Generell Topologi

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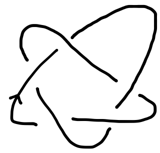
22 Tuesday 9th April

22.1 Skein relations, continued

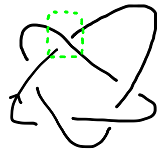
Examples 22.1.

(4) Let us use the skein relations to calculate the Jones polynomial of the knot 5_1, known as the cinquefoil.

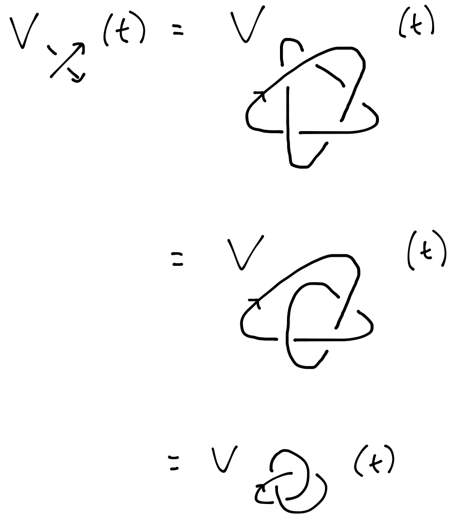


The argument illustrates the inductive nature of a calculation using the skein relations: we apply the skein relations until we encounter an oriented link whose Jones polynomial we already know.

Let us work at the indicated crossing.

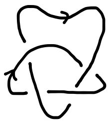


We have the following, appealing to Examples 21.14 (4) for the last equality.

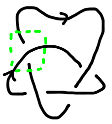


$$= t + t^3 - t^4$$

We also have the following.

V  $(t) = V$  (t)

We have not yet calculated this Jones polynomial. Thus we must interrupt our main calculation in order to use the skein relations to work it out. Let us work at the indicated crossing.



We have the following, appealing to Examples 21.14 (3) for the last equality.

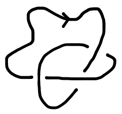
V  $(t) = V$  (t)

$= V$  (t)

$= V$  (t)

$$= -t^{5/2} - t^{1/2}$$

We also have the following, appealing to Examples 21.14 (4) for the last equality.

V  $(t) = V$  (t)

$= V$  (t)

$= t + t^3 - t^4$

Thus by the second skein relation we have the following.

$t^{-1} V$  $(t) - t\left(-t^{5/2} - t^{1/2}\right) = \left(t^{1/2} - t^{-1/2}\right)\left(t + t^3 - t^4\right)$

$\Rightarrow t^{-1} V$  $(t) + t^{7/2} + t^{3/2} = t^{3/2} + t^{7/2} - t^{9/2} - t^{1/2} - t^{5/2} + t^{7/2}$

$\Rightarrow t^{-1} V$  $(t) = -t^{1/2} - t^{5/2} + t^{7/2} - t^{9/2}$

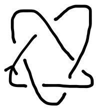
$\Rightarrow V$  $(t) = -t^{3/2} - t^{7/2} + t^{9/2} - t^{11/2}$

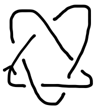
We can now complete our main calculation. We have the following.

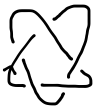
V  $(t) = V$  (t)

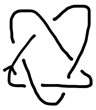
$= -t^{3/2} - t^{7/2} + t^{9/2} - t^{11/2}$

Thus by the second skein relation we have the following.

$t^{-1} V$  $(t) - t(t + t^3 - t^4) = (t^{1/2} - t^{-1/2})(-t^{3/2} - t^{7/2} + t^{9/2} - t^{11/2})$

$\Rightarrow t^{-1} V$  $(t) - t^2 - t^4 + t^5 = -t^2 - t^4 + t^5 - t^6 + t + t^3 - t^4 + t^5$

$\Rightarrow t^{-1} V$  $(t) = t + t^3 - t^4 + t^5 - t^6$

$\Rightarrow V$  $(t) = t^2 + t^4 - t^5 + t^6 - t^7$

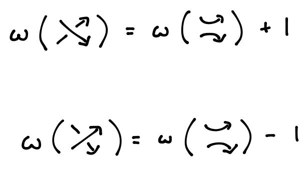
Proposition 22.2. The Jones polynomial for oriented links satisfies the second skein relation of Definition 21.12.

Proof. Throughout this proof we will adopt Notation 20.19. Let us first make several observations.

(1) By definition of $\langle L\rangle$ for a link $(L, \mathcal{O}_L)$ we have the following.

$$\begin{aligned} A\langle \text{⤫} \rangle - A^{-1}\langle \text{⤬} \rangle &= A\left(A\langle \asymp \rangle + A^{-1}\langle)(\rangle\right) \\ &\quad - A^{-1}\left(A\langle)(\rangle + A^{-1}\langle \asymp \rangle\right) \\ &= A^2\langle \asymp \rangle + \langle)(\rangle - \langle)(\rangle - A^{-2}\langle \asymp \rangle \\ &= (A^2 - A^{-2})\langle \asymp \rangle \end{aligned}$$

(2) We have the following.



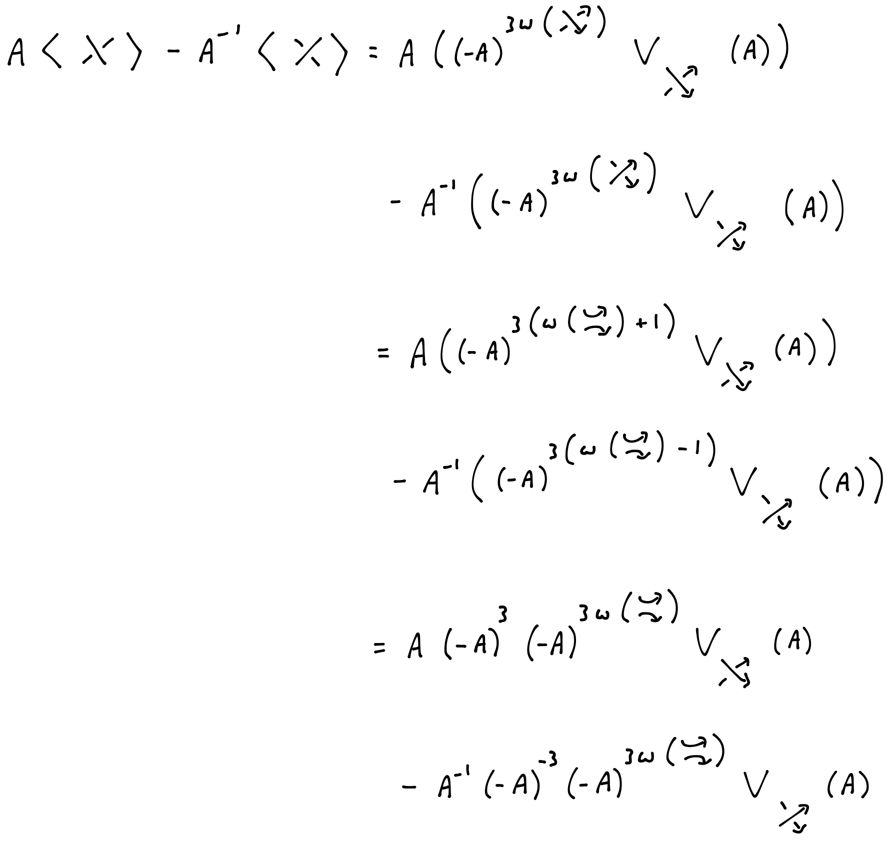
(3) By definition of $V_L(A)$ for an oriented link $(L, \mathcal{O}_L)$ we have that

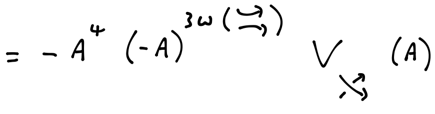
$$V_L(A) = (-A)^{-3w(L)}\langle L\rangle.$$

Thus we have that

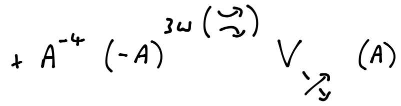
$$\langle L\rangle = (-A)^{3w(L)}V_L(A).$$

We now make the following calculation, appealing to (3) for the first equality and to (2) for the second equality.

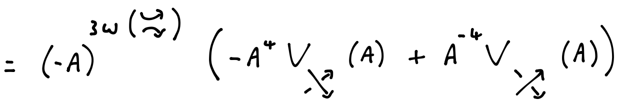




$$= -A^{4}\,(-A)^{3\omega(\;)}\,V\,(A)$$

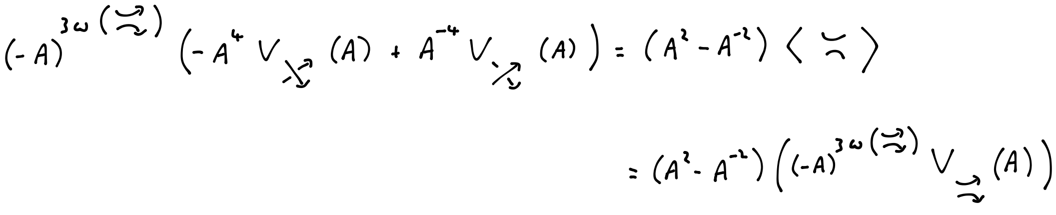


$$+\,A^{-4}\,(-A)^{3\omega(\;)}\,V\,(A)$$



$$= (-A)^{3\omega(\;)}\left(-A^{4}V\,(A) + A^{-4}V\,(A)\right)$$

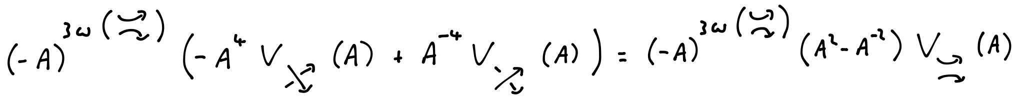
We deduce that the following holds, appealing to (1) for the first equality and to (3) for the second equality.



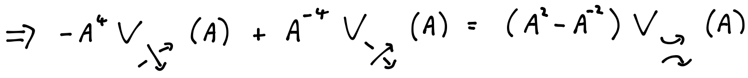
$$(-A)^{3\omega(\;)}\left(-A^{4}V\,(A) + A^{-4}V\,(A)\right) = (A^{2}-A^{-2})\,\langle\;\rangle$$

$$= (A^{2}-A^{-2})\left((-A)^{3\omega(\;)}\,V\,(A)\right)$$

Thus we have the following.

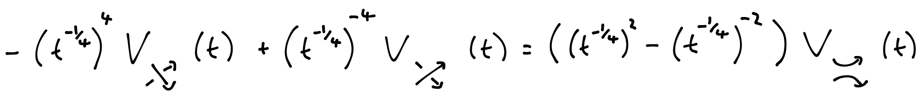


$$(-A)^{3\omega(\;)}\left(-A^{4}V\,(A) + A^{-4}V\,(A)\right) = (-A)^{3\omega(\;)}(A^{2}-A^{-2})\,V\,(A)$$

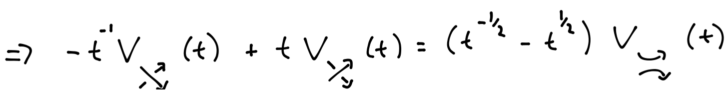


$$\Rightarrow\; -A^{4}V\,(A) + A^{-4}V\,(A) = (A^{2}-A^{-2})\,V\,(A)$$

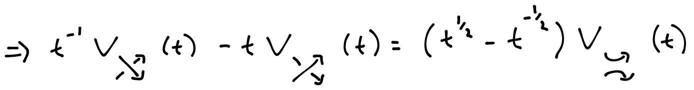
Thus we have the following, as required.



$$-\left(t^{-1/4}\right)^{4}V\,(t) + \left(t^{-1/4}\right)^{-4}V\,(t) = \left((t^{-1/4})^{2}-(t^{-1/4})^{-2}\right)V\,(t)$$



$$\Rightarrow\; -t^{-1}V\,(t) + t\,V\,(t) = (t^{-1/2}-t^{1/2})\,V\,(t)$$



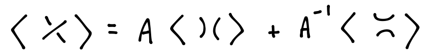
□

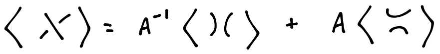
22.2 Jones polynomial of a mirror image

Proposition 22.3. Let $(L, \mathcal{O}_L)$ be a link, and let $(L_m, \mathcal{O}_{L_m})$ be its mirror image. Then

$$V_L(t) = V_{L_m}(t^{-1}).$$

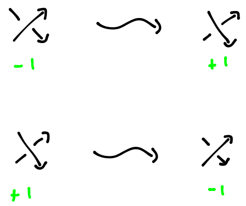
Proof. By definition of $\langle L' \rangle$ we have the following for any link $(L', \mathcal{O}_{L'})$.





Thus $\langle L_m \rangle$ is obtained from $\langle L \rangle$ by replacing A by A^{-1}.

Let us now equip L with a choice of orientation. We have that $w(L) = -w(L_m)$, since passing from L to L_m has the following effect on the sign of a crossing.



Thus we have that

$$(-A)^{-3w(L)} \langle L \rangle = (-A)^{3w(L_m)} = \big(-(A^{-1}) \big)^{-3w(L_m)}.$$

Together with the fact that $\langle L_m \rangle$ is obtained from $\langle L \rangle$ by replacing A by A^{-1}, this implies that $V_L(A) = V_{L_m}(A^{-1})$. We conclude that $V_L(t) = V_{L_m}(t^{-1})$.

□

Terminology 22.4. A Laurent polynomial is *palindromic* if it is not changed by replacing t by t^{-1}.

Corollary 22.5. Let $(L, \mathcal{O}_L)$ be a link, and let $(L_m, \mathcal{O}_{L_m})$ be its mirror image. If L is isotopic to L' then $V_L(t)$ is palindromic.

Proof. Follows immediately from Proposition 22.3. □

Examples 22.6.

(1) By Examples 21.14 (2) and Corollary 22.5 the Hopf link is not isotopic to its mirror image, since its Jones polynomial with respect to the orientation we chose is $-t^{-\frac{5}{2}} - t^{-\frac{1}{2}}$. This is not palindromic!

    We saw by hand in Examples 21.14 (3) that the Jones polynomial of the mirror image of the Hopf link in Examples 21.14 (2) is $-t^{\frac{1}{2}} - t^{\frac{5}{2}}$, as we now know must be true by Proposition 22.3.

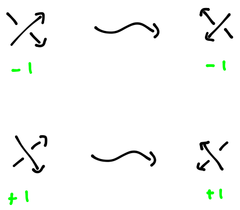
(2) By Examples 21.14 (4) and Corollary 22.5 the trefoil is not isotopic to its mirror image, since its Jones polynomial is $t + t^3 - t^4$. This is not palindromic!

(3) We know by Example 18.2 that the figure of eight knot is isotopic to its mirror image. Its Jones polynomial can be calculated to be $t^{-2} - t^{-1} + 1 - t + t^2$. This is palindromic, as we know by Corollary 22.5 that it must be!

Proposition 22.7. Let $(K, \mathcal{O}_K)$ be a knot. Then $V_K(t)$ does not depend on the choice of orientation of K.

Proof. The definition of $\langle K \rangle$ does not involve an orientation of K, and therefore does not depend on any choice of orientation.

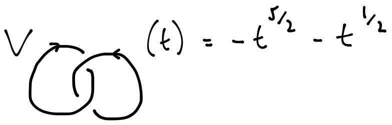
Since a knot has only one component, the only way to change orientation is to reverse it. This has the following effect on the sign of a crossing.



Thus the writhe of K is not changed by reversing its orientation. We deduce that $V_K(A)$ is not changed by reversing the orientation of K. We conclude that $V_K(t)$ is not changed by reversing the orientation of K. ☐

Remark 22.8. Thus we can speak of the Jones polynomial of a knot, rather than only of an oriented knot. In other words, when we calculate the Jones polynomial we are to choose any orientation we wish.

Remark 22.9. Proposition 22.7 does not necessarily hold for a link with more than one component. For example one can prove the following.



$$V(t) = -t^{5/2} - t^{1/2}$$

This is not the same as the Jones polynomial which we calculated in Examples 21.14 (2), where we were working with a different choice of orientation.