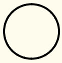
17 Tuesday 12th March

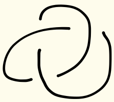
17.1 Knots and links – definitions and examples

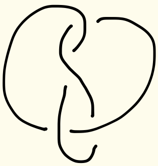
Terminology 17.1 A **subspace** of $\mathbb{R}^3$ is a subset X of $\mathbb{R}^3$ equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$.

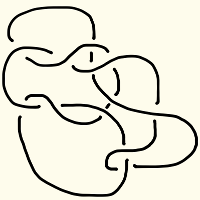
Definition 17.2 A **knot** is a subspace $(K, \mathcal{O}_K)$ of $\mathbb{R}^3$ which is homeomorphic to $(S^1, \mathcal{O}_{S^1})$.

Examples 17.3

(1) Unknot. 

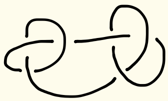
(2) Trefoil. 

(3) Figure of eight. 

(4)  This can in fact be unknotted to  !!!

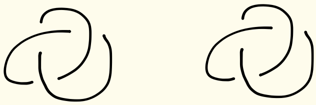
In the language we will introduce shortly, this knot is 'isotopic' to the unknot.

(5) Granny knot.

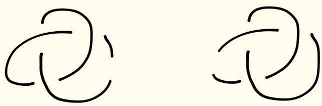


This is the 'connected sum' of two trefoil knots. By a 'connected sum' we mean the following construction.

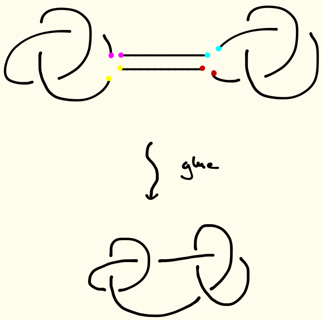
(i) Begin with two trefoils.



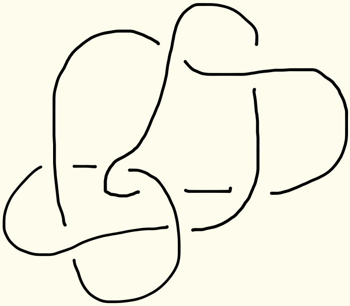
(ii) Cut a small piece out of each trefoil.



(iii) Connect the two trefoils by gluing in a disjoint pair of line segments.



(6) True lovers' knot.



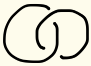
Can you see where the name comes from? Look for two hearts!

Definition 17.4 A **link** is a subspace $(L, \mathcal{O}_L)$ of $\mathbb{R}^3$ which is homeomorphic to a finite disjoint union of copies of $(S^1, \mathcal{O}_{S^1})$.

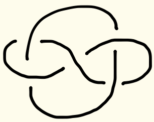
Remark 17.5 In particular, a knot is a link.

Examples 17.6

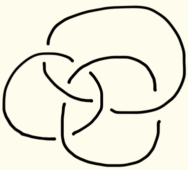
(1) Hopf link.



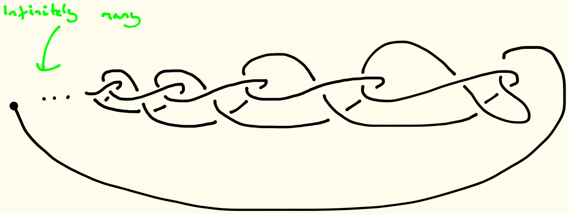
(2) Whitehead link.



(3) Borromean rings.



Remark 17.7 In our study of knots and links we must impose some restrictions to exclude 'wild knots' such as the following.

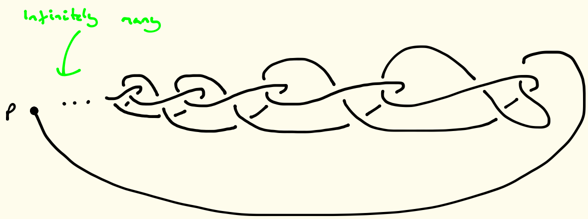


Note this knot would be able to be untied (in the language we are about to introduce, it would be isotopic to the unknot) if it were not for the fact that this would take an infinite length of time!

This kind of issue arises in many places in topology. There are two common ways to exclude this kind of behaviour.

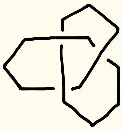
i) We could require that there is a 'smooth' homeomorphism between our knots and $(S^1, \mathcal{O}_{S^1})$, and similarly require that there is a 'smooth' homeomorphism between our links and $(\coprod_n S^1, \mathcal{O}_{\coprod_n S^1})$ for some $n \geq 1$.

This excludes the above example from being a knot, since it is not smooth at the point p indicated below.



Topology under this kind of smoothness assumption is studied in several courses here: MA3402 Analyse på manifoldigheter, TMA4190 Manifoldigheter, MA8402 Lie-grupper og Lie-algebraer.

ii) We could require that our knots and links be able to be obtained by glueing together finitely many line segments. Here are a few examples.



Trefoil

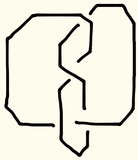
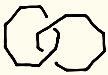


Figure of eight



Hopf link

This excludes our 'wild' example from being a knot, since we can only obtain it by glueing together infinitely many line segments.

Topology under this kind of assumption is known as 'piecewise linear topology'.

I will implicitly take approach (ii), but don't worry about this - just be aware that we have to make some assumption to exclude 'wild' knots and links.

17.2 Isotopy

Definition 17.8 Let (K, θ_K) and $(K', \theta_{K'})$ be knots. Then K is *isotopic* to K' if there exists a continuous map $S^1 \times I \xrightarrow{f} \mathbb{R}^3$ such that the following conditions are satisfied:

(i) for all $t \in I$, the map $S^1 \xrightarrow{f_t} \mathbb{R}^3$ given by $x \mapsto f(x,t)$ is a homeomorphism,

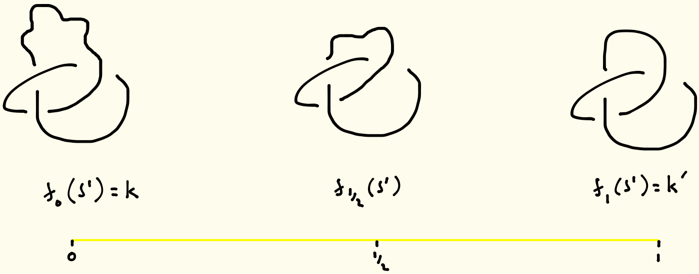
(ii) $f_0(S^1) = K$,

(iii) $f_1(S^1) = K'$.

Here as usual I denotes the unit interval, and $S^1 \times I$ is equipped with the product topology.

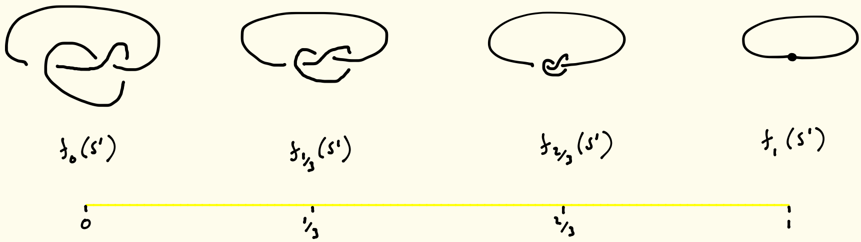
Example 17.9 We can think of an isotopy $S^1 \times I \xrightarrow{f} \mathbb{R}^3$ from a knot (K, θ_K) to a knot $(K', \theta_{K'})$ as a 'movie' of knots beginning with (K, θ_K) and ending with $(K', \theta_{K'})$.

See the next page.



We have seen these kinds of 'movies' earlier in the lectures - look back at Examples 2.13 (2) for example.

Remark 17.10 As in Remark 17.6, we must exclude certain phenomena. For example, we do not wish to allow a knot to be 'pulled tight' as indicated below.



This would mean that every knot would be isotopic to the unknot!

Just as in Remark 17.7, we can exclude this by requiring that our isotopy $S^1 \times I \xrightarrow{f} \mathbb{R}^3$ be 'smooth' or 'piecewise linear'. I will implicitly take f to be piecewise linear.

Don't worry about this! Just as in Remark 17.7, the important thing is to be aware that an additional assumption is necessary.

These remarks apply equally to the following definition.

Definition 17.10 Let $(L, \mathcal{O}_L)$ and $(L', \mathcal{O}_{L'})$ be links. Then L is *isotopic* to L' if for some integer $n \geq 0$ there is a continuous map $\left(\coprod_n S^1\right) \times I \xrightarrow{f} \mathbb{R}^3$ such that the following conditions are satisfied:

(i) for all $t \in I$, the map $\coprod_n S^1 \xrightarrow{f_t} \mathbb{R}^3$ given by $x \mapsto f(x,t)$ is a homeomorphism,

(ii) $f_0\left(\coprod_n S^1\right) = L$,

(iii) $f_1\left(\coprod_n S^1\right) = L'$.

Remark 17.11 Just as in Example 17.9, the isotopy $\left(\coprod_n S^1\right) \times I \xrightarrow{f} \mathbb{R}^3$ can be thought of as a 'movie' of links beginning with L and ending with L'.

Intuitively, two knots or links are isotopic if we can manipulate one to obtain the other in our everyday sense — a knot is isotopic to the unknot if we can untie it in our everday sense!

17.3 Link diagrams and Reidemeister moves

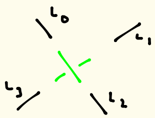
Definition 17.12 A *link diagram* consists of the following data:

(1) A set Arc, whose elements we think of as labels for 'line segments' or 'arcs'.

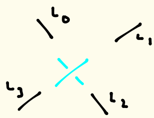
(2) A set OverCrossing, whose elements we think of as labels for 'over crossings'.

(3) A set UnderCrossing, whose elements we think of as labels for 'under crossings'.

(4) A map $\text{OverCrossing} \xrightarrow{d^{\text{Over}}} \text{Arc} \times \text{Arc} \times \text{Arc} \times \text{Arc}$, which we think of as assigning to an over crossing  four arcs (l_0, l_1, l_2, l_3) as follows.

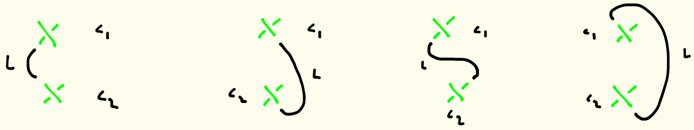


(5) A map $\text{UnderCrossing} \xrightarrow{d^{\text{Under}}} \text{Arc} \times \text{Arc} \times \text{Arc} \times \text{Arc}$, which we think of as assigning to an under crossing  four arcs (l_0, l_1, l_2, l_3) as follows.



We require that for every arc $L \in Arc$, one of the following three possibilities holds:

(i) there are exactly two over crossings $c_1, c_2 \in OverCrossing$ such that L is one of the arcs in $d^{Over}(c_1)$ and L is one of the arcs in $d^{Over}(c_2)$.

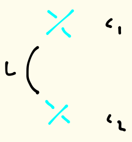


etc.

We allow that $c_1 = c_2$.

etc.

(ii) there are exactly two under crossings $c_1, c_2 \in UnderCrossing$ such that L is one of the arcs in $d^{Under}(c_1)$ and L is one of the arcs in $d^{Under}(c_2)$.

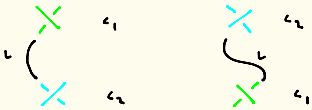


etc.

We allow that $c_1 = c_2$.

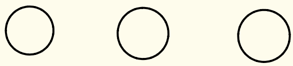
etc.

(iii) there is exactly one crossing $c_1 \in$ OverCrossing and exactly one crossing $c_2 \in$ UnderCrossing such that l is one of the arcs in $\partial^{\mathrm{Over}}(c_1)$ and l is one of the arcs in $\partial^{\mathrm{Under}}(c_2)$.



etc.

We allow both OverCrossings and UnderCrossings to be the empty set. In this case we think of Arc as a set of labels for a disjoint collection of circles.



Remark 17.13 Think of Definition 17.12 as like lego: we have three kinds of 'piece' (arcs, over crossings, and under crossings) which we join together in a prescribed way to build up our link diagram.

In particular, though I've drawn pictures to help our intuition, a link diagram is independent of any particular picture - a computer can understand it, for instance!

Construction 17.14 To any link (L, θ_L) we can associate a link diagram. This construction will not be asked about on the exam, but I will discuss it in a supplementary note for those who are interested.

Intuitively, the diagram associated to a link can be thought of as its projection onto a plane as shown below, together with the information as to whether a crossing is an over crossing or an under crossing.

+ remember
crossing information

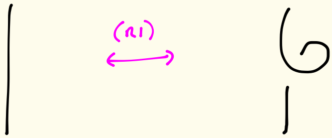
Definition 17.15 We consider three Reidemeister moves upon link diagrams.

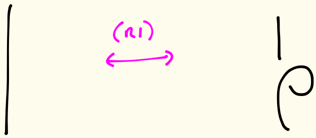
(R1) We replace an arc | by three arcs and an over crossing

or by three arcs and an under crossing

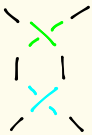
and vice versa.

We depict this as follows.

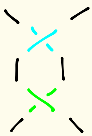




(R2) We replace a pair of arcs || by six arcs, an over crossing, and an under crossing as follows

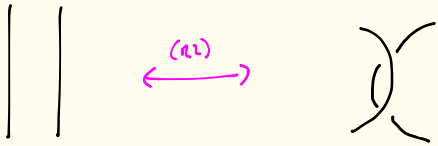


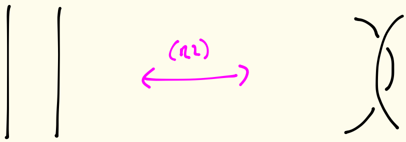
or as follows



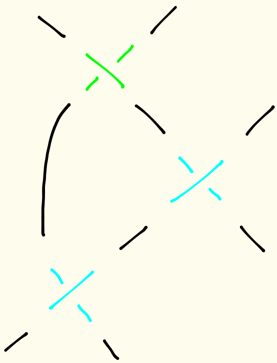
and vice versa.

We depict this as follows.

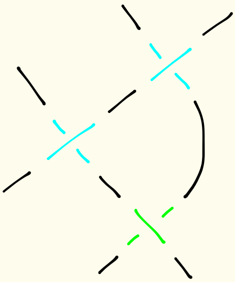




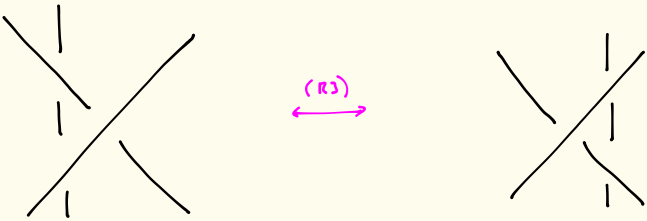
(R3) We replace a configuration of arcs and crossings as follows



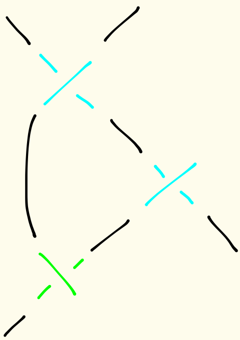
by a configuration of arcs and crossings as follows and vice versa.



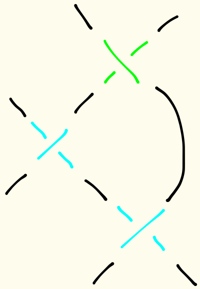
We depict this as follows.



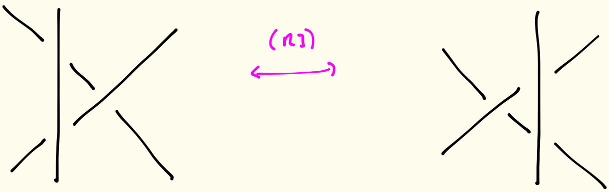
Or we replace a configuration of arcs and crossings as follows



by a configuration of arcs and crossings as follows and vice versa.



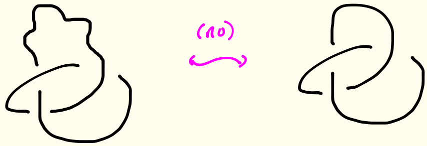
We depict this as follows.



Theorem 17.16 Let $(L, \mathcal{O}_L)$ and $(L', \mathcal{O}_{L'})$ be links, and let D_L and $D_{L'}$ be their associated link diagrams. Then L is isotopic to L' if and only if $D_{L'}$ can be obtained from D_L by a finite sequence of Reidemeister moves.

Remark 17.17 This is a true theorem! A priori all kinds of wild phenomena could occur when working in $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$. A link diagram is a much simpler gadget! The theorem will allow us to define several knot/link 'invariants'. These invariants will allow us to prove for example that various knots cannot be untied, i.e., are not isotopic to the unknot.

Remark 17.18 When drawing knots and links, let us also refer to a change in our picture corresponding to an isotopy which does not affect crossings as an (R_0)-move.



Let us refer to a change in our picture which corresponds to moving an arc under/over the entire rest of our link as an (R_∞)-move.

⚠ Neither an (R_0)-move nor an (R_∞)-move has any meaning for a link diagram! Rather they correspond to the fact that a link diagram is independent of any picture we choose to draw of the link from which it comes, as discussed in Remark 17.13.