Generell Topologi

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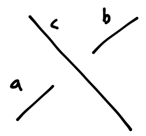
19 Tuesday 19th March

19.1 Colourability

Definition 19.1. Let $(L, \mathcal{O}_L)$ be a link, and let $m \in \mathbb{N}$. Then L is *m-colourable* if we can assign an integer to every arc in L in such a way that the following hold.

(1) At every crossing the assigned integers



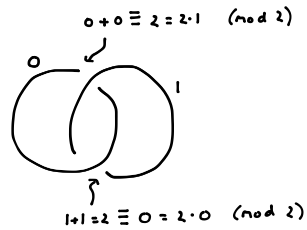
have the property that $a + b \equiv 2c \pmod{m}$.

(2) Not every arc is assigned the same integer mod m.

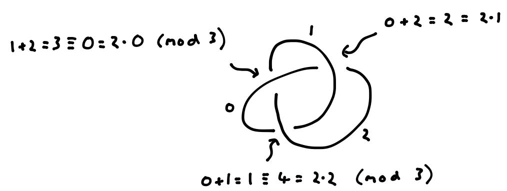
Remark 19.2. No link is 1-colourable! For every integer z we have that $z \equiv 0$ mod 1, and thus condition (2) can never be satisfied for $m = 1$.

Examples 19.3.

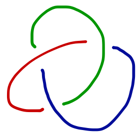
(1) The Hopf link is 2-colourable. An example of a 2-colouring along with a verification that condition (1) holds at both crossings is shown below.



(2) The trefoil knot is 3-colourable. An example of a 2-colouring along with a verification that condition (1) holds at every crossing is shown below.

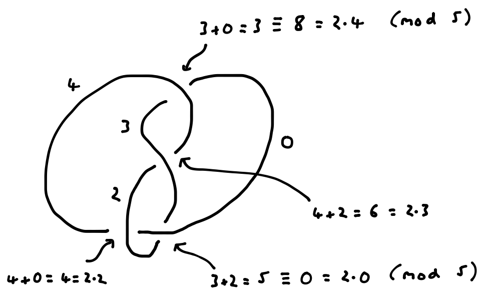


To give a 3-colouring it is equivalent to draw the arcs using three colours, such that every crossing either the three arcs have three different colours or else the three arcs have the same colour.



(3) Here is another example of a 3-colouring.

(4) The figure of eight knot is 5-colourable. An example of a 5-colouring along with the verification that condition (1) holds at every crossing is shown below.

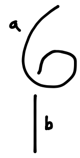


Remark 19.4. There can be many different ways to equip a link with an m-colouring.

Proposition 19.5. Let $(L, \mathcal{O}_L)$ and $(L', \mathcal{O}_{L'})$ be links. If L is isotopic to L' then L is m-colourable for an integer m if and only if L' is m-colourable.

Proof. We know by Theorem 17.16 that two links are isotopic if and only if one can be obtained from the other by a finite sequence of Reidemeister moves. Thus it suffices to prove that whether or not a link is m-colourable is unaffected by applying the Reidemeister moves.

R1 Consider the assigned integers in a part of a link which looks as follows.

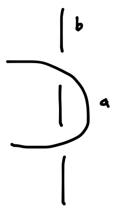


By condition (1) for an m-colouring we have that $a + b \equiv 2a \pmod{m}$. This implies that $a \equiv b \pmod{m}$.

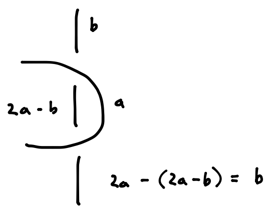
Thus we may replace this part of our link with

without affecting whether or not our link is m-colourable.

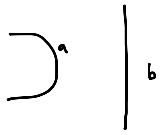
R2 Suppose that in a part of a link which looks as follows we have that two of the assigned integers a and b are as shown.



By condition (1) for an m-colouring with respect to the two crossings, the two remaining arcs must be assigned the following integers mod m.

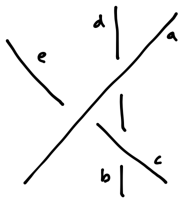


Thus we may replace this part of our link with

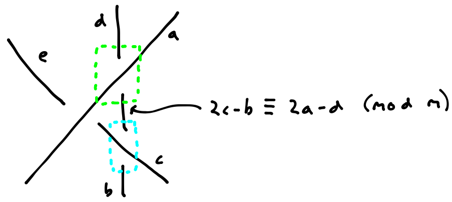


without affecting whether or not our link is m-colourable.

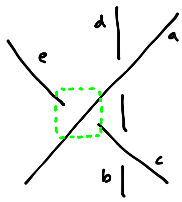
R3 Suppose that in a part of a link which looks as follows we have that five of the assigned integers a, b, c, d, and e are as shown.



By condition (1) for an m-colouring with respect to the two crossings indicated below we have that the integer assigned to the remaining arc must be equal to both $2c - b$ and $2a - d$ mod m.



Moreover by condition (1) for an m-colouring applied to the crossing indicated below we have that $c + e \equiv 2a \pmod{m}$.



Putting these two observations together we have that the following hold mod m.

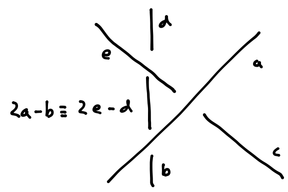
$$2c - b \equiv 2a - d$$
$$c + e \equiv 2a$$

By the second equation we deduce that $c \equiv 2a - e \pmod{m}$. By the first equation we deduce that

$$2 \cdot (2a - e) - b \equiv 2a - d \pmod{m},$$

and hence that $2a - b \equiv 2e - d \pmod{m}$.

Thus we may replace this part of our link with



without affecting whether or not our link is m-colourable.

$\square$

Remark 19.6. This proof is not quite complete. There is another R2 and another R3 move which must be considered. However, just as in the proof of Proposition 18.14 it is the idea that is important. It adapts in a straightforward way to a proof for the other cases.

Examples 19.7.

(1) It is intuitive that the trefoil cannot be unknotted! Proposition 19.5 allows us to give a rigorous proof of this.

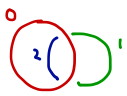
By Examples 19.3 (2) we have that the trefoil is 3-colourable. The unknot cannot be m-colourable for any m, since condition (2) clearly cannot be satisfied! In particular, it is not 3-colourable.

(2) Similarly by Examples 19.3 (4) we have that the figure of eight knot is 5-colourable. Thus it cannot be isotopic to the unknot.

(3) We have to be careful! The unlink with two 2-components is 2-colourable.

Thus we cannot deduce from Examples 19.3 (1) and Proposition 19.5 that the Hopf link is not isotopic to the unlink with two components, which we already proved using its linking number in Examples 18.16.

(4) The unlink with two components is also 3-colourable.



We will see shortly that this will allow us to prove that the Whitehead link is not isotopic to the unlink with two components, which we could not prove using linking numbers.

Lemma 19.8. Let $(L, \mathcal{O}_L)$ be an m-colourable link for $m \in \mathbb{N}$. Let $k \in \mathbb{Z}$. For any arc A of L we can find an m-colouring of L in which A is assigned the integer k.

Proof. Let a, b, c be integers. If $a+b \equiv 2c \pmod{m}$, then for any integer l we have that

$$(a+l)+(b+l) = a+b+2l \equiv 2c+2l = 2(c+l) \pmod{m}.$$

Thus given any m-colouring of L we obtain another m-colouring of by adding l to the integer assigned to every arc.

Let z be the integer assigned to A in a given m-colouring of L. Adding $k-z$ to every arc we obtain another m-colouring. In this m-colouring the integer assigned to A is $z+(k-z)=k$, as required. □

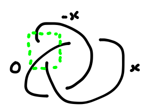
Examples 19.9.

(1) To see the full power of colourability we need to be able to determine for which integers m a given link is m-colourable.

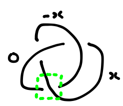
We have already seen that the trefoil is 3-colourable. By part of the argument in the proof of Lemma 19.8 it follows that the trefoil is m-colourable for any $m \in \mathbb{Z}$ with $3 \mid m$. Let us now prove that if the trefoil is m-colourable then $m \equiv 0 \pmod{3}$.

Suppose that we have an m-colouring of the trefoil. By Lemma 19.8 we can fix the integer assigned to one of the arcs to be 0. Let us denote the integer assigned to one of the other arcs by x.

By condition (1) for an m-colouring applied to the indicated crossing the integer assigned to the third arc must be equal to $-x$ mod m.



By condition (1) applied to the indicated crossing we have that $-x \equiv 2x \pmod{m}$, and hence that $3x \equiv 0 \pmod{m}$.



We thus have that $3x = km$ for some $k \in \mathbb{Z}$. Since 3 is prime we deduce that either $3 \mid k$ or $3 \mid m$. If $3 \mid k$ then $x \equiv 0 \pmod{m}$, and hence $-x \equiv 0 \pmod{m}$. Thus our colouring would be

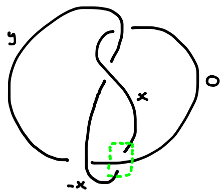
which would contradict condition (2) for an m-colouring.

We deduce that $3 \mid m$.

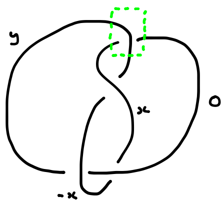
(2) Let us prove in a similar way that the figure of eight knot is m-colourable if and only if $m \equiv 0 \pmod{5}$.

Suppose that we have an m-colouring of the figure of eight knot. By Lemma 19.8 we can fix the integer assigned to one of the arcs to be 0. Let us denote the integer assigned to two of the other arcs by x and y.

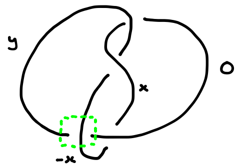
By condition (1) for an m-colouring applied to the crossing indicated below, the integer assigned to the arc shown must be equal to $-x$ mod m.



By condition (1) for an m-colouring applied to the crossing indicated below, we must have that $x \equiv 2y \pmod{m}$.

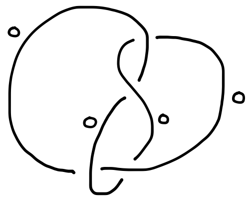


By condition (1) for an m-colouring applied to the crossing indicated below, we have that $y \equiv -2x \pmod{m}$.



Together the fact that $x \equiv 2y \pmod{m}$ and $y \equiv -2x \pmod{m}$ implies that $x \equiv -4x \pmod{m}$, and hence that $5x \equiv 0 \pmod{m}$.

We thus have that $5x = km$ for some $k \in \mathbb{Z}$. Since 5 is prime we deduce that either $5 \mid k$ or $5 \mid m$. If $5 \mid k$ then $x \equiv 0 \pmod{m}$, and hence $-x \equiv 0 \pmod{m}$ and $y \equiv 2 \cdot 0 = 0 \pmod{m}$. Thus our colouring would be



which would contradict condition (2) for an m-colouring.

We deduce that $5 \mid m$.

(3) By (1), (2), and Proposition 19.5 we can conclude that the trefoil is not isotopic to the figure of eight knot.