subset W$. We have that

$$\begin{aligned} X' &= \bigcup_{x' \in X'} \{x'\} \\ &\subset \bigcup_{x' \in X'} U'_{x'}. \end{aligned}$$

Thus $X = \bigcup_{x' \in X'} U'_{x'}$ and we have that $\{U'_{x'}\}_{x' \in X}$ is an open covering of X'.

Since $(X', \mathcal{O}_{X'})$ is compact there is a finite subset J of X' such that $\{U'_{x'}\}_{x' \in J}$ is an open covering of X'. Let $U = \bigcap_{x' \in J} U_{x'}$. We make the following observations.

(1) Since J is finite we have that $U \in \mathcal{O}_X$.

(2) Since $x \in U_{x'}$ for all $x' \in X'$, we in particular have that $x \in U_{x'}$ for all $x' \in J$. Thus $x \in U$.

(3) For any $x' \in J$ we have that $U \times U'_{x'} \subset U_{x'} \times U'_{x'} \subset W$. Thus we have that

$$\begin{aligned} U \times X' &= U \times \Big(\bigcup_{x' \in J} U'_{x'} \Big) \\ &= \bigcup_{x' \in J} (U \times U'_{x'}) \\ &\subset \bigcup_{x' \in J} W \\ &= W. \end{aligned}$$

□

Remark 14.2. Proposition 14.1 is sometimes known as the *tube lemma*. It does not necessarily hold if X' is not compact. Let us illustrate this by an example.

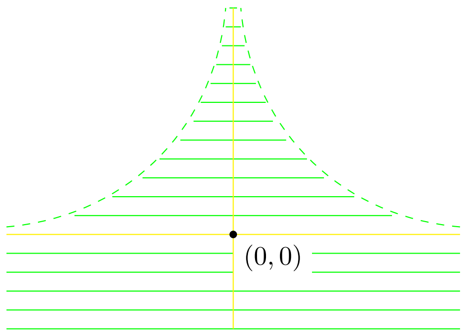
(1) Let $(X, \mathcal{O}_X) = (\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

(2) Let $(X', \mathcal{O}_{X'}) = (\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

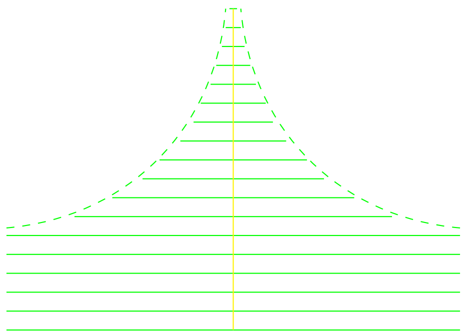
(3) Let $x = 0$.

(4) Let $W = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0 \text{ and } y < \left|\frac{1}{x}\right|\} \cup \{(0,y) \mid y \in \mathbb{R}\}$.

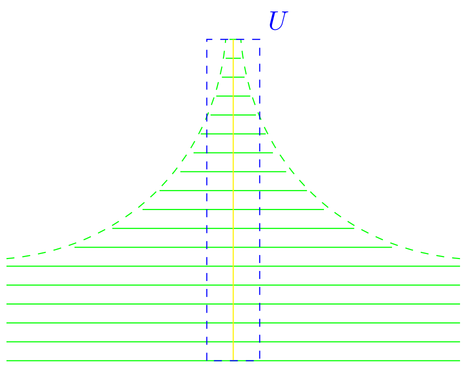
In the picture below W is the shaded green area. The two dashed green curves do not themselves belong to W.



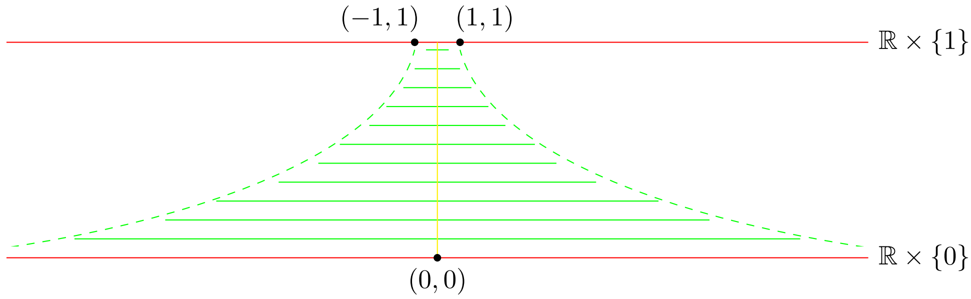
Then W is a neighbourhood of $\{x\} \times \mathbb{R} = \{0\} \times \mathbb{R}$ in $(X \times X', \mathcal{O}_{X \times X'}) = (\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. The set $\{0\} \times \mathbb{R}$, or in other words the y-axis, is depicted in yellow below.



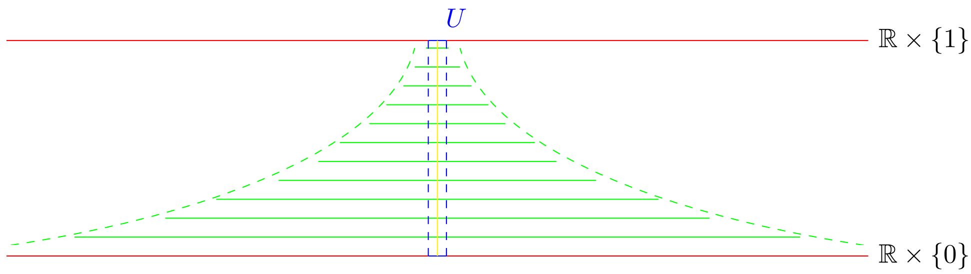
There is no neighbourhood U of x in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ such that $U \times \mathbb{R} \subset W$.



This is due to the 'infinitesimal narrowing' of W. Proposition 14.1 establishes that this kind of behaviour cannot occur if $(X', \mathcal{O}_{X'})$ is compact. For instance, suppose that we instead let $(X', \mathcal{O}_{X'}) = (I, \mathcal{O}_I)$. By Proposition 13.2 we have that $(I, \mathcal{O}_I)$ is compact. The restriction of W to $\mathbb{R} \times I$ is pictured below.



We can find a neighbourhood U of $\{0\} \times \mathbb{R}$ such that $U \subset W$.



Remark 14.3. The role of compactness in the proof of Proposition 14.1 is very similar to its role in the proof of Lemma 13.9, which was discussed in Remark 13.10. The key to the proof is observation (1), that if J is finite then $U \in \mathcal{O}_X$.

The idea of the proof of Proposition 14.1 is that since $(X', \mathcal{O}_{X'})$ is compact we can find a finite set of points $x' \in X'$ and a neighbourhood $U_{x'} \times U'_{x'} \subset W$ of (x, x') for each of these points x' such that $\{x\} \times X'$ is contained in the union of the sets $U_{x'} \times U'_{x'}$.

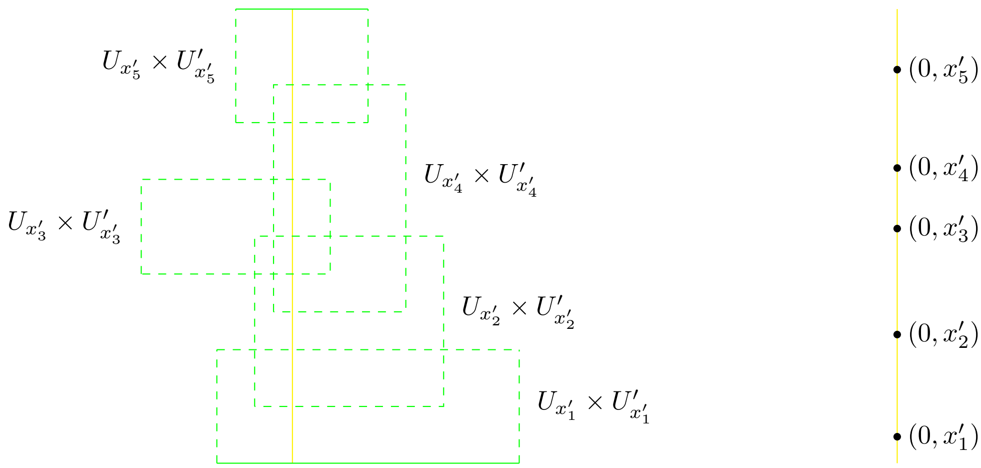
An example is depicted below when we have the following.

(1) $(X, \mathcal{O}_X) = (\mathbb{R}, \mathcal{O}_{\mathbb{R}})$,

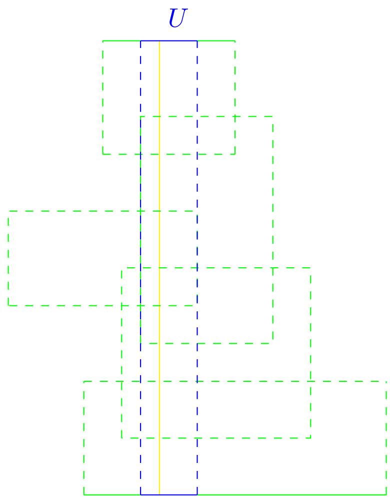
(2) $(X', \mathcal{O}_{X'}) = (I, \mathcal{O}_I)$,

(3) $x = 0$.

A set W is not drawn, but should be thought of as an open subset of $\mathbb{R} \times I$ which contains all the green rectangles. To avoid cluttering the picture, possible locations for the points $(0, x'_1), \ldots, (0, x'_5)$ are indicated seperately to the right.



The open set U in the proof of Proposition 14.1 is the intersection of all the neighbourhoods $U_{x'} \times U'_{x'}$.



Proposition 14.4. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be compact topological spaces. Then the topological space $(X \times Y, \mathcal{O}_{X \times Y})$ is compact.

Proof. Let $\{W_j\}_{j \in J}$ be an open covering of $X \times Y$. For any $x \in X$, let $\{x\} \times Y$ be equipped with its subspace topology $\mathcal{O}_{\{x\} \times Y}$ with respect to $(X \times Y, \mathcal{O}_{X \times Y})$. Then

$$\left\{ W_j \cap \left(\{x\} \times Y \right) \right\}_{j \in J}$$

is an open covering of $\{x\} \times Y$.

By Lemma 7.12 we have that $\left(\{x\} \times Y, \mathcal{O}_{\{x\} \times Y} \right)$ is homeomorphic to $(Y, \mathcal{O}_Y)$. Since Y is compact we have by Corollary 12.11 that $\left(\{x\} \times Y, \mathcal{O}_{\{x\} \times Y} \right)$ is compact. We deduce that there is a finite subset J_x of J such that

$$\left\{ W_j \cap \left(\{x\} \times Y \right) \right\}_{j \in J_x}$$

is an open covering of $\{x\} \times Y$.

Let $W_x = \bigcup_{j \in J_x} W_j$. Then

$$\begin{aligned}\{x\} \times Y \subset W_x &= \bigcup_{j \in J_x} \Big(W_j \cap \big(\{x\} \times Y \big) \Big) \\ &\subset \bigcup_{j \in J_x} W_j \\ &= W_x.\end{aligned}$$

Since Y is compact we deduce by Proposition 14.1 that there is a neighbourhood U_x of x in $(X, \mathcal{O}_X)$ such that $U_x \times Y \subset W_x$. We have that

$$\begin{aligned}X &= \bigcup_{x \in X} \{x\} \\ &\subset \bigcup_{x \in X} U_x.\end{aligned}$$

Hence $X = \bigcup_{x \in X} U_x$. Thus $\{U_x\}_{x \in X}$ is an open covering of X. Since $(X, \mathcal{O}_X)$ is compact we deduce that there is a finite subset K of X such that $\{U_x\}_{x \in K}$ is an open covering of X.

Let $J' = \bigcup_{x \in K} J_x$. We make the following observations.

(1) We have that J_x is finite for every $x \in X$. In particular, J_x is finite for every $x \in K$. Thus J' is finite.

(2) We have that

$$\begin{aligned}X \times Y &= \Big(\bigcup_{x \in K} U_x \Big) \times Y \\ &= \bigcup_{x \in K} \big(U_x \times Y \big) \\ &\subset \bigcup_{x \in K} W_x.\end{aligned}$$

Hence $X \times Y = \bigcup_{x \in K} W_x$. We deduce that

$$\begin{aligned}X \times Y &= \bigcup_{x \in K} W_x \\ &= \bigcup_{x \in K} \Big(\bigcup_{j \in J_x} W_j \Big) \\ &= \bigcup_{j \in J'} W_j.\end{aligned}$$

We conclude that $\{W_j\}_{j \in J'}$ is a finite subcovering of $\{W_j\}_{j \in J}$.

□

Examples 14.5. By Proposition 14.4 we have that $(I^2, \mathcal{O}_{I^2})$ is compact. Thus by Proposition 13.14 we have that all the topological spaces of Examples 3.9 (1) – (5) are compact.

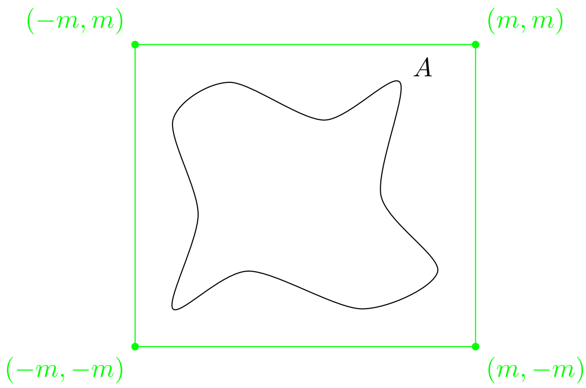
Moreover we have by Examples 4.10 (3) that $D^2 \cong I^2$. By Corollary 12.11 we deduce that $(D^2, \mathcal{O}_{D^2})$ is compact. Hence by Proposition 13.14 we have that the topological space $(S^2, \mathcal{O}_{S^2})$ constructed in Examples 3.9 (6) is compact.

14.2 Characterisation of compact subsets of $\mathbb{R}^n$

Terminology 14.6. Let A be a subset of $\mathbb{R}^n$. Then A is *bounded* if there is an $m \in \mathbb{N}$ such that

$$A \subset \underbrace{[-m, m] \times \ldots \times [-m, m]}_{n}$$

where $[-m, m]$ denotes the closed interval in $\mathbb{R}$ from $-m$ to m.



Remark 14.7. Roughly speaking a subset A of $\mathbb{R}^n$ is bounded if we can enclose it in a box. There are many ways to express this, all of which are equivalent to the definition of Terminology 14.6.

Notation 14.8. Let $m \in \mathbb{N}$. We denote the subset

$$\underbrace{[-m, m] \times \ldots \times [-m, m]}_{n}$$

of $\mathbb{R}^n$ by $[-m, m]^n$.

Let $(-m, m)$ denote the open interval from $-m$ to m in $\mathbb{R}$. We denote the subset

$$\underbrace{(-m, m) \times \ldots \times (-m, m)}_{n}$$

of $\mathbb{R}^n$ by $(-m, m)^n$.

Proposition 14.9. Let A be a subset of $\mathbb{R}^n$ equipped with its subspace topology $\mathcal{O}_A$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. Then $(A, \mathcal{O}_A)$ is compact if and only if A is bounded and closed in $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.

Proof. Suppose that A is bounded and closed in $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. Since A is bounded there is an $m \in \mathbb{N}$ such that $A \subset [-m, m]^n$. Let $\mathcal{O}_{[-m,m]^n}$ denote the subspace topology on $[-m, m]^n$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. We make the following observations.

(1) The subspace topology $\mathcal{O}_A$ on A with respect to $(X, \mathcal{O}_X)$ is equal to the subspace topology on A with respect to $\big([-m, m]^n, \mathcal{O}_{[-m,m]^n}\big)$.

(2) By Corollary 13.5 we have that $[-m, m]$ is a compact subset of $\mathbb{R}$. By Proposition 14.4 and induction we deduce that $[-m, m]^n$ is a compact subset of $\mathbb{R}^n$.

(3) Since A is closed in $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ we have by Question 1 (c) of Exercise Sheet 4 that A is closed in $\big([-m, m]^n, \mathcal{O}_{[-m,m]^n}\big)$.

We deduce by Proposition 13.7 that $(A, \mathcal{O}_A)$ is compact.

Suppose instead now that $(A, \mathcal{O}_A)$ is compact. By Examples 11.7 (1) and Proposition 11.11 we have that $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is Hausdorff. We deduce by Proposition 13.11 that A is closed.

The set $\big\{A \cap (-m, m)^n\big\}_{m \in \mathbb{N}}$ is an open covering of A. Thus since $(A, \mathcal{O}_A)$ is compact there is a finite subset J of $\mathbb{N}$ such that $\big\{A \cap (-m, m)^n\big\}_{m \in J}$ is an open covering of A. Let m' be the largest natural number which belongs to J. Then

$$\begin{aligned} A &= \bigcup_{m \in J} \big(A \cap (-m, m)^n\big) \\ &\subset \bigcup_{m \in J} (-m, m)^n \\ &\subset \bigcup_{m \in J} (-m', m')^n \\ &= (-m', m')^n. \end{aligned}$$

Thus A is bounded. □

Corollary 14.10. Let $(X, \mathcal{O}_X)$ be a compact topological space. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. There is an $x \in X$ such that $f(x) = \inf f(X)$ and an $x' \in X$ such that $f(x') = \sup f(X)$. Equivalently we have that

$$f(x) \leq f(x'') \leq f(x')$$

for all $x'' \in X$.

Proof. Since $(X, \mathcal{O}_X)$ is compact we have by Proposition 12.10 that $f(X)$ is a compact subset of $\mathbb{R}$. By Proposition 14.9 we deduce that $f(X)$ is closed in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ and bounded.

Since $f(X)$ is bounded we have that $\sup f(X) \in \mathbb{R}$ and $\inf f(X) \in \mathbb{R}$. Thus by Question 5 (a) and (b) on Exercise Sheet 4 we have that $\sup f(X)$ and $\inf f(X)$ are limit points of $f(X)$ in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Since $f(X)$ is closed in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ we deduce by Proposition 5.7 that $\sup f(X)$ belongs to $f(X)$ and that $\inf f(X)$ belongs to $f(X)$. □

Remark 14.11. You will have met Corollary 14.10 when X is a closed interval in $\mathbb{R}$ in real analysis/calculus, where it is sometimes known as the 'extreme value theorem'!

14.3 Locally compact topological spaces

Definition 14.12. A topological space $(X, \mathcal{O}_X)$ is *locally compact* if for every $x \in X$ and every neighbourhood U of x in $(X, \mathcal{O}_X)$ there is a neighbourhood U' of x in $(X, \mathcal{O}_X)$ such that the following hold.

(1) The closure $\overline{U'}$ of U in $(X, \mathcal{O}_X)$ is a compact subset of X.

(2) We have that $\overline{U'} \subset U$.

Examples 14.13.

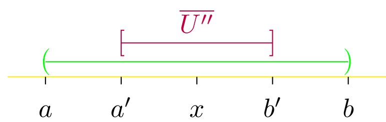
(1) The topological space $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally compact, whereas in Examples 12.9 (2) we saw that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not compact. Let us prove that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally compact.

Let $x \in \mathbb{R}$ and let U be a neighbourhood of x in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By definition of $\mathcal{O}_{\mathbb{R}}$ we have that

$$\{(a,b) \mid a, b \in \mathbb{R}\}$$

is a basis of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Question 3 (a) of Exercise Sheet 2 we deduce that there is an open interval (a,b) in $\mathbb{R}$ such that $x \in (a,b)$ and $(a,b) \subset U$.

Let $a' \in \mathbb{R}$ be such that $a < a' < x$. Let $b' \in \mathbb{R}$ be such that $x < b' < b$. Let $U' = (a', b')$. By Question 5 (c) of Exercise Sheet 4 we have that $\overline{U'} = [a', b']$. Here $\overline{U'}$ denotes the closure of U' in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. In particular we have that $\overline{U} \subset (a,b) \subset U$.



By Corollary 13.5 we have that $[a,b]$ is a compact subset of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Putting everything together we have that:

(i) $U' \in \mathcal{O}_{\mathbb{R}}$

(ii) $x \in U'$

(iii) $\overline{U'} \subset U$

(iv) $\overline{U'}$ is a compact subset of $\mathbb{R}$.

(2) Let X be a set and let $\mathcal{O}^{\text{disc}}$ be the discrete topology on X. Then $(X, \mathcal{O}^{\text{disc}})$ is locally compact. Let us prove this.

Let $x \in X$ and let U be a neighbourhood of x in $(X, \mathcal{O}^{\text{disc}})$. We make the following observations.

(i) $\{x\} \subset U$.

(ii) $\{x\}$ is open in $(X, \mathcal{O}^{\text{disc}})$.

(iii) $\{x\}$ is closed in $(X, \mathcal{O}^{\text{disc}})$ since $X \setminus \{x\}$ is open in $(X, \mathcal{O}^{\text{disc}})$. By Proposition 5.7 we deduce that $\overline{\{x\}} = \{x\}$.

Thus $(X, \mathcal{O}^{\text{disc}})$ is locally compact.

By contrast, if X is infinite then $(X, \mathcal{O}_X)$ is not compact. For

$$\big\{\{x\}\big\}_{x \in X}$$

is an open covering of X which if X is infinite has no finite subcovering.

(3) A product of locally compact topological spaces is locally compact. This is left as an exercise. Thus $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is locally compact.

(4) Let $(X, \mathcal{O}_X)$ be a locally compact topological space. Let U be an open subset of X equipped with its subspace topology $\mathcal{O}_U$ with respect to $(X, \mathcal{O}_X)$. Then $(U, \mathcal{O}_U)$ is locally compact. This is left as an exercise.

By (3) we conclude that any 'open blob' in $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is locally compact.

(5) Let $(X, \mathcal{O}_X)$ be a locally compact topological space. Let A be a closed subset of X equipped with its subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Then $(A, \mathcal{O}_A)$ is locally compact. This is left as an exercise.

Proposition 14.14. Let $(X, \mathcal{O}_X)$ be a compact Hausdorff topological space. Then $(X, \mathcal{O}_X)$ is locally compact.

Proof. Let $x \in X$ and let U be a neighbourhood of x in $(X, \mathcal{O}_X)$. Since U is open in $(X, \mathcal{O}_X)$ we have that $X \setminus U$ is closed in $(X, \mathcal{O}_X)$. Since $(X, \mathcal{O}_X)$ is compact we deduce by Proposition 13.7 that $X \setminus U$ is a compact subset of X.

Since $(X, \mathcal{O}_X)$ is Hausdorff we deduce by Lemma 13.9 that there are open subsets U' and U'' of X with the following properties:

(1) $X \setminus U \subset U'$,

(2) $x \in U''$,

(3) $U' \cap U'' = \emptyset$.

By (3) and Question 1 (f) of Exercise Sheet 4 we have that $U' \cap \overline{U''} = \emptyset$. Here $\overline{U''}$ is the closure of U'' in $(X, \mathcal{O}_X)$. We deduce by appeal to (1) that

$$\begin{aligned}\overline{U''} &\subset X \setminus U' \\ &\subset X \setminus (X \setminus U) \\ &= U.\end{aligned}$$

By Proposition 13.7 we have that $\overline{U''}$ is closed in $(X, \mathcal{O}_X)$. Since $(X, \mathcal{O}_X)$ is compact we deduce by Proposition 13.7 that $\overline{U''}$ is a compact subset of X. Putting everything together we have the following.

(1) $U'' \in \mathcal{O}_X$.

(2) $x \in U''$.

(3) $\overline{U''} \subset U$.

(4) $\overline{U''}$ is a compact subset of X.

Thus $(X, \mathcal{O}_X)$ is locally compact. □

Example 14.15. Let $n \geq 1$. We make the following observations.

(1) By Proposition 13.2, Proposition 14.4, and induction we have that $(I^n, \mathcal{O}_{I^n})$ is compact.

(2) By Examples 11.7 (1) we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. We deduce that $(I, \mathcal{O}_I)$ is Hausdorff by Proposition 11.10. Thus by Proposition 11.11 we have that $(I^n, \mathcal{O}_{I^n})$ is Hausdorff.

We conclude by Proposition 14.14 that $(I^n, \mathcal{O}_{I^n})$ is locally compact. We can also see this by appealing to Examples 14.13 (1), (3), and (5).

14.4 Topological spaces which are not locally compact

Example 14.16. Let $\mathbb{Q}$ be equipped with its subspace topology $\mathcal{O}_{\mathbb{Q}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not locally compact. Let us prove this.

Let $q \in \mathbb{Q}$ and let U be a neighbourhood of q in $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$. By definition of $\mathcal{O}_{\mathbb{Q}}$ there is a $U' \in \mathcal{O}_{\mathbb{R}}$ such that $U = \mathbb{Q} \cap U'$. By definition of $\mathcal{O}_{\mathbb{R}}$ we have that

$$\big\{(a, b) \mid a, b \in \mathbb{R}\big\}$$

is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. We deduce by Question 3 (a) of Exercise Sheet 2 that there are $a, b \in \mathbb{R}$ such that $q \in (a, b)$ and $(a, b) \subset U'$.

Suppose that $\overline{U}$ is a compact subset of $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$. The subspace topology on $\overline{U}$ with respect to $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is equal to the subspace topology on $\overline{U}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Thus we have that $\overline{U}$ is a compact subset of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

By Examples 11.7 (1) we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. We deduce by Proposition 13.11 that $\overline{U}$ is closed in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Since $(a, b) \subset U'$ we have that

$$\overline{\mathbb{Q} \cap (a, b)} \subset \overline{\mathbb{Q} \cap U'} = \overline{U},$$

where $\overline{\mathbb{Q} \cap (a, b)}$ denotes the closure of $\mathbb{Q} \cap (a, b)$ in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

By Question 5 (d) of Exercise Sheet 4 we have that $\overline{\mathbb{Q} \cap (a, b)} = [a, b]$. We deduce that $[a, b] \subset \overline{U}$. Since $[a, b]$ contains irrational numbers this contradicts the fact that $\overline{U} \subset \mathbb{Q}$.

We conclude that no neighbourhood of q in $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is compact. Hence $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not locally compact.