Generell Topologi

Richard Williamson

May 27, 2013

Contents

1 Tuesday 15th January

1.1 Topological spaces — definition, terminology, finite examples

Definition 1.1. A *topological space* is a pair $(X, \mathcal{O})$ of a set X and a set $\mathcal{O}_X$ of subsets of X, such that the following conditions are satisfied.

(1) The empty set $\emptyset$ belongs to $\mathcal{O}$.

(2) The set X itself belongs to $\mathcal{O}$.

(3) Let U be a (possibly infinite) union of subsets of X belonging to $\mathcal{O}$. Then U belongs to $\mathcal{O}$.

(4) Let U and U' be subsets of X belonging to $\mathcal{O}$. Then the set $U \cap U'$ belongs to $\mathcal{O}$.

Remark 1.2. By induction, the following condition is equivalent to (4).

(4') Let $\{U_j\}_{j \in J}$ be a finite set of subsets of X belonging to $\mathcal{O}$. Then $\bigcap_j U_j$ belongs to $\mathcal{O}$.

Terminology 1.3. Let $(X, \mathcal{O})$ be a topological space. We refer to $\mathcal{O}$ as a *topology* on X.

A set may be able to be equipped with many different topologies! See Examples 1.7.

Convention 1.4. Nevertheless, a topological space $(X, \mathcal{O})$ is often denoted simply by X. To avoid confusion, we will not make use of this convention, at least in the early part of the course.

Notation 1.5. Let X be a set. We will write $A \subset X$ to mean that A is a subset of X, allowing that A may be equal to X. In the past you may instead have written $A \subseteq X$.

Terminology 1.6. Let $(X, \mathcal{O})$ be a topological space. If $U \subset X$ belongs to $\mathcal{O}$, we say that U is an *open subset* of X with respect to $\mathcal{O}$, or simply that U is *open* in X with respect to $\mathcal{O}$.

If $V \subset X$ has the property that $X \setminus V$ is an open subset of X with respect to $\mathcal{O}$, we say that V is a *closed subset* of X with respect to $\mathcal{O}$, or simply that V is *closed* in X with respect to $\mathcal{O}$.

Examples 1.7.

(1) We can equip any set X with the following two topologies.

   (i) *Discrete topology.* Here we define $\mathcal{O}$ to be the set of all subsets of X. In other words, $\mathcal{O}$ is the power set of X.

(ii) *Indiscrete topology.* Here we define $\mathcal{O}$ to be the set $\{\emptyset, X\}$. By conditions (1) and (2) of Definition 1.1, any topology on X must include both $\emptyset$ and X. Thus $\mathcal{O}$ is the smallest topology with which X may be equipped.

(2) Let $X = \{a\}$ be a set with one element. Then X can be equipped with exactly one topology, $\mathcal{O} = \{\emptyset, X\}$. In particular, the discrete topology on X is the same as the indiscrete topology on X.

The topological space $(X, \mathcal{O})$ is important! It is known as the *point*.

(3) Let $X = \{a, b\}$ be a set with two elements. We can define exactly four topologies upon X.

(i) Discrete topology. $\mathcal{O} := \big\{\emptyset, \{a\}, \{b\}, X\big\}$.

(ii) $\mathcal{O} := \big\{\emptyset, \{a\}, X\big\}$.

(iii) $\mathcal{O} := \big\{\emptyset, \{b\}, X\big\}$.

(iv) Indiscrete topology. $\mathcal{O} := \big\{\emptyset, X\big\}$.

Up to the bijection

$$X \xrightarrow{f} X$$

given by $a \mapsto b$ and $b \mapsto a$, or in other words up to relabelling the elements of X, the topologies of (ii) and (iii) are the same.

The topological space $(X, \mathcal{O})$ where $\mathcal{O}$ is defined as in (ii) or (iii) is known as the *Sierpiński interval* or *Sierpiński space.*

(4) Let $X = \{a, b, c\}$ be a set with three elements. We can define exactly 29 topologies upon X! Again, up to relabelling, many of these topologies are the same.

(i) For instance,

$$\mathcal{O} := \big\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\big\}$$

defines a topology on X.

(ii) But $\mathcal{O} := \big\{\emptyset, \{a\}, \{c\}, X\big\}$ does not define a topology on X. This is because

$$\{a\} \cup \{c\} = \{a, c\}$$

does not belong to $\mathcal{O}$, so condition (3) of Definition 1.1 is not satisfied.

(iii) Also, $\mathcal{O} := \big\{\emptyset, \{a, b\}, \{a, c\}, X\big\}$ does not define a topology on X. This is because $\{a, b\} \cap \{b, c\} = \{b\}$ does not belong to $\mathcal{O}$, so condition (4) of Definition 1.1 is not satisfied.

There are quite a few more 'non-topologies' on X.

1.2 Towards a topology on $\mathbb{R}$ — recollections on completeness of $\mathbb{R}$

Notation 1.8. Let $a, b \in \mathbb{R}$.

(1) We refer to a subset of $\mathbb{R}$ of one of the following four kinds as an *open interval*.

   (i) $(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$.

   (ii) $(a, \infty) := \{x \in \mathbb{R} \mid x > a\}$.

   (iii) $(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$.

   (iv) $\mathbb{R}$, which we may sometimes also denote by $(-\infty, \infty)$.

(2) We refer to a subset of $\mathbb{R}$ of the following kind as a *closed interval*.

$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

(3) We refer to a subset of $\mathbb{R}$ of one of the following four kinds as a *half open interval*.

   (i) $[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}$.

   (ii) $(a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}$.

   (iii) $[a, \infty) := \{x \in \mathbb{R} \mid x \geq a\}$.

   (iv) $(-\infty, b] := \{x \in \mathbb{R} \mid x \leq b\}$.

Recollection 1.9. The key property of $\mathbb{R}$ is *completeness*. There are many equivalent characterisations of this property — Theorem 1.10 and Theorem 1.15 are the two characterisations that are of importance to us here.

Theorem 1.10. Let $\{x_j\}_{j \in J}$ be a (possibly infinite) set of real numbers. Suppose that there exists a $b \in \mathbb{R}$ such that $x_j \leq b$ for all $j \in J$. Then there exists a $b' \in \mathbb{R}$ such that:

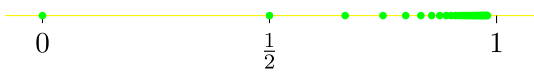
(i) $x_j \leq b'$ for all $j \in J$,

(ii) if $b'' \in \mathbb{R}$ has the property that $x_j \leq b''$ for all $j \in J$, then $b'' \geq b'$.

Remark 1.11. In other words, if $\{x_j\}_{j \in J}$ has an upper bound b, then $\{x_j\}_{j \in J}$ has an upper bound b' which is less than or equal to any upper bound b'' of $\{x_j\}_{j \in J}$.

Terminology 1.12. Let $\{x_j\}_{j \in J}$ be a set of real numbers which admits an upper bound. We refer to the corresponding least upper bound b' of $\{x_j\}_{j \in J}$ that the completeness of $\mathbb{R}$ in the form of Theorem 1.10 gives us as the *supremum* of $\{x_j\}_{j \in J}$. We denote it by $\sup x_j$.

Recollection 1.13. Recall from your early courses in real analysis some examples of a supremum. For instance, the supremum of the set $\{1 - \frac{1}{n}\}_{n \in \mathbb{N}}$ is 1.



The picture shows the elements of $\{1 - \frac{1}{n}\}_{n \in \mathbb{N}}$ for $1 \leq n \leq 50$, getting closer and closer to 1 without reaching it!

Notation 1.14. Let $\{x_j\}_{j\in J}$ be a set of real numbers such that for every $b \in \mathbb{R}$ there is a $k \in J$ with the property that $x_k > b$. In other words, we assume that $\{x_j\}_{j\in J}$ is not bounded above. In this case, we write $\sup x_j = \infty$.

Theorem 1.15. Let $\{x_j\}_{j\in J}$ be a (possibly infinite) set of real numbers. Suppose that there exists a $b \in \mathbb{R}$ such that $x_j \geq b$ for all $j \in J$. Then there exists a $b' \in \mathbb{R}$ such that:

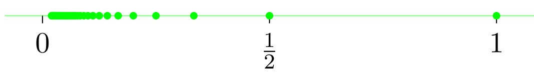
(i) $x_j \geq b'$ for all $j \in J$,

(ii) if $b'' \in \mathbb{R}$ has the property that $x_j \geq b''$ for all $j \in J$, then $b'' \leq b'$.

Remark 1.16. In other words, if $\{x_j\}_{j\in J}$ has a lower bound b, then $\{x_j\}_{j\in J}$ has a lower bound b' which is greater than or equal to any lower bound b'' of $\{x_j\}_{j\in J}$.

Terminology 1.17. Let $\{x_j\}_{j\in J}$ be a set of real numbers which admits a lower bound. We refer to the corresponding greatest upper bound b' of $\{x_j\}_{j\in J}$ that the completeness of $\mathbb{R}$ in the form of Theorem 1.15 gives us as the *infimum* of $\{x_j\}_{j\in J}$. We denote it by $\inf x_j$.

Recollection 1.18. Recall from your early courses in real analysis some examples of an infimum. For instance, the infimum of the set $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is 0.



The picture shows the elements of $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ for $1 \leq n \leq 50$, getting closer and closer to 0 without reaching it!

Notation 1.19. Let $\{x_j\}_{j\in J}$ be a set of real numbers such that for every $b \in \mathbb{R}$ there is a $k \in J$ with the property that $x_k < b$. In other words, we assume that $\{x_j\}_{j\in J}$ is not bounded below. In this case, we write $\inf x_j = -\infty$.

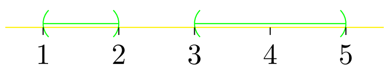
Goal 1.20. To equip $\mathbb{R}$ with a a topology to which the open intervals in $\mathbb{R}$ belong.

Observation 1.21. Let $a, b, a', b' \in \mathbb{R}$. Then

$$(a,b) \cap (a',b') = \begin{cases} \big(\sup\{a,a'\}, \inf\{b,b'\}\big) & \text{if } \sup\{a,a'\} < \inf\{b,b'\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Remark 1.22. Thus condition (4) of Definition 1.1 is satisfied for $\mathcal{O}' := \{\text{open intervals in } \mathbb{R}\}$.

However, condition (3) of Definition 1.1 is not satisfied for $\mathcal{O}' := \{\text{open intervals in } \mathbb{R}\}$. Indeed, take any two open intervals in $\mathbb{R}$ which do not intersect. For example, $(1,2)$ and $(3,5)$. The union of these two open intervals is disjoint, and in particular is not an open interval.



Idea 1.23. Observing this, we might try to enlarge $\mathcal{O}'$ to include disjoint unions of (possibly infinitely many) open intervals in $\mathbb{R}$. This works! The set

$$\mathcal{O} := \{\bigsqcup_{j \in J} U_j \mid U_j \text{ is an open interval in } \mathbb{R}\}$$

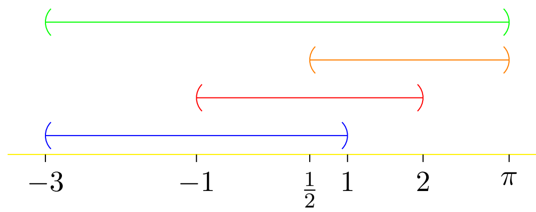
does equip $\mathbb{R}$ with a topology.

We will not prove this now. It will be more convenient for us to build a topology on $\mathbb{R}$ by a formal procedure — the topology 'generated by' open intervals in $\mathbb{R}$. We will see this in the next lecture, as Definition 2.5. Later on, we will prove that this topology is exactly $\mathcal{O}$.

Observation 1.24. However, we can already appreciate one of the two key aspects of the proof. Suppose that we have a set $\{(a_j, b_j)\}_{j \in J}$ of (possibly infinitely many) open intervals in $\mathbb{R}$. Suppose that $\bigcup_{j \in J}(a_j, b_j)$ cannot be obtained as a disjoint union of any pair of subsets of $\mathbb{R}$. Then

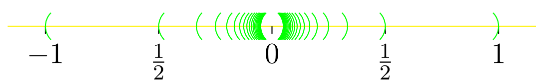
$$\bigcup_{j \in J}(a_j, b_j) = (\inf a_j, \sup b_j).$$

Remark 1.25. Observation 1.24 expresses the intuition that a 'chain of overlapping open intervals' is an open interval. For instance, the union of $\{(-3, 1), (-1, 2), (\frac{1}{2}, \pi)\}$ is $(-3, \pi)$.



Remark 1.26. By contrast with Observation 1.21, Observation 1.24 relies on the full strength of the completeness of $\mathbb{R}$ as expressed in Theorem 1.10 and Theorem 1.15.

An intersection of open intervals, even a 'chain of overlapping open intervals', need not be an open interval. For instance, $\bigcap_{n \in \mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$, and the set $\{0\}$ is not an open interval in $\mathbb{R}$!



The picture shows the suprema and infima of the intervals $(-\frac{1}{n}, \frac{1}{n})$ for $1 \leq n \leq 20$.

Summary 1.27.

(1) A union of (possibly infinitely many) open intervals in $\mathbb{R}$ is an open interval, if these open intervals 'overlap sufficiently nicely'.

(2) An intersection of a pair of open intervals in $\mathbb{R}$ which overlap is an open interval.

(3) An intersection of infinitely many open intervals in $\mathbb{R}$ need not be an open interval, even if these open intervals 'overlap sufficiently nicely'.

Remark 1.28. These three facts together motivate the requirement in condition (3) of Definition 1.1 that unions of possibly infinitely many subsets of X belonging to $\mathcal{O}$ belong to $\mathcal{O}$, by contrast with condition (4) of Definition 1.1, in which an intersection of only a pair of subsets of X belonging to $\mathcal{O}$ is required to belong to $\mathcal{O}$.

Remark 1.29. In Exercise Sheet 1 we will explore topological spaces $(X, \mathcal{O})$ with the property that an intersection of any set of subsets of X, possibly infinitely many, belonging to $\mathcal{O}$ belongs to $\mathcal{O}$. These topological spaces are known as *Alexandroff spaces.*

1.3 Canonical constructions of topological spaces — subspace topologies, product topologies, examples

Assumption 1.30. For now let us assume that we have equipped $\mathbb{R}$ with a topology $\mathcal{O}_{\mathbb{R}}$ to which every open interval in $\mathbb{R}$ belongs. As indicated in Idea 1.23, will construct $\mathcal{O}_{\mathbb{R}}$ in the next lecture.

Theme 1.31. Given $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, we can construct many topological spaces in a 'canonical way'.

Preview 1.32. Over the next few lectures, we will become acquainted with four tools:

(1) subspace topologies,

(2) product topologies,

(3) quotient topologies,

(4) coproduct topologies.

We will investigate (1) and (2) now. In Lecture 3, we will investigate (3). Later, we will investigate (4).

Proposition 1.33. Let $(Y, \mathcal{O}_Y)$ be a topological space. Let X be a subset of Y. Then

$$\mathcal{O}_X := \{X \cap U \mid U \in \mathcal{O}_Y\}$$

defines a topology on X.

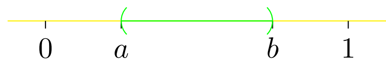
Proof. Exercise Sheet 1. □

Terminology 1.34. Let $(Y, \mathcal{O}_Y)$ be a topological space. Let X be a subset of Y. We refer to the topology $\mathcal{O}_X$ on X defined in Proposition 1.33 as the *subspace topology* on X.

Example 1.35. Let I denote the closed interval $[0, 1]$ in $\mathbb{R}$. Let $\mathcal{O}_I$ denote the subspace topology on I with respect to the topological space $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. We refer to the topological space $(I, \mathcal{O}_I)$ as the *unit interval*.

Explicitly, $\mathcal{O}_I$ consists of subsets of I of the following three kinds, in addition to $\emptyset$ and I itself.

(1) Open intervals (a, b) with $a, b \in \mathbb{R}$, $a > 0$, and $b < 1$.



(2) Half open intervals $[0, b)$ with $0 < b < 1$.

(3) Half open intervals $(a, 1]$ with $0 < a < 1$.

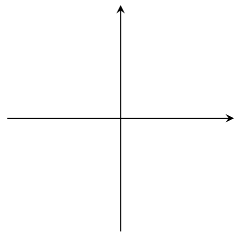
Proposition 1.36. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\mathcal{O}_{X \times Y}$ denote the set of subsets W of $X \times Y$ such that for every $(x, y) \in W$ there exists $U \in \mathcal{O}_X$ and $U' \in \mathcal{O}_Y$ with $x \in U$, $y \in U'$, and $U \times U' \subset W$. Then $\mathcal{O}_{X \times Y}$ defines a topology on $X \times Y$.

Proof. Exercise Sheet 1. $\square$

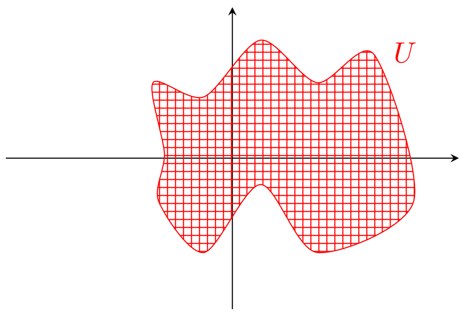
Terminology 1.37. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. We refer to the topology $\mathcal{O}_{X \times Y}$ on $X \times Y$ defined in Proposition 1.36 as the *product topology* on $X \times Y$.

Examples 1.38.

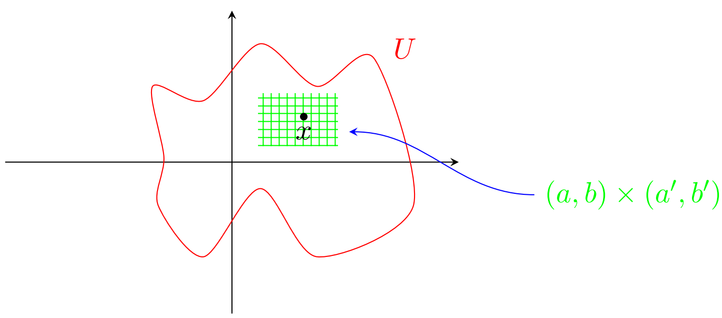
(1) $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, equipped with the product topology $\mathcal{O}_{\mathbb{R} \times \mathbb{R}}$.



A typical example of a subset of $\mathbb{R}^2$ belonging to $\mathcal{O}_{\mathbb{R}\times\mathbb{R}}$ is an 'open blob' U.

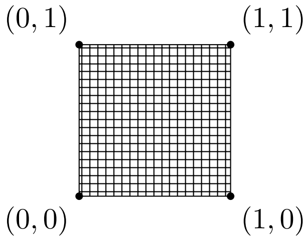


Indeed by the completeness of $\mathbb{R}$ we have that for any $x \in \mathbb{R}$ belonging to U there is an 'open rectangle' contained in U to which x belongs. By an 'open rectangle' we mean a product of an open interval (a, b) with an open interval (a', b'), for some $a, b, a', b' \in \mathbb{R}$.

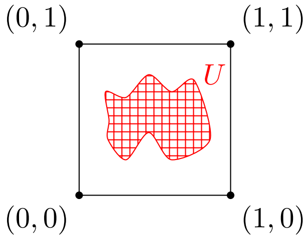


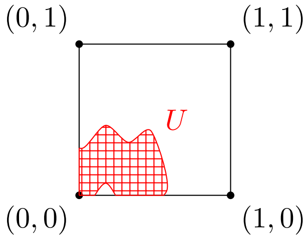
The boundary of U in the last two pictures is not to be thought of as belonging to U.

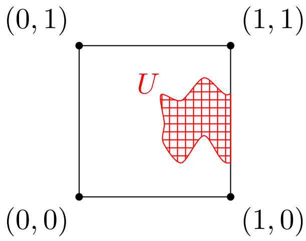
(2) $I^2 := I \times I$, equipped with the product topology $\mathcal{O}_{I\times I}$. We refer to the topological space $(I^2, \mathcal{O}_{I\times I})$ as the *unit square*.

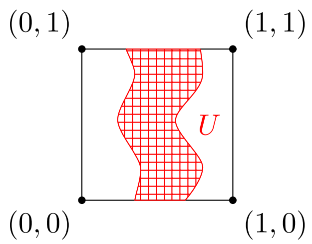


A typical example of a subset U of I^2 belonging to $\mathcal{O}_{I\times I}$ is an intersection with I^2 of an 'open blob' in $\mathbb{R}^2$.



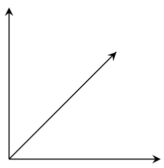






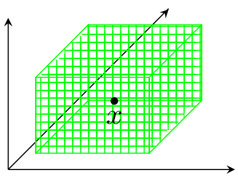

In the first figure, the boundary of U is not to be thought of as belonging to U. In the last three figures, the part of the boundary of U which intersects the boundary of the square belongs to U, but the remainder of the boundary of U is not to be thought of as belonging to U.

(3) $\mathbb{R}^3 := \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.



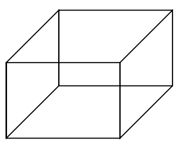
A typical example of a subset U of $\mathbb{R}^3$ belonging to $\mathcal{O}_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}}$ is a '3-dimensional open blob'. I leave it to your imagination to visualise one of these!

By the completeness of $\mathbb{R}$, for any $x \in U$ there is an 'open rectangular cuboid' contained in U to which x belongs.




Our notation $\mathcal{O}_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}}$ is potentially ambiguous, since we may cook up a product topology on $\mathbb{R}^3$ either by viewing $\mathbb{R}^3$ as $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ or by viewing $\mathbb{R}^3$ as $\mathbb{R}\times(\mathbb{R}\times\mathbb{R})$. However, these two topologies coincide, and the same is true in general.

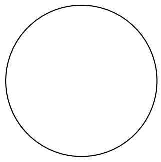
(4) $I^3 := I \times I \times I$, equipped with the product topology $\mathcal{O}_{I\times I\times I}$. We refer to the topological space $(I^3, \mathcal{O}_{I\times I\times I})$ as the *unit cube*.



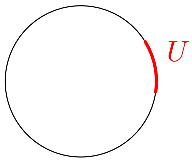
A typical example of a subset of I^3 belonging to $\mathcal{O}_{I\times I\times I}$ is the intersection of a '3-dimensional open blob' in $\mathbb{R}^3$ with I^3. Again I leave the visualisation of such a subset to your imagination!

(5) Examples (1) and (3) generalise to a product topology upon $\mathbb{R}^n := \underbrace{\mathbb{R}\times\ldots\times\mathbb{R}}_{n}$ for any $n \in \mathbb{N}$. Examples (2) and (4) generalise to a product topology upon $I^n := \underbrace{I\times\ldots\times I}_{n}$ for any $n \in \mathbb{N}$.

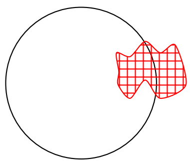
(6) $S^1 := \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\| = 1\}$, equipped with the subspace topology $\mathcal{O}_{S^1}$ with respect to the topological space $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. We refer to $(S^1, \mathcal{O}_{S^1})$ as the *circle*.



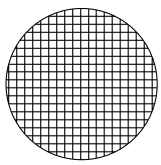
A typical subset of S^1 belonging to $\mathcal{O}_{S^1}$ is the intersection of an 'open blob' in $\mathbb{R}^2$ with S^1. For instance, the subset U of S^1 pictured below belongs to $\mathcal{O}_{S^1}$.



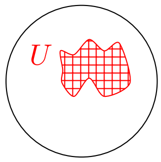
Indeed, U is the intersection with S^1 of the 'open blob' in the picture below.

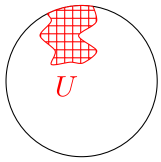


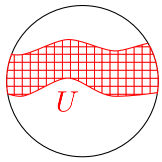
(7) $D^2 := \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\| \leq 1\}$, equipped with the subspace topology $\mathcal{O}_{D^2}$ with respect to the topological space $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. We refer to $(D^2, \mathcal{O}_{D^2})$ as the *disc*.



A typical example of a subset of D^2 belonging to $\mathcal{O}_{D^2}$ is an intersection of an 'open blob' in $\mathbb{R}^2$ with D^2.

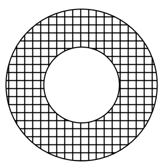




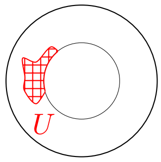


In the first figure, the boundary of U is not to be thought of as belonging to U. In the last two figures, the part of the boundary of U which intersects the boundary of the disc belongs to U, but the remainder of the boundary of U is not to be thought of as belonging to U.

(8) For any $k \in \mathbb{R}$ with $0 < k < 1$, $A_k := \{(x,y) \in \mathbb{R}^2 \mid k \leq \|(x,y)\| \leq 1\}$, equipped with the subspace topology $\mathcal{O}_{A_k}$ with respect to the topological space $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. We refer to $(A_k, \mathcal{O}_{A_k})$ as an *annulus*.



A typical example of a subset of A_k belonging to $\mathcal{O}_{A_k}$ is an intersection of an 'open blob' in $\mathbb{R}^2$ with A_k.



(9) $S^1 \times I$, equipped with the product topology $\mathcal{O}_{S^1 \times I}$. We refer to $(S^1 \times I, \mathcal{O}_{S^1 \times I})$ as the *cylinder*.

(10) $D^2 \times I$, equipped with product topology $\mathcal{O}_{D^2 \times I}$. We refer to $(D^2 \times I, \mathcal{O}_{D^2 \times I})$ as the *solid cylinder*.

2 Thursday 17th January

2.1 Basis of a topological space — generating a topology with a specified basis — standard topology on $\mathbb{R}$ — examples

Definition 2.1. Let $(X, \mathcal{O})$ be a topological space. A *basis* for $(X, \mathcal{O})$ is a set $\mathcal{O}'$ of subsets of X belonging to $\mathcal{O}$ such that every subset U of X belonging to $\mathcal{O}$ may be obtained as a union of subsets of X belonging to $\mathcal{O}'$.

Proposition 2.2. Let X be a set, and let $\mathcal{O}'$ be a set of subsets of X such that the following conditions are satisfied.

(1) X can be obtained as a union of (possibly infinitely many) subsets of X belonging to $\mathcal{O}'$.

(2) Let U and U' be subsets of X belonging to $\mathcal{O}'$. Then $U \cap U'$ belongs to $\mathcal{O}'$.

Let $\mathcal{O}$ denote the set of subsets U of X which may be obtained as a union of (possibly infinitely many) subsets of $\mathcal{O}'$. Then $(X, \mathcal{O})$ is a topological space, with basis $\mathcal{O}'$.

Proof. We verify conditions (1)-(4) of Definition 1.1.

(1) We think of $\emptyset$ an an 'empty union' of subsets of X belonging to $\mathcal{O}'$, so that $\emptyset \in \mathcal{O}$. If you are not comfortable with this, just change the definition of $\mathcal{O}$ to include $\emptyset$ as well.

(2) We have that $X \in \mathcal{O}$ by definition of $\mathcal{O}$ together with the fact that $\mathcal{O}'$ atsifies condition (1) in the statement of the proposition.

(3) Let $\{U_j\}_{j \in J}$ be a set of subsets of X belonging to $\mathcal{O}$. For every $j \in J$, by definition of $\mathcal{O}$ we have that $X = \bigcup_{k \in K_j} U'_k$ for a set K_j, where $U'_k \in \mathcal{O}'$. Then

$$\begin{aligned}\bigcup_{j \in J} U_j &= \bigcup_{j \in J} \Big(\bigcup_{k \in K_j} U'_k \Big) \\ &= \bigcup_{r \in \left(\bigcup_{j \in J} K_j \right)} U'_r.\end{aligned}$$

Thus $\bigcup_{j \in J} U_j$ is a union of subsets of X belonging to $\mathcal{O}'$, and hence $\bigcup_{j \in J} U_j$ belongs to $\mathcal{O}$.

(4) Let U and U' be subsets of X which belong to $\mathcal{O}$. By definition of $\mathcal{O}$, we have that $U = \bigcup_{j \in J} U_j$ where $U_j \in \mathcal{O}'$ for all $j \in J$, and that $U' = \bigcup_{j' \in J'} U'_j$, where $U'_j \in \mathcal{O}'$ for all $j' \in J'$. Then

$$\begin{aligned}U \cap U' &= \Big(\bigcup_{j \in J} U_j \Big) \cap \Big(\bigcup_{j' \in J'} U'_{j'} \Big) \\ &= \bigcup_{(j,j') \in J \times J'} U_j \cap U_{j'}.\end{aligned}$$

Since $\mathcal{O}'$ satisfies condition (2) of the proposition, we have that $U_j \cap U_{j'}$ belongs to $\mathcal{O}'$ for every $(j, j') \in J \times J'$. Thus $U \cap U'$ belongs to $\mathcal{O}'$.

By construction of $\mathcal{O}$, we have that $\mathcal{O}'$ is a basis for $(X, \mathcal{O})$. $\square$

Terminology 2.3. Let X be a set, and let $\mathcal{O}'$ be a set of subsets of X satisfying conditions (1) and (2) of Proposition 2.2. Let $\mathcal{O}$ denote the set of unions of subsets of X belonging to $\mathcal{O}'$, which by Proposition 2.2 defines a topology on X. We refer to $\mathcal{O}$ as the topology on X which is *generated* by $\mathcal{O}'$.

Observation 2.4. Let $\mathcal{O}' := \{(a, b) \mid a, b \in \mathbb{R}\}$. Then $\mathcal{O}'$ satisfies condition (1) of Proposition 2.2 with respect to $\mathbb{R}$, since for example $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$. By Observation 1.21, we have that $\mathcal{O}'$ satisfies condition (2) of Proposition 2.2.

Definition 2.5. The *standard topology* on $\mathbb{R}$ is the topology $\mathcal{O}_{\mathbb{R}}$ generated by $\mathcal{O}'$.

Observation 2.6. All open intervals in $\mathbb{R}$ belong to $\mathcal{O}_{\mathbb{R}}$. We have the following cases.

(1) If $a, b \in \mathbb{R}$, then by definition of $\mathcal{O}$ and $\mathcal{O}'$ we have that $(a, b) \in \mathcal{O}_{\mathbb{R}}$.

(2) If $a \in \mathbb{R}$, we have that $(a, \inf) = \bigcup_{n \in \mathbb{N}} (a, a + n)$. Since $(a, a + n)$ belongs to $\mathcal{O}'$ for every $n \in \mathbb{N}$, we deduce that $(a, \inf) \in \mathcal{O}_{\mathbb{R}}$.

(3) If $b \in \mathbb{R}$, we have that $(-\inf, b) = \bigcup_{n \in \mathbb{N}} (b - n, b)$. Since $(b - n, b)$ belongs to $\mathcal{O}'$ for every $n \in \mathbb{N}$, we deduce that $(-\inf, b) \in \mathcal{O}_{\mathbb{R}}$.

(4) We noted in Observation 2.4 that $\mathbb{R} \in \mathcal{O}_{\mathbb{R}}$.

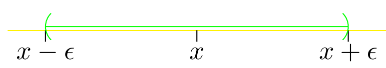
Remark 2.7. As mentioned in Idea 1.23, we will prove later in the course that $\mathcal{O}_{\mathbb{R}}$ consists exactly of disjoint unions of (possibly infinitely many) open intervals.

Observation 2.8. Let $(X, \mathcal{O})$ be a topological space, and let $\mathcal{O}'$ be a basis for $(X, \mathcal{O})$. Let $\mathcal{O}''$ be a set of subsets of X. If every $U \subset X$ such that $U \in \mathcal{O}'$ can be obtained as a union of subsets of $\mathcal{O}''$, then $\mathcal{O}''$ defines a basis for $(X, \mathcal{O})$.

Examples 2.9.

(1) For $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$, and for any $x \in \mathbb{R}$, let

$$B_\epsilon(x) := \{y \in \mathbb{R} \mid x - \epsilon < y < x + \epsilon\}.$$



In other words, $B_\epsilon(x)$ is the open interval $(x - \epsilon, x + \epsilon)$. Then

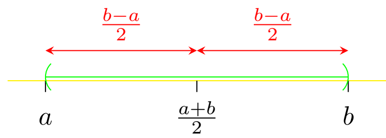
$$\mathcal{O}'' := \{B_\epsilon(x) \mid \epsilon \in \mathbb{R} \text{ and } \epsilon > 0, \text{ and } x \in \mathbb{R}\}$$

is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Proof. By Observation 2.8, it suffices to prove that for every $a, b \in \mathbb{R}$ we can obtain the open interval (a, b) as a union of subsets of $\mathbb{R}$ belonging to $\mathcal{O}''$. In fact, (a, b) itself already belongs to $\mathcal{O}''$. Indeed

$$(a, b) = B_{\frac{b-a}{2}}\Big(\frac{a+b}{2}\Big).$$

$\square$



In particular, we see that a topological space may admit more than one basis.

(2) Let $X = \{a, b, c, d, e\}$, and let $\mathcal{O}$ denote the topology on X given by

$$\big\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{d, e\}, \{a, b, c\}, \{b, d, e\}, \{a, b, d, e\}, \{b, c, d, e\}, X\big\}.$$

Then

$$\mathcal{O}_1 := \big\{\{b\}, \{a, b\}, \{b, c\}, \{d, e\}\big\}$$

is a basis for $(X, \mathcal{O})$.

The same holds for any set $\mathcal{O}_2$ of subsets of X such that $\mathcal{O}_1 \subset \mathcal{O}_2$. No other set of subsets of X is a basis for $(X, \mathcal{O})$. For example,

$$\mathcal{O}_3 := \big\{\{a, b\}, \{b, c\}, \{d, e\}\big\}$$

is not a basis for $(X, \mathcal{O})$, since $\{b\}$ cannot be obtained as a union of subsets of X belonging to $\mathcal{O}''$. Similarly

$$\mathcal{O}_4 := \big\{\{b\}, \{a, b\}, \{d, e\}\big\}$$

is not a basis for $\mathcal{O}''$, since $\{b, c\}$ cannot be obtained as a union of subsets of X belonging to $\mathcal{O}_4$.

(3) Let

$$\mathcal{O}' := \{(-\infty, b) \mid b \in \mathbb{R}\}.$$

Then $\mathcal{O}'$ is not a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, since for example we cannot obtain the open interval $(0, 1)$ as a union of open intervals of the form $(-\infty, b)$.

But $\mathcal{O}'$ satisfies the conditions of Proposition 2.2, and thus generates a topology $\mathcal{O}$ on $\mathbb{R}$. In the manner of Observation 1.24, one can prove that $\mathcal{O} = \mathcal{O}' \cup \{\emptyset, \mathbb{R}\}$.

2.2 Continuous maps — examples — continuity of inclusion maps, compositions of continuous maps, and constant maps

Notation 2.10. Let X and Y be sets, and let

$$X \xrightarrow{f} Y$$

be a map. Let U be a subset of Y. We define $f^{-1}(U)$ to be $\{x \in X \mid f(x) \in U\}$.

Definition 2.11. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is *continuous* if for every $U \in \mathcal{O}_Y$ we have that $f^{-1}(U)$ belongs to $\mathcal{O}_X$.

Remark 2.12. A map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous with respect to the standard topology on both copies of $\mathbb{R}$ if and only if it is continuous in the $\epsilon - \delta$ sense that you know from real analysis/calculus. See the Exercise Sheet.

Examples 2.13.

(1) Let $X := \{a, b\}$, and let $\mathcal{O}$ denote the topology on X given by $\big\{\emptyset, \{b\}, X\big\}$, so that $(X, \mathcal{O})$ is the Sierpiński interval. Let $X' := \{a', b', c'\}$, and let $\mathcal{O}'$ denote the topology on X' given by

$$\big\{\emptyset, \{a'\}, \{c'\}, \{a', c'\}, \{b', c'\}, X'\big\}.$$

Let

$$X \xrightarrow{f} Y$$

be given by $a \mapsto b'$ and $b \mapsto c'$. Then f is continous.

Proof. We verify that $f^{-1}(U) \in \mathcal{O}_X$ for every $U \in \mathcal{O}_Y$, as follows.

(1) $f^{-1}(\emptyset) = \emptyset$

(2) $f^{-1}\big(\{a'\}\big) = \emptyset$

(3) $f^{-1}\big(\{c'\}\big) = \{b'\}$

(4) $f^{-1}\big(\{a', c'\}\big) = \{b\}$

(5) $f^{-1}(Y) = X$.

Let

$$Y \xrightarrow{g} X$$

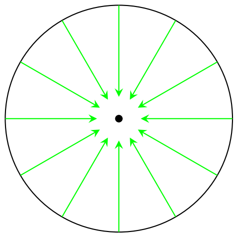
be given by $a \mapsto c'$ and $b \mapsto b'$. Then g is not continuous, since for example $g^{-1}(\{c'\}) = \{a\}$, which does not belong to $\mathcal{O}_X$. $\square$

(2) Let

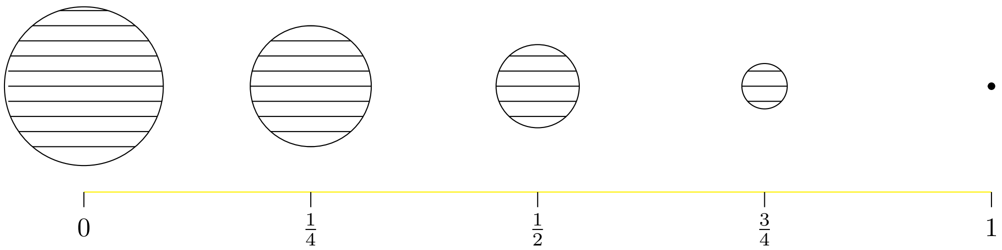
$$D^2 \times I \xrightarrow{f} D^2$$

be given by $(x, y, t) \mapsto \big((1-t)x, (1-t)y\big)$. We will prove on the Exercise Sheet that f is continuous.

We may think of f as a 'shrinking of D^2 onto its centre', as t moves from 0 to 1.



We can picture the image of $D^2 \times \{t\}$ under f as follows as t moves from 0 to 1.

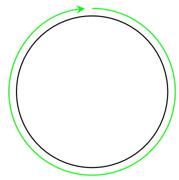


(3) Fix $k \in \mathbb{R}$. Let

$$I \xrightarrow{f} S^1$$

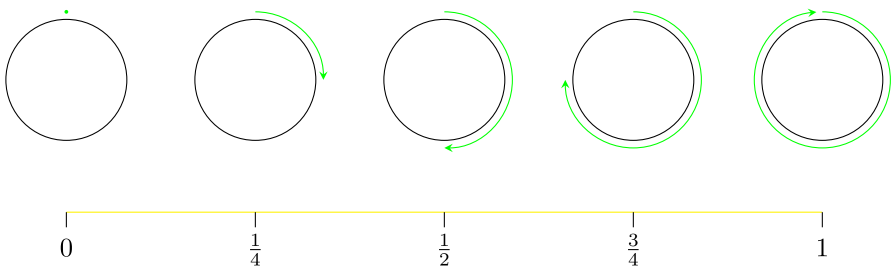
be given by $t \mapsto \phi(kt)$, where ϕ is the continuous map of Question 8 of Exercise Sheet 3. Let us picture f for a few values of k.

(1) Let $k = 1$. In words, f begins at the point $(0, 1)$, and travels exactly once around S^1.

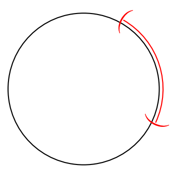


Don't be misled by the picture — the path really travels around the circle, not slightly outside it.

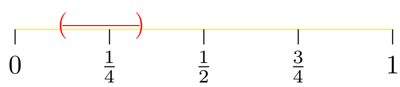
We may picture $f\big([0, t]\big)$ as t moves from 0 to 1 as follows.



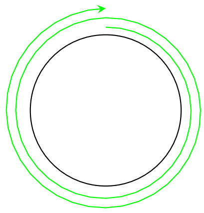
Recall from Examples 1.38 (6) that a typical open subset U of S^1 is as depicted below.



Then $f^{-1}(U)$ is as depicted below. In particular, $f^{-1}(U)$ is open in I. Thus intuitively we can believe that f is continuous!

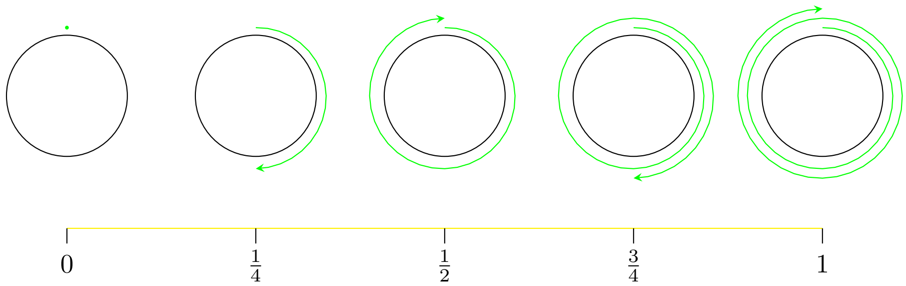


(2) Let $k = 2$. In words, f begins at the point $(0, 1)$, and travels exactly twice around S^1.

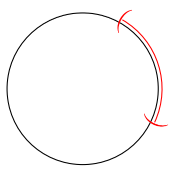


Again, don't be misled by the picture — the path really travels twice around the circle, thus passing through every point on the circle twice, not in a spiral outside the circle as drawn.

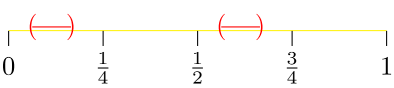
We may picture $f\big([0, t]\big)$ as t moves from 0 to 1 as follows.



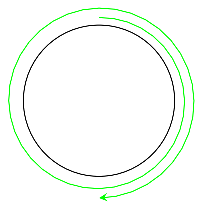
Let $U \subset S^1$ be the open subset depicted below.



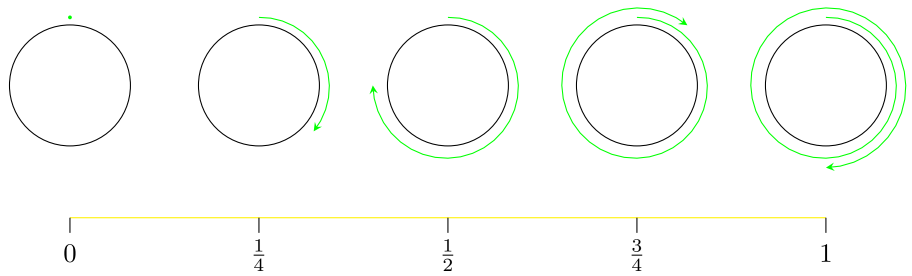
Then $f^{-1}(U)$ a disjoint union of open intervals as depicted below, so is open in I.



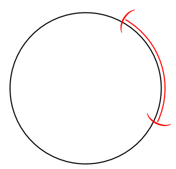
(3 Let $k = \frac{3}{2}$. In words, f begins at the point $(0, 1)$, and travels exactly one and a half times around S^1.



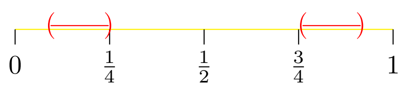
We may picture $f\big([0, t]\big)$ as t moves from 0 to 1 as follows.



Let $U \subset S^1$ be the open subset depicted below.



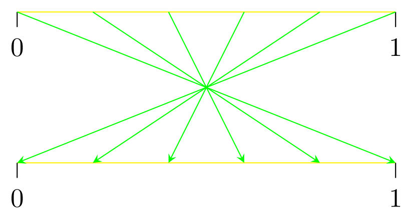
Then $f^{-1}(U)$ is a disjoint union of open subsets of I as depicted below, so is open in I.



(4) Let

$$I \xrightarrow{f} I$$

be given by $t \mapsto 1 - t$. We will prove on the Exercise Sheet that f is continuous. We may depict f as follows.



Let $U \subset I$ be the open subset depicted below.

Then $f^{-1}(U)$ is as depicted below. In particular, $f^{-1}(U)$ is open in I.

(5) Let

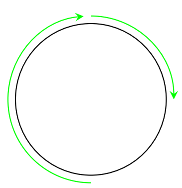
$$I \xrightarrow{f} S^1$$

be the map given by

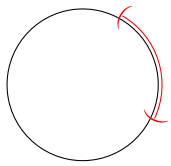
$$t \mapsto \begin{cases} \phi\left(\frac{1}{2}t\right) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \phi(t) & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

As in (3), ϕ is the map of Question 8 of Exercise Sheet 3.

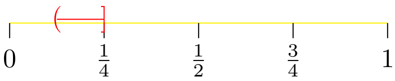
We may depict f as follows.



Then f is not continuous. Indeed, consider an open subset U of S^1 as depicted below.



Then $f^{-1}(U)$ is a half open interval as depicted below.

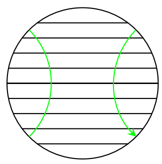


In particular, $f^{-1}(U)$ is not an open subset of I.

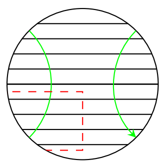
(5) Consider a map

$$I \xrightarrow{f} D^2$$

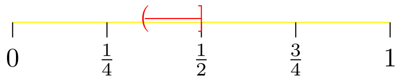
as depicted below. A precise definition of this map is not important here — the path should be interpreted as beginning on the top left of the disc, moving to the bottom left, jumping to the top right, and then moving to the bottom right.



Let $U \subset D^2$ be an open subset of D^2 depicted as a dashed rectangle below.



Then $f^{-1}(U)$ is a half open interval in I as depicted below. In particular, $f^{-1}(U)$ is not open in I.



Terminology 2.14. Let X be a set, and let A be a subset of X. The *inclusion map* with respect to A and X is the map

$$A \longrightarrow X$$

given by $a \mapsto a$. We will often denote it by

$$A \hookrightarrow X.$$

Proposition 2.15. Let $(X, \mathcal{O}_X)$ be a topological space. Let $A \subset X$ be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Then the inclusion map

$$A \overset{i}{\hookrightarrow} X$$

is continuous.

Proof. Let U be a subset of X belonging to $\mathcal{O}_X$. Then $i^{-1}(U) = A \cap U$. By definition of $\mathcal{O}_A$, we have that $A \cap U$ belongs to $\mathcal{O}_A$. Hence $i^{-1}(U)$ belongs to $\mathcal{O}_A$. $\square$

Proposition 2.16. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

and

$$Y \xrightarrow{g} Z$$

be continuous maps. Then the map

$$X \xrightarrow{g \circ f} Z$$

is continuous.

Proof. Let U be a subset of Z belonging to $\mathcal{O}_Z$. Then

$$\begin{aligned}(g \circ f)^{-1}(U) &= \{x \in X \mid g\big(f(x)\big) \in U\} \\ &= \{x \in X \mid f(x) \in g^{-1}(U)\} \\ &= f^{-1}\big(g^{-1}(U)\big).\end{aligned}$$

Since g is continuous, we have that $g^{-1}(U) \in \mathcal{O}_Y$. Hence, since f is continuous, we have that $f^{-1}\big(g^{-1}(U)\big) \in \mathcal{O}_X$. Thus $(g \circ f)^{-1}(U) \in \mathcal{O}_X$. $\square$

Terminology 2.17. Let X and Y be sets. A map

$$X \xrightarrow{f} Y$$

is *constant* if $f(x) = f(x')$ for all $x, x' \in X$.

Proposition 2.18. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a constant map. Then f is continuous.

Proof. Since f is constant, $f(x) = y$ for some $y \in Y$ and all $x \in X$. Let $U \in \mathcal{O}_Y$. If $y \notin U$, then $f^{-1}(U) = \emptyset$, which belongs to $\mathcal{O}_X$. If $y \in U$, then $f^{-1}(U) = X$, which also belongs to $\mathcal{O}_X$. $\square$

3 Tuesday 22nd January

3.1 Projection maps are continuous — pictures versus rigour

Notation 3.1. Let X and Y be sets. We denote by

$$X \times Y \xrightarrow{p_1} X$$

the map given by $(x, y) \mapsto x$. We denote by

$$X \times Y \xrightarrow{p_2} Y$$

the map given by $(x, y) \mapsto y$.

Proposition 3.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $X \times Y$ be equipped with the product topology $\mathcal{O}_{X \times Y}$. Then

$$X \times Y \xrightarrow{p_1} X$$

and

$$X \times Y \xrightarrow{p_2} Y$$

define continuous maps.

Proof. Suppose that $U \subset X$ belongs to $\mathcal{O}_X$. Then $p_1^{-1}(U) = U \times Y$, which belongs to $\mathcal{O}_{X \times Y}$.

Suppose that $U' \subset Y$ belongs to $\mathcal{O}_Y$. Then $p_2^{-1}(U') = X \times U'$, which belongs to $\mathcal{O}_{X \times Y}$. $\square$

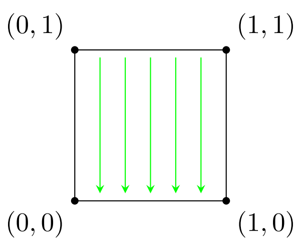
Remark 3.3. It is often helpful to our intuition to picture p_1 and p_2. Let us consider

$$I \times I \xrightarrow{p_1} I$$

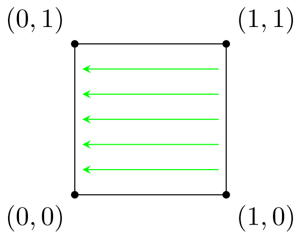
and

$$I \times I \xrightarrow{p_2} I.$$

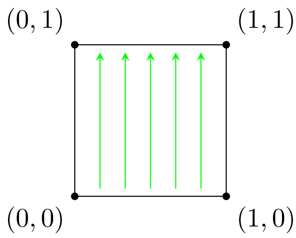
Up to a bijection between I and $I \times \{0\} = \{(x, 0) \mid x \in [0, 1]\}$, we may think of p_1 as the map $(x, y) \mapsto (x, 0)$.



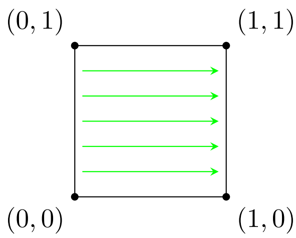
Up to a bijection between I and $\{0\} \times I = \{(0, y) \mid y \in [0, 1]\}$, we may think of p_2 as the map $(x, y) \mapsto (0, y)$.



It is important to note, though, that there will typically be many good ways that we may picture p_1 and p_2. In this example, we may for instance equally think of p_1 as the map given by $(x, y) \mapsto (x, 1)$



and/or think of p_2 as the map given by $(x, y) \mapsto (1, y)$.



The moral to draw from this is that pictures help our intuition, often profoundly. In topology we often see a proof before we can write it down!

But we must never forget that it is with rigorous definitions and proofs — which are independent of any particular picture — that we must ultimately be able to capture our intuition.

3.2 Quotient topologies

Notation 3.4. Let X be a set, and let $\sim$ be an equivalence relation on X. We denote by $X/\sim$ the set

$$\{[x] \mid x \in X\}$$

of equivalences classes of X with respect to $\sim$. We denote by

$$X \xrightarrow{\pi} X/\sim$$

the map given by $x \mapsto [x]$.

Proposition 3.5. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\sim$ be an equivalence relation on X. Then

$$\mathcal{O}_{X/\sim} := \{U \in X/\sim \mid \pi^{-1}(U) \in \mathcal{O}_X\}$$

defines a topology on $X/\sim$.

Proof. Exercise. $\square$

Terminology 3.6. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\sim$ be an equivalence relation on X. We refer to $\mathcal{O}_{X/\sim}$ as the *quotient topology* upon $X/\sim$.

Observation 3.7. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\sim$ be an equivalence relation on X. Let $X/\sim$ be equipped with the quotient topology. Then

$$X \xrightarrow{\pi} X/\sim$$

is continuous. Indeed, $\mathcal{O}_{X/\sim}$ is defined exactly so as to ensure this.

Notation 3.8. In Examples 3.9 we will adopt the following notation. Let X be a set, and let $\approx$ be a transitive relation on X. We denote by $\sim$ the equivalence relation on X defined by

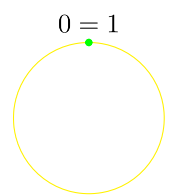
$$x \sim x' \Leftrightarrow \begin{cases} x \approx x' \text{ or } x' \approx x \text{ or } x = x' & \text{if } x, x' \in X', \\ x = x' & \text{otherwise.} \end{cases}$$

We refer to $\sim$ as the equivalence relation on X which is *generated* by $\approx$.

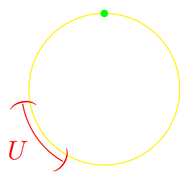
Examples 3.9.

(1) Define $\approx$ on I by $0 \approx 1$.

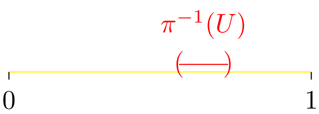
Then $I/\sim$ is obtained by glueing 0 to 1.



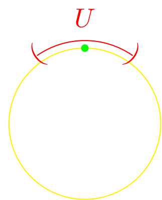
Let us explore subsets U of $I/\sim$ which belong to $\mathcal{O}_{I/\sim}$. Let $U \subset I/\sim$ be as depicted below, and suppose that $[0] = [1] \notin U$.



Then $\pi^{-1}(U)$ is an open interval in I, and thus $U \in \mathcal{O}_{I/\sim}$.

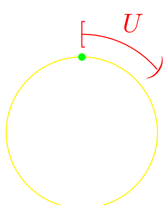


Suppose now that $[0] = [1] \in U$.

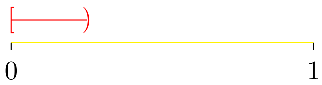


Then $\pi^{-1}(U)$ is a disjoint union of subsets of I which belong to $\mathcal{O}_I$, and thus $U \in \mathcal{O}_{I/\sim}$.

Do not be misled by this into thinking that subsets of $I/\sim$ belonging to are exactly images under π of subsets of I belonging to $\mathcal{O}_I$. Indeed, suppose that U is as depicted below.



This is the image under π of a half open interval as depicted below, which belongs to $\mathcal{O}_I$.



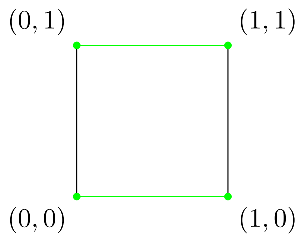
Then $\pi^{-1}(U)$ is the disjoint union of the half open interval depicted above with the singleton set $\{1\}$. Thus $\pi^{-1}(U) \notin \mathcal{O}_I$.

We see that $I/\sim$ looks like the circle S^1, which we equipped with a topology $\mathcal{O}_{S^1}$ in a different way in Examples 1.38 (6). Moreover, the subsets of $I/\sim$ which belong to $\mathcal{O}_{I/\sim}$ seem very similar to the subsets of S^1 which belong to $\mathcal{O}_{S^1}$.

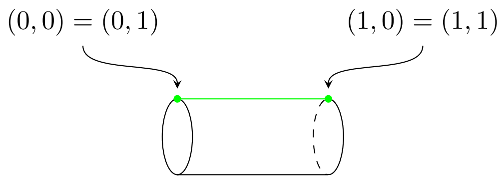
Question 3.10. Are $(I/\sim, \mathcal{O}_{I/\sim})$ and $(S^1, \mathcal{O}_{S^1})$ the same topological space, in an appropriate sense?

Answer 3.11. Yes! The appropriate notion of sameness for topological spaces will be defined at the end of this lecture. In a later lecture we will prove that $(I/\sim, \mathcal{O}_{I/\sim})$ and $(S^1, \mathcal{O}_{S^1})$ are the same in this sense.

(2) Define $\approx$ on I^2 by $(x, 1) \approx (x, 0)$ for all $x \in [0, 1]$.



Then $I/\sim$ is obtained by glueing the upper horizontal edge of I^2 to the lower horizontal edge.

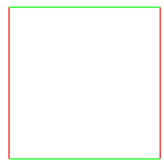


In a later lecture we will see a way to prove that $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ is the same, in the appropriate sense, to the cylinder $(S^1 \times I, \mathcal{O}_{S^1\times I})$ which was defined in a different way in Examples 1.38 (9).

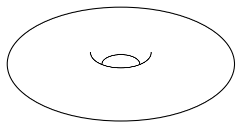
(3) Define $\approx$ on I^2 by

$$\begin{cases} (x,1) \approx (x,0) & \text{for all } x \in [0,1], \\ (1,y) \approx (0,y) & \text{for all } y \in [0,1]. \end{cases}$$

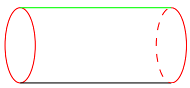
Then $I^2/\sim$ is obtained by glueing together the two horizontal edges of I^2 and glueing together the two vertical edges of I^2.



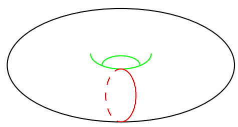
We may picture $I^2/\sim$ as follows.



Indeed we may for example first glue the horizontal edges together as in (2), obtaining a cylinder.

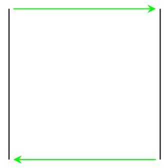


We then glue the two red circles together.

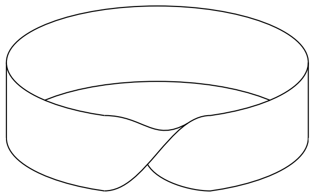


We refer to $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ as the *torus*, and denote it by T^2.

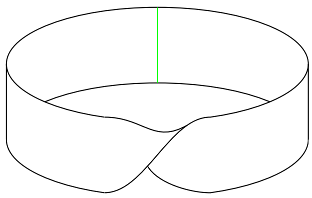
(4) Define $\approx$ on I^2 by $(x, 1) \approx (1 - x, 0)$ for all $x \in [0, 1]$. Then $I^2/\sim$ is obtained by glueing together the two horizontal edges of I^2 with a twist, indicated by the arrows in the picture below.



We may picture $I^2/\sim$ as follows.



In this picture, the glued horizontal edges of I^2 can be thought of as a line in $I^2/\sim$.

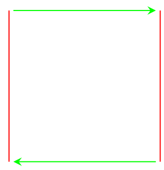


We refer to $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ as the *Möbius band*, and denote it by M^2.

(5) Define $\approx$ on I^2 by

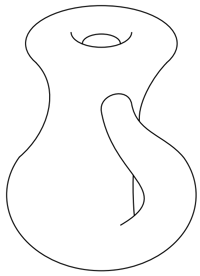
$$\begin{cases} (x, 1) \approx (1 - x, 0) & \text{for all } x \in [0, 1], \\ (1, y) \approx (0, y) & \text{for all } y \in [0, 1]. \end{cases}$$

Then $I^2/\sim$ is obtained by glueing together the two vertical edges of I^2 and glueing together the two horizontal edges of I^2 with a twist.

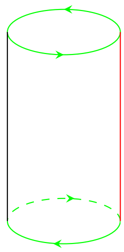


We refer to $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ as the *Klein bottle*, and denote it by K^2.

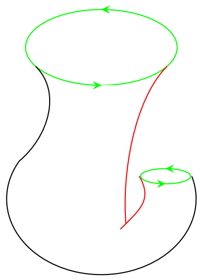
We cannot truly picture K^2 in $\mathbb{R}^3$. Nevertheless we can gain an intuitive feeling for K^2 through a picture as follows.



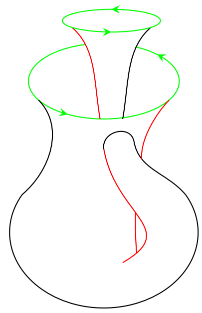
Indeed we may for example first glue the vertical edges to obtain a cylinder.



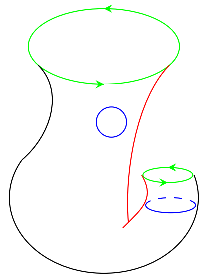
We then bend this cylinder so that the directions of the arrows on the circles at its ends match up.



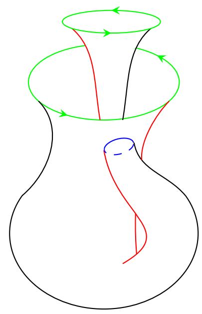
Next we push the cylinder through itself.



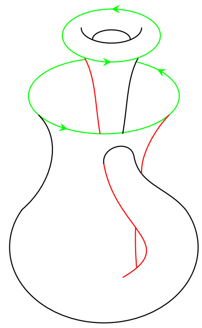
It is this step that is not possible in a true picture of K^2. It can be thought of as the glueing of two circles: a cross-section of the cylinder, and a circle on the side of the cylinder.



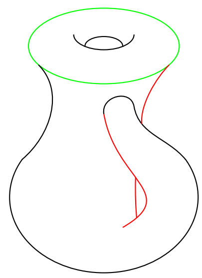
This is not specified by $\sim$. The circle obtained after glueing these two circles is indicated below.



Next we fold back one of the ends of the cylinder, giving a 'mushroom with a hollow stalk'.



Finally we glue the ends of the cylinder together, as specified by $\sim$.



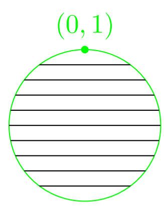
A rite of passage when learning about topology for the first time is to be confronted with the following limerick — I'm sure that I remember Colin Rourke enunciating it during the lecture in which I first met the Klein bottle!

> A mathematician named Klein
> Thought the Möbius band was divine.
> Said he: "If you glue
> The edges of two,
> You'll get a weird bottle like mine!"

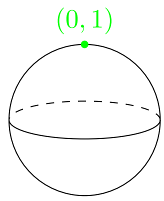
We will investigate the meaning of this in Exercise Sheet 4!

Colin Rourke also had a glass model of the topological space depicted above — I'm sorry that I could not match up to this!

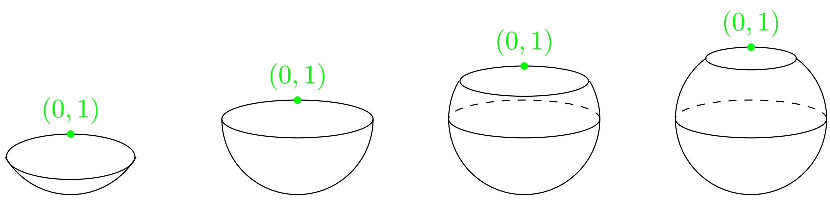
(6) Define $\approx$ on D^2 by $(x, y) \approx (0, 1)$ for all $(x, y) \in S^1$.



Then $D^2/\sim$ can be depicted as a hollow ball, as follows.



We think of $D^2/\sim$ as obtained by 'contracting the boundary of D^2 to the point $(0,1)$'. For instance, think of the boundary circle of D^2 as a loop of fishing line, and suppose that we have a reel at the point $(0,1)$. Then $D^2/\sim$ is obtained by reeling in tight all of our fishing line.



We refer to $(D^2/\sim, \mathcal{O}_{D^2/\sim})$ as the *2-sphere*, and denote it by S^2. It can be proven that $(S^2, \mathcal{O}_{S^2})$ is the same — in the appropriate sense, which we are about to introduce — as

$$\{x \in \mathbb{R}^3 \mid \|x\| = 1\}$$

equipped with the subspace topology with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}})$.

We could choose any single point on S^1 instead of $(0,1)$ in the definition of $\approx$.

3.3 Homeomorphisms

Notation 3.12. Let X be a set. We denote by id_X the identity map

$$X \longrightarrow X,$$

namely the map given by $x \mapsto x$.

Recollection 3.13. The following definitions of a bijective map

$$X \xrightarrow{f} Y$$

are equivalent.

(1) There is a map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$.

(2) The map f is both injective and surjective.

We leave (1) $\Rightarrow$ (2) as an exercise. For (2) $\Rightarrow$ (1), observe that if f is both injective and surjective, then $x \mapsto f^{-1}(x)$ gives a well-defined map

$$Y \longrightarrow X.$$

with the required properties.

Definition 3.14. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is a *homeomorphism* if:

(1) f is continuous,

(2) there is a continuous map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$.

Proposition 3.15. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is a homeomorphism if and only if:

(1) f is bijective,

(2) for every $U \subset X$, we have that $f(U) \in \mathcal{O}_Y$ if and only if $U \in \mathcal{O}_X$.

Proof. Exercise. $\square$

4 Thursday 24th January

4.1 Homeomorphisms — continued

Definition 4.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is *open* if $f(U) \in \mathcal{O}_Y$ for every $U \in \mathcal{O}_X$.

Remark 4.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. In our new terminology, Proposition 3.15 gives us that a map

$$X \xrightarrow{f} Y$$

is a homeomorphism if and only if it is bijective, continuous, and open.

Observation 4.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. If a map

$$X \xrightarrow{f} Y$$

is a homeomorphism, then

$$Y \xrightarrow{f^{-1}} X$$

is a homeomorphism. We can take the required map

$$X \xrightarrow{g} Y$$

of condition (2) of Definition 3.14 such that $g \circ f^{-1} = id_Y$ and $f^{-1} \circ g = id_X$ to be f.

Proposition 4.4. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

and

$$Y \xrightarrow{f'} Z$$

be homeomorphisms. Then

$$X \xrightarrow{f' \circ f} Z$$

is a homeomorphism.

Proof. Since f is a homeomorphism, there is a map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$. Since f' is a homeomorphism, there is a map

$$Y \xrightarrow{g'} X$$

such that $g' \circ f' = id_Y$ and $f' \circ g' = id_Z$. By Proposition 2.16, we have that $f' \circ f$ and $g \circ g'$ are continuous. Moreover

$$\begin{aligned}(g \circ g') \circ (f' \circ f) &= g \circ (g' \circ f') \circ f \\ &= g \circ id_Y \circ f \\ &= g \circ f \\ &= id_X\end{aligned}$$

and

$$\begin{aligned}(f' \circ f) \circ (g \circ g') &= f' \circ (f \circ g) \circ g' \\ &= f' \circ id_Y \circ g' \\ &= g' \circ f' \\ &= id_X.\end{aligned}$$

□

Definition 4.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then $(X, \mathcal{O}_X)$ is *homeomorphic* to $(Y, \mathcal{O}_Y)$ if there exists a homeomorphism

$$X \longrightarrow Y.$$

Notation 4.6. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. If $(X, \mathcal{O}_X)$ is homeomorphic to $(Y, \mathcal{O}_Y)$, we write $X \cong Y$.

Examples 4.7.

(1) Let $X = \{a, b, c\}$. Define

$$X \xrightarrow{f} X$$

by $a \mapsto b$, $b \mapsto c$, and $c \mapsto a$. We have that f is a bijection. Let

$$\mathcal{O} := \big\{\emptyset, \{a\}, \{b, c\}, X\big\}$$

and let

$$\mathcal{O}' := \big\{\emptyset, \{a,c\}, \{b\}, X\big\}.$$

We have that

$$\begin{aligned} f^{-1}\big(\emptyset\big) &= \emptyset \in \mathcal{O} \\ f^{-1}\big(\{a,c\}\big) &= \{b,c\} \in \mathcal{O} \\ f^{-1}\big(\{b\}\big) &= \{a\} \in \mathcal{O} \\ f^{-1}\big(X\big) &= X \in \mathcal{O}. \end{aligned}$$

Thus f defines a continuous map from $(X, \mathcal{O})$ to $(X, \mathcal{O}')$.

Moreover, we have that

$$\begin{aligned} f\big(\emptyset\big) &= \emptyset \in \mathcal{O}' \\ f\big(\{a\}\big) &= \{b\} \in \mathcal{O}' \\ f\big(\{b,c\}\big) &= \{a,c\} \in \mathcal{O}' \\ f\big(X\big) &= Y \in \mathcal{O}'. \end{aligned}$$

Thus f defines an open map from $(X, \mathcal{O})$ to $(X, \mathcal{O}')$. Putting everything together, we have that f defines a homeomorphism between $(X, \mathcal{O})$ and $(X, \mathcal{O}')$.

Let

$$\mathcal{O}'' := \big\{\emptyset, \{a,b\}, \{c\}, X\big\}.$$

Then f does not define a continuous map from $(X, \mathcal{O})$ to $(X, \mathcal{O}'')$, since $f^{-1}\big(\{a\}\big) = \{b\} \notin \mathcal{O}$. Thus f is not a homeomorphism.

Nevertheless, $(X, \mathcal{O})$ and $(X, \mathcal{O}'')$ are homeomorphic. Indeed, let

$$X \xrightarrow{g} Y$$

be given by $a \mapsto c$, $b \mapsto b$, and $c \mapsto a$. Then

$$\begin{aligned} g^{-1}\big(\emptyset\big) &= \emptyset \in \mathcal{O} \\ g^{-1}\big(\{a,b\}\big) &= \{b,c\} \in \mathcal{O} \\ g^{-1}\big(\{c\}\big) &= \{a\} \in \mathcal{O} \\ g^{-1}\big(Y\big) &= X \in \mathcal{O}. \end{aligned}$$

Thus g defines a continuous map from $(X, \mathcal{O})$ to $(X, \mathcal{O}'')$.

Moreover, we have that

$$\begin{aligned} g\big(\emptyset\big) &= \emptyset \in \mathcal{O}'' \\ g\big(\{a\}\big) &= \{c\} \in \mathcal{O} \\ g\big(\{b,c\}\big) &= \{a,b\} \in \mathcal{O}'' \\ g\big(X\big) &= Y \in \mathcal{O}''. \end{aligned}$$

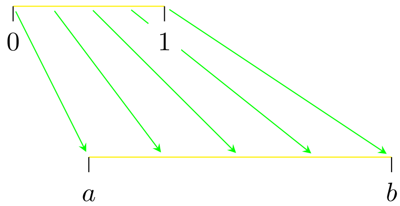
Let $\mathcal{O}''' := \big\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}\big\}$. Then f defines a continuous bijection from $(X, \mathcal{O}''')$ to $(X, \mathcal{O}')$, but f is not a homeomorphism. Indeed, $f\big(\{b\}\big) = \{c\} \notin \mathcal{O}'$.

More generally, two homeomorphic spaces whose underlying sets are finite must have the same number of open sets, so $(X, \mathcal{O}''')$ is not homeomorphic to $(X, \mathcal{O}')$.

(2) For any $a, b \in \mathbb{R}$ with $a < b$, the open interval (a,b) equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is homeomorphic to the open interval $(0,1)$ equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Indeed, let

$$(0,1) \xrightarrow{f} (a,b)$$

denote the map given by $t \mapsto a(1-t) + bt$. By Question 3 (f) of Exercise Sheet 3, we have that f is continuous. We can think of f as a 'stretching/shrinking and translation' of $(0,1)$.



A continuous inverse

$$(a,b) \xrightarrow{g} (0,1)$$

to f is defined by $t \mapsto \frac{t-a}{b-a}$. Again, that g is continuous is established by Question 3 (f) of Exercise Sheet 3. Thus f is a homeomorphism.

(3) By Proposition 4.4, we deduce from (2) that for any $a, a', b, b' \in \mathbb{R}$ with $a < b$ and $a' < b'$, the open interval (a,b) equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is homeomorphic to (a', b') equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Intuitively, we can 'stretch/shrink' and 'translate' any open interval into any other open interval.

(4) Similarly, for any $a, b \in \mathbb{R}$ with $a < b$, the closed interval $[a,b]$ equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is homeomorphic to $(I, \mathcal{O}_I)$. Indeed, the map

$$I \xrightarrow{f} [a,b]$$

given by $t \mapsto a(1-t)+bt$ again defines a homeomorphism (we just have a different source and target), with a continuous inverse

$$[a,b] \xrightarrow{g} I$$

given by $t \mapsto \frac{t-a}{b-a}$.

It is crucial here that we assume that $a < b$, and do not allow that $a = b$. Indeed the point, which we introduced in Examples 1.7 (2), is not homeomorphic to $(I, \mathcal{O}_I)$, since there is no bijection between a set with one element and I. Note that our argument above breaks down if $a = b$, since then g is not a well-defined map.

(5) By Proposition 4.4, we deduce from (4) that for any $a, a', b, b' \in \mathbb{R}$ with $a < b$ and $a' < b'$, the closed interval $[a,b]$ equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is homeomorphic to $[a',b']$ equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Again, intuitively we can 'stretch/shrink' and 'translate' any closed interval into any other closed interval. The same arguments adapt to prove that any two half open intervals are homeomorphic.

(6) Let the open interval $(-1,1)$ be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. The map

$$(-1,1) \xrightarrow{f} \mathbb{R}$$

defined by $t \mapsto \frac{t}{1-|t|}$ is continuous by Questions 3 (a) and (f) of Exercise Sheet 3 and Proposition 2.16 — check that you understand how to apply these results to deduce this! A continuous inverse

$$\mathbb{R} \xrightarrow{g} (-1,1)$$

is defined by $x \mapsto \frac{x}{1+|x|}$. Thus f is a homeomorphism.

By Proposition 4.4, we deduce from this and (3) that the open interval (a,b) equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Remark 4.8. We do not yet have any tools for proving that two topological spaces are *not* homeomorphic. For any particular map between two topological spaces, we can hope to verify whether or not it defines a homeomorphism. But to show that two topological spaces are not homeomorphic, we have to be able to prove that we cannot find *any* homeomorphism between them.

To be able to do this, we first need to develop some machinery. After this, we will in a later lecture be able to prove that for any $a, b \in \mathbb{R}$ with $a < b$, the open interval (a, b) is not homeomorphic to the closed interval $[a, b]$.

Proposition 4.9. Let $(X, \mathcal{O}_X)$, $(X', \mathcal{O}_{X'})$, $(Y, \mathcal{O}_Y)$, and $(Y', \mathcal{O}_{Y'})$ be topological spaces, and let

$$X \xrightarrow{f} Y$$

and

$$X' \xrightarrow{f'} Y'$$

be homeomorphisms. Then the map

$$X \times X' \xrightarrow{f \times f'} Y \times Y'$$

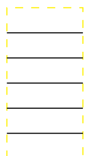
given by $(x, x') \mapsto \big(f(x), f'(x')\big)$ is a homeomorphism.

Proof. Exercise. $\square$

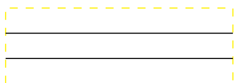
Examples 4.10.

(1) Let the open intervals (a, b), (c, d), (a', b'), and (c', d') be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $(a, b) \times (c, d)$ and $(a', b') \times (c', d')$be equipped with the product topologies.

By Proposition 4.9, we deduce from Examples 4.7 (3) that $(a, b) \times (c, d)$ is homeomorphic to $(a', b') \times (c', d')$.



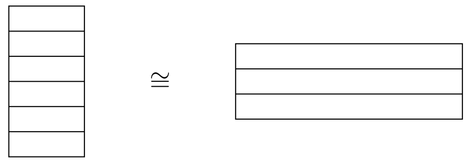
$\cong$



Intuitively, we can squash, stretch, and translate any open rectangle into any other.

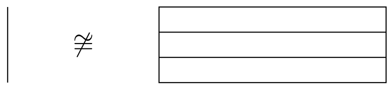
(2) Similarly, suppose that we have closed intervals $[a, b]$, $[c, d]$, $[a', b']$, and $[c', d']$ equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that $a < b$, $c < d$, $a' < b'$, and $c' < d'$.

Let $[a, b] \times [c, d]$ and $[a', b'] \times [c', d']$ be equipped with the product topologies. By Proposition 4.9 we deduce from Examples 4.7 (5) that $[a, b] \times [c, d]$ is homeomorphic to $[a', b'] \times [c', d']$.

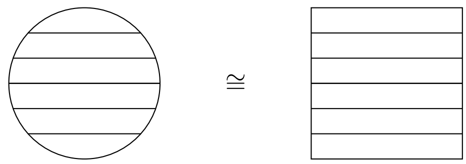


Intuitively, we can squash, stretch, and translate any open rectangle into any other. We can similarly deduce that rectangles $[a,b] \times (c,d)$ and $[a',b'] \times (c',d')$ are homeomorphic, and so on.

As in Examples 4.7 (4), note that these arguments to do not prove that a line $\{x\} \times [c,d]$ is homeomorphic to a rectangle $[a,b] \times [c,d]$. Indeed, we will in a later lecture be able to prove that these two topological spaces are not homeomorphic.



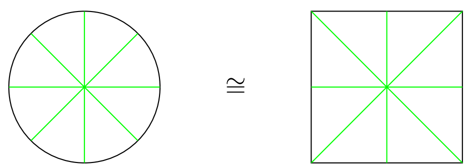
(3) We have that $D^2 \cong I^2$.



We can construct a homeomorphism

$$D^2 \xrightarrow{f} I^2$$

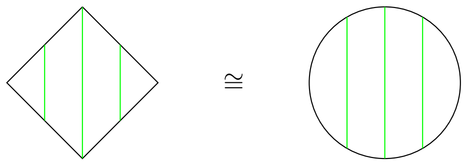
by stretching each line through the origin in D^2 to a line through the origin in I^2.



Alternatively we can for instance construct a homeomorphism

$$I^2 \xrightarrow{g} D^2$$

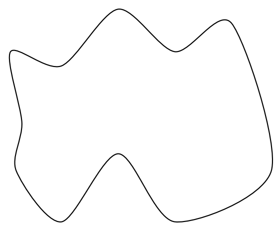
by stretching vertical lines I^2 to vertical lines in D^2.



See Exercise Sheet 4.

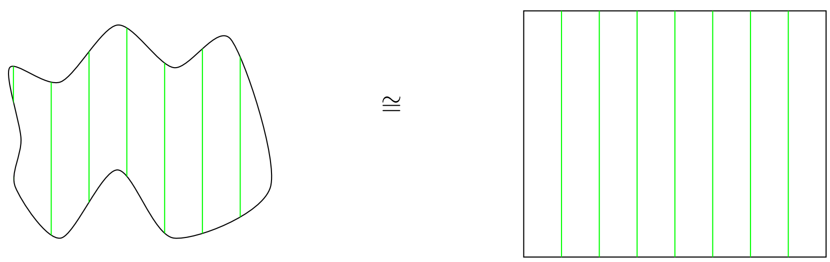
Don't be confused here: g is not inverse to f, just a different homeomorphism!

(4) Let X be a 'blob' in $\mathbb{R}^2$.



By similar ideas to those of (3) one can prove that X equipped with the subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is homeomorphic to I^2.

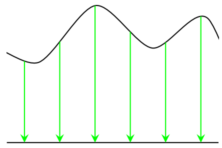
Roughly speaking one cuts X into strips with the property that there is a point in each strip to which every other point in the strip can be joined by a straight line. This property is known as *star convexity* — the strip is said to be *star shaped*.



As in (3) one proves that each strip is homeomorphic to D^2. Glueing two copies of D^2 which intersect in an arc is again homeomorphic to D^2. By induction one deduces that X is homeomorphic to D^2, and hence to I^2.

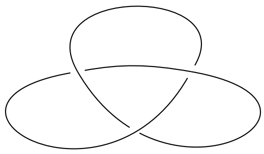
See Exercise Sheet 4.

(5) A ‘squiggle’ in $\mathbb{R}^2$ is homeomorphic to I.



See Exercise Sheet 4.

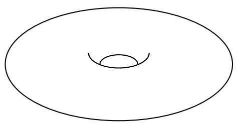
(6) We define a *knot* to be a subset of $\mathbb{R}^3$ which, equipped with the subspace topology with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$, is homeomorphic to S^1. For now we will not work rigorously with knots, but an example known as the ‘trefoil knot’ is pictured below.



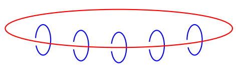
Intuitively, both the trefoil knot and S^1 may be obtained from a piece of string by glueing together the ends — we may bend, twist, and stretch the string as much as we wish before we glue the ends together.

We will look at the theory of knots later in the course.

(7) We have that $S^1 \times S^1 \cong T^2$, where $S^1 \times S^1$ is equipped with the product topology with respect to $\mathcal{O}_{S^1}$.



We will prove this in a later lecture. Intuitively, the idea is that T^2 can be obtained as a ‘circle of circles’.



Remark 4.11. Let us summarise these examples. Intuitively, two topological spaces are homeomorphic if we can bend, stretch, twist, compress, and otherwise ‘manipulate in a continuous manner’ each of these topological spaces so as to obtain the other!

4.2 Neighbourhoods and limit points

Definition 4.12. Let $(X, \mathcal{O}_X)$ be a topological space, and let $x \in X$. A *neighbourhood* of x is a subset U of X such that $x \in U$ and $U \in \mathcal{O}_X$.

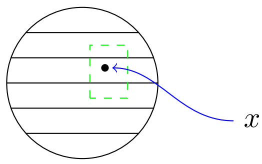
Examples 4.13.

(1) Let $X = \{a, b, c, d\}$, and let

$$\mathcal{O} := \big\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}, X\big\}.$$

The neighbourhoods of a are $\{a\}$, $\{a, b\}$, $\{a, c, d\}$, $\{a, b, d\}$, and X. The neighbourhoods of b are $\{b\}$, $\{a, b\}$, $\{a, b, d\}$, $\{b, c, d\}$, and X. The neighbourhoods of c are $\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, and X. The neighbourhoods of d are $\{c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{b, c, d\}$, and X.

(2) Let $x \in D^2$. A typical example of a neighbourhood of x is an open rectangle in D^2 containing x.



Definition 4.14. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. A *limit point* of A in X is an element $x \in X$ such that every neighbourhood of x in $(X, \mathcal{O}_X)$ contains at least one point of A.

5 Tuesday 29th January

5.1 Limits points, closure, boundary — continued

Definition 5.1. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. The *closure* of A in X is the set of limit points of A in X.

Remark 5.2. This choice of terminology will be explained by Proposition 5.7.

Notation 5.3. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. We denote the closure of A in X by $\overline{A}$.

Definition 5.4. Let $(X, \mathcal{O}_X)$ be a topological space. A subset A of X is *dense* in X if $X = \overline{A}$.

Observation 5.5. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. Every $a \in A$ is a limit point of A, so $A \subset \overline{A}$.

Examples 5.6.

(1) Let $X = \{a, b\}$, and let $\mathcal{O} := \big\{\emptyset, \{b\}, X\big\}$. In other words, $(X, \mathcal{O})$ is the Sierpiński interval. Let $A := \{b\}$. We have that a is a limit point of A. Indeed, X is the only neighbourhood of a in X, and it contains b. Thus $\overline{A} = X$, and we have that A is dense in X.

(2) Let $X = \{a, b, c, d, e\}$, and let $\mathcal{O}$ denote the topology on X given by

$$\big\{\emptyset, \{a\}, \{b\}, \{c, d\}, \{a, b\}, \{a, c, d\}, \{b, e\}, \{b, c, d\}, \{b, c, d, e\}, \{a, b, c, d\}, \{a, b, e\}, X\big\}.$$

Let $A := \{d\}$. Then c is a limit point of A, since the neighbourhoods of $\{c\}$ in X are $\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{b, c, d, e\}$, $\{a, b, c, d\}$, and X, all of which contain d.

But b is not a limit point of A, since $\{b\}$ is a neighbourhood of b, and $\{b\} \cap A = \emptyset$. Similarly, a is not a limit point of A.

Also, $\{e\}$ is not a limit point of A, since the neighbourhood $\{b, e\}$ of e does not contain d. Thus $\overline{A} = \{c, d\}$.

Let $A' := \{b, d\}$. Then c is a limit point of A', since every neighbourhood of c in X contains d, as we already observed.

In addition, e is a limit point of A, since the neighbourhoods of e in X are $\{b, e\}$, $\{b, c, d, e\}$, $\{a, b, e\}$, and X, all of which contain either b or d, or both.

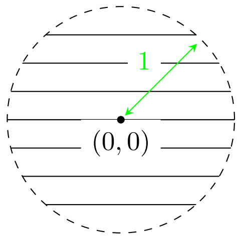
But a is not a limit point of A', since $\{a\} \cap A' = \emptyset$. Thus $\overline{A'} = \{b, c, d, e\}$.

(3) Let $A := [0, 1)$. Then 1 is a limit point of A in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. See Exercise Sheet 4.

(4) Let $A := \mathbb{Q}$, the set of rational numbers. Then every $x \in \mathbb{R}$ is a limit point of A in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Indeed, for every open interval (a, b) such that $a, b \in \mathbb{R}$ and $x \in (a, b)$, there is a rational number q with $a < q < x$. Thus $\overline{A} = \mathbb{R}$, and we have that $\mathbb{Q}$ is dense in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

(5) Let $A := \mathbb{Z}$, the set of integers. Then no real number which is not an integer is a limit point of A in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Indeed, let $x \in \mathbb{R}$, and suppose that $x \notin \mathbb{Z}$. Then $(\lfloor x \rfloor, \lceil x \rceil)$ is a neighbourhood of x not containing any integer. Thus $\overline{A} = \mathbb{Z}$. Here $\lfloor x \rfloor$ is the floor of x, namely the largest integer z such that $z \leq x$, and $\lceil x \rceil$ is the roof of x, namely the smallest integer z such that $z \geq x$.

(6) Let $A := \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| < 1\}$, the *open disc* around 0 in $\mathbb{R}^2$ of radius 1.



Then $(x, y) \in \mathbb{R}^2$ is a limit point of A in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$ if and only if $(x, y) \in D^2$. Let us prove this. If $(x, y) \notin D^2$, then $\|(x, y)\| > 1$. Let $\epsilon \in \mathbb{R}$ be such that

$$0 < \epsilon \leq |x| - \frac{|x|}{\|(x, y)\|},$$

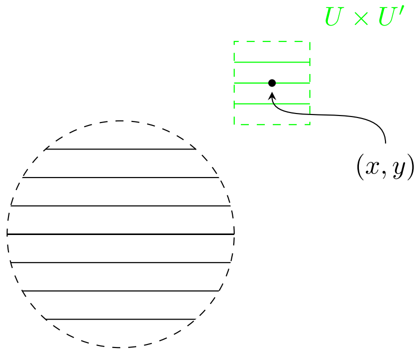
and let $\epsilon' \in \mathbb{R}$ be such that

$$0 < \epsilon' \leq |y| - \frac{|y|}{\|(x, y)\|}.$$

GLet $U := (x - \epsilon, x + \epsilon)$, and let $U' := (y - \epsilon', y + \epsilon')$. By definition of $\mathcal{O}_{\mathbb{R}\times\mathbb{R}}$, $U \times U' \in \mathcal{O}_{\mathbb{R}\times\mathbb{R}}$. Moreover, for every $(u, u') \in U \times U'$, we have that

$$\begin{aligned}
\|(u, u')\| &= \|(|u|, |u'|)\| \\
&> \|(|x| - \epsilon, |y| - \epsilon')\| \\
&\geq \|\frac{1}{\|(|x|, |y|)\|}(x, y)\| \\
&= 1.
\end{aligned}$$

Thus $U \times U'$ is a neighbourhood of (x, y) in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$ with the property that $A \cap (U \times U') = \emptyset$. We deduce that (x, y) is not a limit point of A.

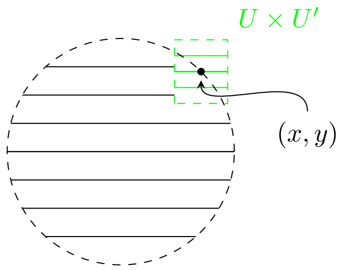


Suppose now that $(x, y) \in S^1$. Let W be a neighbourhood of (x, y) in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. By definition of $\mathcal{O}_{\mathbb{R}\times\mathbb{R}}$, there is an open interval U in $\mathbb{R}$ and an open interval U' in $\mathbb{R}$ such that $x \in U$, $y \in U'$, and $U \times U' \subset W$.

Let us denote the open interval $\{|u| \mid u \in U\}$ in $\mathbb{R}$ by (a, b) for $a, b \in \mathbb{R}$, and let us denote the open interval $\{|u'| \mid u' \in U'\}$ in $\mathbb{R}$ by (a', b') for $a', b' \in \mathbb{R}$. Let $x' \in U$ be such that $a < |x'| < |x|$, and let $y' \in U$ be such that $a' < |y'| < |y|$. Then we have that

$$\begin{aligned}
\|(x', y')\| &= \|(|x'|, |y'|)\| \\
&< \|(|x|, |y|)\| \\
&= 1.
\end{aligned}$$

Thus $(x', y') \in A \cap (U \times U')$, and hence $(x', y') \in A \cap W$. Thus (x, y) is a limit point of A.



Putting everything together, we conclude that $\overline{A} = D^2$.

Proposition 5.7. Let $(X, \mathcal{O}_X)$ be a topological space, and let V be a subset of X. Then V is closed in $(X, \mathcal{O}_X)$ if and only if $V = \overline{V}$.

Proof. Suppose that V is closed. By definition, $X \setminus V$ is then open. Thus, for any $x \in X$ such that $x \notin V$, we have that $X \setminus V$ is a neighbourhood of x. Moreover, by definition, $X \setminus V$ does not contain any element of V. Thus x is not a limit point of V in X. We conclude that $V = \overline{V}$.

Suppose now that $V = \overline{V}$. Then for every $x \notin V$ there is a neighbourhood of x which does not contain any element of V. Let us denote this neighbourhood by U_x. We make three observations.

(1) $X \setminus V \subset \bigcup_{x \in X \setminus V} U_x$, since $x \in U_x$.

(2) $\bigcup_{x \in X \setminus V} U_x \subset X \setminus V$, since

$$V \cap \Big(\bigcup_{x \in X \setminus V} U_x \Big) = \bigcup_{x \in X \setminus V} (U_x \cap V) = \bigcup_{x \in X \setminus V} \emptyset = \emptyset.$$

(3) $\bigcup_{x \in X \setminus V} U_x \in \mathcal{O}_X$, since $U_x \in \mathcal{O}_X$ for all $x \in X \setminus V$.

Putting (1) and (2) together, we have that $\bigcup_{x \in X \setminus V} U_x = X \setminus V$. Hence, by (3), $X \setminus V \in \mathcal{O}_X$. Thus V is closed. □

Remark 5.8. In other words, a subset V of a topological space $(X, \mathcal{O}_X)$ is closed if and only if every limit point of V belongs to V.

Proposition 5.9. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. Suppose that V is a closed subset of X with $A \subset V$. Then $\overline{A} \subset V$.

Proof. See Exercise Sheet 4. □

Remark 5.10. In other words, $\overline{A}$ is the smallest closed subset of X containing A.

Corollary 5.11. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. Then

$$\overline{A} = \bigcap_{V} V,$$

where the intersection is taken over all closed subsets V of X with the property that $A \subset V$.

Proof. Follows immediately from Proposition 5.9. □

Definition 5.12. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. The *boundary* of A in X is the set $x \in X$ such that every neighbourhood of x in X contains at least one element of A and at least one element of $X \setminus A$.

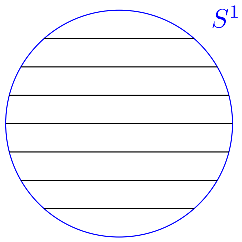
Notation 5.13. We denote the boundary of A in X by $\partial_X A$.

Observation 5.14. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. Every limit point of A which does not belong to A belongs to $\partial_X A$.

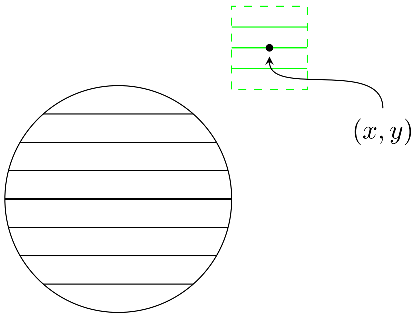
Terminology 5.15. The boundary of A in X is also known as the *frontier* of A in X.

Examples 5.16.

(1) Let $X := (\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$, and let $A := D^2$. Then $\partial_X A = S^1$.



Let us prove this. By exactly the argument of the first part of the proof in Examples 5.6 (6), every $(x, y) \in \mathbb{R}^2 \setminus D^2$ is not a limit point of D^2. Thus $\partial_A X \subset D^2$.



Suppose that $(x, y) \in D^2$, but that $(x, y) \notin S^1$. Then $\|(x, y)\| < 1$. Let $\epsilon \in \mathbb{R}$ be such that

$$0 < \epsilon \leq \frac{|x|}{\|(x, y)\|} - |x|,$$

and let $\epsilon' \in \mathbb{R}$ be such that

$$0 < \epsilon' \leq \frac{|y|}{\|(x, y)\|} - |y|.$$

Let $U := (x - \epsilon, x + \epsilon)$, and let $U' := (y - \epsilon', y + \epsilon')$. By definition of $\mathcal{O}_{\mathbb{R}\times\mathbb{R}}$, $U \times U' \in \mathcal{O}_{\mathbb{R}\times\mathbb{R}}$. Moreover, for every $(u, u') \in U \times U'$, we have that

$$\begin{aligned}
\|(u, u')\| &= \|(|u|, |u'|)\| \\
&< \|(|x| + \epsilon, |y| + \epsilon')\| \\
&\leq \|\frac{1}{\|(x, y)\|}(|x|, |y|)\| \\
&= 1.
\end{aligned}$$

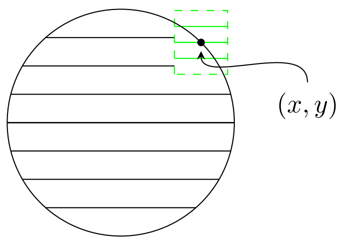
Thus $U \times U'$ is a neighbourhood of (x, y) in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$ with the property that $(\mathbb{R}^2 \setminus D^2) \cap (U \times U') = \emptyset$. We deduce that $(x, y) \notin \partial_A X$.

We now have that $\partial_X A \subset S^1$. Suppose that $(x, y) \in S^1$, and let W be a neighbourhood of (x, y) in $\mathbb{R}^2$. By definition of $\mathcal{O}_{\mathbb{R}\times\mathbb{R}}$, there is an open interval U in $\mathbb{R}$ and an open interval U' in $\mathbb{R}$ such that $x \in U$, $y \in U'$, and $U \times U' \subset W$.

Let us denote the open interval $\{|u| \mid u \in U\}$ in $\mathbb{R}$ by (a, b) for $a, b \in \mathbb{R}$, and let us denote the open interval $\{|u'| \mid u' \in U'\}$ in $\mathbb{R}$ by (a', b') for $a', b' \in \mathbb{R}$. Let $x' \in U$ be such that $|x| < |x'| < b$, and let $y' \in U$ be such that $|y| < |y'| < b'$. Then we have that

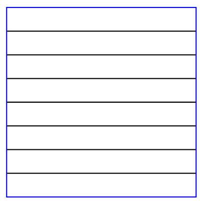
$$\begin{aligned} \|(x', y')\| &= \|(|x'|, |y'|)\| \\ &> \|(|x|, |y|)\| \\ &= 1. \end{aligned}$$

Thus $(x', y') \in (\mathbb{R}^2 \setminus D^2) \cap (U \times U')$, and hence $(x', y') \in (\mathbb{R}^2 \setminus D^2) \cap W$. In addition, (x, y) belongs to both D^2 and W. We deduce that $(x, y) \in \partial_X A$, and conclude that $S^1 \subset \partial_X A$.



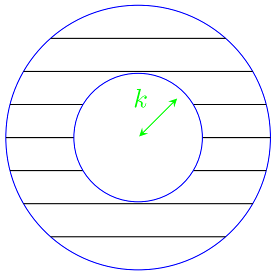
Putting everything together, we have that $\partial_A X = S^1$. Alternatively, this may be deduced from Example (2) below, via a homeomorphism between D^2 and I^2.

(2) Let $X := (\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$, and let $A := I^2$. Then $\partial_A X$ is as indicated in blue below.



We have at least three ways to prove this. Firstly, as a corollary of Example (1), via a homeomorphism between I^2 and D^2. Secondly directly, by an argument similar to that in Example (1). Thirdly as a corollary of Example (4) below, using a general result on the boundary of a product of topological spaces which we will prove in Exercise Sheet 4.

(3) Let $X := (\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$, and let $A := A_k$, an annulus, for some $k \in \mathbb{R}$ with $0 < k < 1$. Then $\partial_X A$ is as indicated in blue below. This may be proven by an argument similar to that in Example (1).



(4) Let $X := (\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then $\partial_X(0,1) = \partial_X\,(0,1] = \partial_X\,[0,1) = \partial_X\,[0,1] = \{0,1\}$. See Exercise Sheet 4.

(5) Let $X := \{a,b,c,d,e\}$, and let $\mathcal{O}$ denote the topology

$$\big\{\emptyset, \{a\}, \{b\}, \{c,d\}, \{a,b\}, \{a,c,d\}, \{b,e\}, \{b,c,d\}, \{b,c,d,e\}, \{a,b,c,d\}, \{a,b,e\}, X\big\}$$

on X, as in Examples 5.6 (2). Let $A := \{b,d\}$.

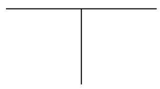
We saw in Examples 5.6 (2) that the limit points of A which do not belong to A are $\{c\}$ and $\{e\}$. Also $d \in \partial_X A$. Indeed, the neighbourhoods of d in X are $\{c,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, $\{b,c,d,e\}$, $\{a,b,c,d\}$, and X. Each of these neighbourhoods contains c, which does not belong to A.

But b does not belong to $\partial_X A$, since $\{b\}$ is a neighbourhood of b in X, and $\{b\}$ does not contain an element of $X \setminus A$. Thus $\partial_X A = \{c,d,e\}$.

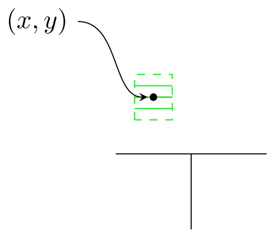
(6) Let A denote the letter T, viewed as the subset

$$\{(0,y) \mid 0 \leq y \leq 1\} \cup \{(x,1) \mid -1 \leq x \leq 1\}$$

of $\mathbb{R}^2$.



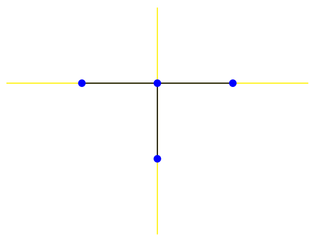
Let $X := (\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. Then $\partial_X \mathsf{T} = \mathsf{T}$. Indeed, for every $(x,y) \notin \mathsf{T}$, there exists a neighbourhood $U \times U' \subset \mathbb{R}^2$ of (x,y) such that $(U \times U') \cap \mathsf{T} = \emptyset$.



Instead, let X denote the subset

$$\{(0, y) \mid y \in \mathbb{R}\} \cup \{(x, 1) \mid x \in \mathbb{R}\}$$

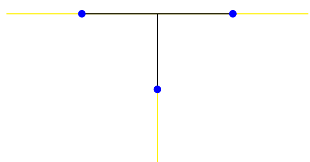
of $\mathbb{R}^2$, equipped with the subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. Then $\partial_X \mathsf{T}$ consists of the four elements of T indicated in blue in the following picture, in which X is drawn in yellow.



Now let X denote the subset

$$\{(0, y) \mid y \leq 1\} \cup \{(x, 1) \mid x \in \mathbb{R}\}$$

of $\mathbb{R}^2$, equipped with the subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. Then $\partial_X \mathsf{T}$ consists of the three elements of T indicated in blue in the following picture, in which again X is drawn in yellow.



As Examples 5.16 (6) illustrates, a set A may have a different boundary depending upon which topological space it is regarded as a subset of.

5.2 Coproduct topology

Recollection 5.17. Let X and Y be sets. The *disjoint union* of X and Y is the set $(X \times \{0\}) \cup (Y \times \{1\})$.

Let

$$X \xrightarrow{i_X} X \sqcup Y$$

denote the map given by $x \mapsto (x, 0)$, and let

$$Y \xrightarrow{i_Y} X \sqcup Y$$

denote the map given by $y \mapsto (y, 1)$.

Terminology 5.18. A disjoint union is also known as a *coproduct*.

Proposition 5.19. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\mathcal{O}_{X \sqcup Y}$ be the set of subsets U of $X \sqcup Y$ such the following conditions are satisfied.

(1) $i_X^{-1}(U) \in \mathcal{O}_X$.

(2) $i_Y^{-1}(U) \in \mathcal{O}_Y$.

Then $\mathcal{O}_{X \sqcup Y}$ defines a topology on $X \sqcup Y$.

Proof. Exercise. $\square$

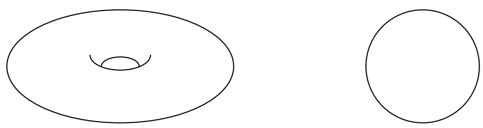
Terminology 5.20. We refer to $\mathcal{O}_{X \sqcup Y}$ as the *coproduct topology* on $X \sqcup Y$.

Observation 5.21. It is immediate from the definition of $\mathcal{O}_{X \sqcup Y}$ that i_X and i_Y are continuous.

Examples 5.22.

(1) $T^2 \sqcup T^2$.

(2) $T^2 \sqcup S^1$.



The disjoint union of two sets is very different from the union. Indeed, $T^2 \cup T^2 = T^2$. Two doughnuts are very different from one doughnut!

6 Thursday 31st January

6.1 Connected topological spaces — equivalent conditions, an example, and two non-examples

Observation 6.1. Both X and Y are open in $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$. Indeed, $i_X^{-1}(X \sqcup Y) = X$, and $X \in \mathcal{O}_X$. Similarly, $i_Y^{-1}(X \sqcup Y) = Y$, and $Y \in \mathcal{O}_Y$.

Moreover, $(X \sqcup Y) \setminus X = Y$. Thus, since Y is open in $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$, we have that X is closed in $X \sqcup Y$. Similarly, $(X \sqcup Y) \setminus Y = X$. Since X is open in $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$, we have that Y is closed in $X \sqcup Y$.

Putting everything together, we have that X and Y are each both open and closed in $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$.

Definition 6.2. A space $(X, \mathcal{O}_X)$ is *connected* if there do not exist $X_0, X_1 \subset X$ such that the following all hold.

(1) $X = X_0 \sqcup X_1$.

(2) $X_0 \in \mathcal{O}_X$, and $X_0 \neq \emptyset$.

(3) $X_1 \in \mathcal{O}_X$, and $X_1 \neq \emptyset$.

Proposition 6.3. Let $(X, \mathcal{O}_X)$ be a topological space. Then X is connected if and only if the only subsets of X which are both open and closed in $(X, \mathcal{O}_X)$ are $\emptyset$ and X.

Proof. Suppose that there exists a subset A of X which is both open and closed. Then A and $X \setminus A$ are both open in X, and also $X = A \sqcup (X \setminus A)$. If X is connected, A must therefore be $\emptyset$ or X.

Suppose now that the only subsets of X which are both open and closed are $\emptyset$ and X, and that $X = A \sqcup A'$, with both A and A' open. Then since $X \setminus A = A'$, $X \setminus A$ is open in X, and thus A is closed in X. Thus A is $\emptyset$ or X. Hence X is connected. $\square$

Observation 6.4. Let X be a set, and let A and A' be subsets of X. Then $X = A \sqcup A'$ if and only if the following conditions are satisfied.

(1) $X = A \cup A'$.

(2) $A \cap A' = \emptyset$.

Proposition 6.5. Let $\{0,1\}$ be equipped with the discrete topology $\mathcal{O}^{\text{disc}}_{\{0,1\}}$. A topological space $(X, \mathcal{O}_X)$ is connected if and only if there does not exist a surjective continuous map

$$X \longrightarrow \{0,1\}.$$

Proof. Suppose that there exists a surjective continuous map

$$X \xrightarrow{f} \{0,1\}.$$

We make the following observations.

(1) $f^{-1}(0)$ and $f^{-1}(1)$ are both open in X, since f is continuous and $\{0\}$ and $\{1\}$ both belong to $\mathcal{O}^{\text{disc}}_{\{0,1\}}$.

(2) $f^{-1}(0)$ and $f^{-1}(1)$ are both non-empty in X, since f is surjective.

(3) $f^{-1}(0) \cup f^{-1}(1) = f^{-1}(\{0,1\}) = X$.

(4) $f^{-1}(0) \cap f^{-1}(1) = \{x \in X \mid f(x) = 0 \text{ and } f(x) = 1\} = \emptyset$, since f is a well-defined map.

By (3) and (4) and Observation 6.4, $X = f^{-1}(0) \sqcup f^{-1}(1)$. Thus, by (1) and (2), X is not connected.

Suppose now that X is not connected. Thus we have $X = A \sqcup A'$ for a pair of open subsets A and A' of X. Define

$$X \xrightarrow{f} \{0,1\}$$

by

$$\begin{cases} x \mapsto 0 & \text{if } x \in A, \\ x \mapsto 1 & \text{if } x \in A'. \end{cases}$$

Then $f^{-1}(0) = A$ and $f^{-1}(1) = A'$, and thus f is continuous. □

Examples 6.6.

(1) Let $X = \{a,b\}$ be a set with two elements, and let $\mathcal{O} := \{\emptyset, \{b\}, X\}$. In other words, $(X, \mathcal{O})$ is the Sierpiński interval. Then $(X, \mathcal{O})$ is connected, since the only way to express X as a disjoint union is $X = \{a\} \sqcup \{b\}$, but $\{a\} \notin \mathcal{O}$.

(2) Take $X = \{a,b,c,d,e\}$. Let $\mathcal{O}$ be the topology

$$\{\emptyset, \{a\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{c,d,e\}, \{a,b,c,d\}, \{a,c,d,e\}, X\}$$

on X. Then $(X, \mathcal{O})$ is not connected, since $X = \{a,b\} \sqcup \{c,d,e\}$, and we have that both $\{a,b\}$ and $\{c,d,e\}$ belong to $\mathcal{O}$.

(3) Equip $\mathbb{Q}$ with the subspace topology $\mathcal{O}_{\mathbb{Q}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not connected. Indeed, pick any irrational $x \in \mathbb{R}$, such as $x = \sqrt{2}$. Then

$$\mathbb{Q} = \big(\mathbb{Q} \cap (-\infty, x)\big) \sqcup \big(\mathbb{Q} \cap (x, \infty)\big),$$

and since both $(-\infty, x)$ and (x, ∞) belong to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, we have that both $\mathbb{Q} \cap (-\infty, x)$ and $\mathbb{Q} \cap (x, \infty)$ belong to $\mathcal{O}_{\mathbb{Q}}$.

6.2 Connectedness of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$

Lemma 6.7. Let A be a subset of $\mathbb{R}$ which is bounded below. Let b denote the greatest lower bound of A, which exists by the completeness of $\mathbb{R}$, as expressed in Theorem 1.10. Then $b \in \overline{A}$, and for every $x \in \overline{A}$, we have that $b < x$. Here $\overline{A}$ is the closure of A with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Proof. Let U be a neighbourhood of b in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By definition of $\mathcal{O}_{\mathbb{R}}$, U is a union of open intervals in $\mathbb{R}$. One of these open intervals must contain b. Let us denote it by (x, x'). There exists $b' \in A$ such that $b \leq b' < x'$, since otherwise x' would be a lower bound of A with the property that $x' > b$. We thus have that $b' \in (x, x')$, and since $(x, x') \subset U$, we deduce that $b' \in U$.

We have now shown that $b' \in A \cap U$. We conclude that b is a limit point of A in $\mathbb{R}$, and thus that $b \in \overline{A}$.

Suppose now that $a \in \overline{A}$. If $a < b$, let $\epsilon := b - a$. Since $\epsilon > 0$, we have that $(a - \epsilon, a + \epsilon)$ is a neighbourhood of a in $\mathbb{R}$. Since a is a limit point of A in $\mathbb{R}$, we deduce that there exists $a' \in \mathbb{R}$ with $a' \in A \cap (a - \epsilon, a + \epsilon)$. But then $a' < a + \epsilon$, and since $a + \epsilon = b$, we have that $a' < b$. Together with the fact that $a' \in A$, this contradicts our assumption that b is a lower bound of A.

We therefore have that $a \geq b$, as required.

$\square$

Example 6.8. For a prototypical illustration of Lemma 6.7, let A denote the open interval $(0, 1)$.

Then 0 is the greatest lower bound of A, and $\overline{A} = [0, 1]$.

Proposition 6.9. The topological space $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.

Proof. Suppose that there exists a subset U of $\mathbb{R}$ which is both open and closed in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, and such that $U \neq \mathbb{R}$ and $U \neq \emptyset$. Let $x \in \mathbb{R} \setminus U$. Since $U \neq \emptyset$, either $U \cap [x, \infty) \neq \emptyset$ or $U \cap (-\infty, x] \neq \emptyset$. Suppose that $U \cap [x, \infty) \neq \emptyset$, and let us denote $U \cap [x, \infty)$ by A. Then

$$\begin{aligned}\mathbb{R} \setminus A &= \mathbb{R} \setminus \big(U \cap [x, \infty)\big) \\ &= \big(\mathbb{R} \setminus U\big) \cup \big(\mathbb{R} \setminus [x, \infty)\big) \\ &= (\mathbb{R} \setminus U) \cup (-\infty, x).\end{aligned}$$

Since U is closed in $\mathbb{R}$, $\mathbb{R} \setminus U$ is open in $\mathbb{R}$. Also, $(-\infty, x)$ is open in $\mathbb{R}$. Thus $\mathbb{R} \setminus A$ is open in $\mathbb{R}$, and hence A is closed in $\mathbb{R}$.

In addition, since $x \notin U$, we have that $A = U \cap (x, \infty)$. Since U is open in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, and since (x, ∞) is also open in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, we have that A is open in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

By definition of A, x is a lower bound of A. Thus, by the completeness of $\mathbb{R}$ as expressed in Theorem 1.10, A admits a greatest lower bound. Let us denote it by $b \in \mathbb{R}$. Since A is closed in $\mathbb{R}$, by Lemma 6.7 and Proposition 5.7, we have that $b \in A$, and that for every $a \in A$, $b \leq a$.

But since A is open in $\mathbb{R}$, it is a union of open intervals in $\mathbb{R}$, one of which must contain b. Let us denote it by (a', a''). Then $a' < b$, which since $a' \in A$ contradicts that $b \leq a$ for all $a \in A$.

The proof in the case that $U \cap (-\infty, x] \neq \emptyset$ is entirely analogous. $\square$

7 Tuesday 5th February

7.1 Characterisation of connected subspaces of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$

Proposition 7.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let

$$X \xrightarrow{f} Y$$

be a continuous map. Suppose that $(X, \mathcal{O}_X)$ is connected. Let $f(X)$ be equipped with the subspace topology $\mathcal{O}_{f(X)}$ with respect to $(Y, \mathcal{O}_Y)$. Then $\big(f(X), \mathcal{O}_{f(X)}\big)$ is connected.

Proof. Suppose that $f(X) = U_0 \sqcup U_1$, and that U_0 and U_1 are open in $f(X)$. By definition, $U_0 = f(X) \cap Y_0$ for an open subset Y_0 of Y, and $U_1 = f(X) \cap Y_1$ for an open subset Y_1 of Y. We make the following observations.

(1) Since f is continuous, $f^{-1}(Y_0)$ is open in X. We have that

$$\begin{aligned} f^{-1}(U_0) &= f^{-1}\big(f(X) \cap Y_0\big) \\ &= f^{-1}\big(f(X)\big) \cap f^{-1}(Y_0) \\ &= X \cap f^{-1}(Y_0) \\ &= f^{-1}(Y_0). \end{aligned}$$

Thus $f^{-1}(U_0)$ is open in X.

(2) By an analogous argument, $f^{-1}(U_1)$ is open in X.

(3) We have that $f^{-1}(U_0) \cap f^{-1}(U_1) = f^{-1}(U_0 \cap U_1)$. Since $U_0 \cap U_1 = \emptyset$, we deduce that $f^{-1}(U_0 \cap U_1) = \emptyset$. Thus $f^{-1}(U_0) \cap f^{-1}(U_1) = \emptyset$.

(4) We have that $f^{-1}(U_0) \cup f^{-1}(U_1) = f^{-1}(U_0 \cup U_1)$. Since $U_0 \cup U_1 = f(X)$, and since $f^{-1}\big(f(X)\big) = X$, we deduce that $f^{-1}(U_0) \cup f^{-1}(U_1) = X$.

By (3) and (4), we have that $X = f^{-1}(U_0) \sqcup f^{-1}(U_1)$. Thus, by (1), (2), and the fact that $(X, \mathcal{O}_X)$ is connected, we must have that either $f^{-1}(U_0) = X$ or that $f^{-1}(U_0) = \emptyset$. Hence either $U_0 = f(X)$ or $U_0 = \emptyset$. $\square$

Remark 7.2. We will sometimes refer to the conclusion of Proposition 7.1 as: 'the continuous image of a connected topological space is connected'.

Corollary 7.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let

$$X \xrightarrow{f} Y$$

be a homeomorphism. If $(X, \mathcal{O}_X)$ is connected, then $(Y, \mathcal{O}_Y)$ is connected.

Proof. By Proposition 3.15 we have that f is surjective, or in other words we have that $f(X) = Y$. It follows immediately from Proposition 7.1 that Y is connected. $\square$

Proposition 7.4. Let $(X, \mathcal{O}_X)$ be a connected topological space, and let A and A' be subsets of X. Let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$, and let A' be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Suppose that $(A', \mathcal{O}_{A'})$ is connected, and that $A' \subset A \subset \overline{A'}$. Then $(A, \mathcal{O}_A)$ is connected.

Proof. Suppose that $A = U_0 \sqcup U_1$, where both U_0 and U_1 belong to $\mathcal{O}_A$. By definition of $\mathcal{O}_A$, we have that $U_0 = A \cap U_0'$ for an open subset U_0' of X, and that $U_1 = A \cap U_1'$ for an open subset U_1' of X.

We have that

$$\begin{aligned}(A' \cap U_0) \cup (A' \cap U_1) &= A' \cap (U_0 \cup U_1) \\ &= A' \cap A \\ &= A',\end{aligned}$$

where for the final equality we appeal to the fact that $A' \subset A$.

Moreover, we have that

$$\begin{aligned}(A' \cap U_0) \cap (A' \cap U_1) &= A' \cap (U_0 \cap U_1) \\ &= A' \cap \emptyset \\ &= \emptyset.\end{aligned}$$

Putting the last two observations together, we have that $A' = (A' \cap U_0) \sqcup (A' \cap U_1)$. Moreover, we have that

$$\begin{aligned}A' \cap U_0 &= A' \cap (A \cap U_0') \\ &= (A' \cap A) \cap U_0' \\ &= A' \cap U_0',\end{aligned}$$

and that

$$\begin{aligned}A' \cap U_1 &= A' \cap (A \cap U_1') \\ &= (A' \cap A) \cap U_1' \\ &= A' \cap U_1'.\end{aligned}$$

For the last equality in each case we appeal again to the fact that $A' \subset A$. By definition of $\mathcal{O}_{A'}$, we thus have that $A' \cap U_0 = A' \cap U_0'$ and $A' \cap U_1 = A' \cap U_1'$ are open in A'. Since $(A', \mathcal{O}_{A'})$ is connected, we deduce that either $A' \cap U_0 = \emptyset$ or $A' \cap U_1 = \emptyset$.

Suppose that $A' \cap U_0 = \emptyset$. Since $A' \cap U_0 = A' \cap U_0'$, we then have that $A' \cap U_0' = \emptyset$. Thus $A' \subset X \setminus U_0'$. Since U_0' is open in X, we have that $X \setminus U_0'$ is closed in X. By Proposition 5.9, we deduce that $\overline{A'} \subset X \setminus U_0'$.

By assumption we have that $A \subset \overline{A'}$. Thus $U_0 = A \cap U_0' \subset \overline{A'} \cap X_0 = \emptyset$, so that $U_0 = \emptyset$.

An entirely analogous argument gives that if $A' \cap U_1 = \emptyset$, then $U_1 = \emptyset$. Putting everything together, we have proven that if $A = U_0 \sqcup U_1$, where both U_0 and U_1 are open in $(A, \mathcal{O}_A)$, then either $U_0 = \emptyset$ or $U_1 = \emptyset$. Thus $(A, \mathcal{O}_A)$ is connected.

□

Remark 7.5. We will sometimes refer to Proposition 7.4 as the 'sandwich proposition'.

Corollary 7.6. *Let $(X, \mathcal{O}_X)$ be a connected topological space, and let A be a subset of X. Let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$, and let $\overline{A}$ be equipped with the subspace topology $\mathcal{O}_{\overline{A}}$ with respect to $(X, \mathcal{O}_X)$. Suppose that $(A, \mathcal{O}_A)$ is connected. Then $(\overline{A}, \mathcal{O}_{\overline{A}})$ is connected.*

Proof. Follows immediately from Proposition 7.4, taking both A and A' to be A. □

Lemma 7.7. *Let $(X, \mathcal{O})$ be a topological space. If X is empty or consists of exactly one element, then $(X, \mathcal{O})$ is connected.*

Proof. If X is empty or consists of exactly one element, the only ways to express X as a disjoint union of subsets are $X = \emptyset \sqcup X$ and $X = X \sqcup \emptyset$. For $X = \emptyset \sqcup X$, condition (2) of Definition 6.2 is not satisfied. For $X = X \sqcup \emptyset$, condition (3) of Definition 6.2 is not satisfied. □

Lemma 7.8. *Let X be a non-empty subset of $\mathbb{R}$. Then X is an open interval, a closed interval, or a half open interval if and only if for every $x, x' \in X$ and $y \in \mathbb{R}$ with $x < y < x'$ we have that $y \in X$.*

Proof. Suppose that $X = [a, b]$, for $a, b \in \mathbb{R}$ with $a \leq b$. If $x, x' \in X$, then by definition of $[a, b]$ we have that $a \leq x \leq b$ and $a \leq x' \leq b$. Suppose that $x < x'$ and that $y \in \mathbb{R}$ has the property that $x < y < x'$. Then we have that $a \leq x < y < x'' \leq b$, and in particular $a \leq y \leq b$. Thus, by definition of $[a, b]$, we have that $y \in [a, b]$.

If X is an open interval or a half open interval, an entirely analogous argument proves that if $x, x' \in X$ and $y \in \mathbb{R}$ have the property that $x < y < x'$, then $y \in X$.

Conversely, suppose that for every $x, x' \in X$ and $y \in \mathbb{R}$ with $x < y < x'$ we have that $y \in X$. Let $a = \inf X$ and let $b = \sup X$. As in Lecture 1, if X is not bounded below we adopt the convention that $\inf X = -\infty$, and if X is not bounded above we adopt the convention that $\sup X = \infty$.

Suppose that $y \in X$ has the property that $a < y < b$. Since $y > a$ there is an $x \in X$ with $x < y$, since otherwise y would be a lower bound of X, contradicting the fact that a is by definition the greatest lower bound of X. Since $y < b$ there is an $x' \in X$ with $y < x'$, since otherwise y would be an upper bound of X, contradicting the fact that b is by definition the least upper bound of X. We have shown that $x < y < x'$, with $x, x' \in X$. By assumption, we deduce that $y \in X$.

We have now shown that for all $y \in X$ such that $a < y < b$, we have that $y \in X$. To complete the proof, there are four cases to consider.

(1) If $a, b \in X$, we have by definition of $[a, b]$ that $X = [a, b]$.

(2) If $a \in X$ and $b \notin X$, we have by definition of $[a, b)$ that $X = [a, b)$.

(3) If $a \notin X$ and $b \in X$, we have by definition of $(a, b]$ that $X = (a, b]$.

(4) If $a \notin X$ and $b \notin X$, we have by definition of (a, b) that $X = (a, b)$.

□

Proposition 7.9. Let X be a subset of $\mathbb{R}$, and let X be equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then X is connected if and only if X is an open interval, a closed interval, a half open interval, or $\emptyset$.

Proof. If X is not an open interval, a closed interval, a half open interval, or $\emptyset$, then by Lemma 7.8 there are $x, x' \in X$ and $y \in \mathbb{R} \setminus X$ with $x < y < x'$. Let $U_0 := X \cap (-\infty, y)$, and let $U_1 := X \cap (y, \infty)$. By definition of $\mathcal{O}_X$, both U_0 and U_1 are open in X. Moreover, $X = X_0 \sqcup X_1$. Thus X is not connected. This completes one direction of the proof.

Conversely, let us prove that if X is an open interval, a closed interval, a half open interval, or $\emptyset$, then X is connected. By Lemma 7.7, if $X = \emptyset$ or if X consists of exactly one element, then X is connected.

Suppose instead that X is an open interval. By Examples 4.7 (2) we have that X is homeomorphic to $\mathbb{R}$. By Proposition 6.9 and Corollary 7.3, we deduce that X is connected.

Suppose now that X is either a closed interval or a half open interval, with more than one element. Let us denote the open interval $X \setminus \partial_X \mathbb{R}$ by X'. We have by earlier in the proof that X' is connected. Moreover we have that $X' \subset X \subset \overline{X'}$. By Proposition 7.4, we deduce that X is connected. □

Corollary 7.10. Let $(X, \mathcal{O}_X)$ be a connected topological space, and let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Let x and x' be elements of X with $f(x) \leq f(x')$. Then for every $y \in \mathbb{R}$ such that $f(x) \leq y \leq f(x')$, there is an $x'' \in X$ such that $f(x'') = y$.

Proof. By Proposition 7.1, $f(X)$ is connected. By Proposition 7.9 we deduce that $f(X)$ is an open interval, a closed interval, or a half open interval. By Lemma 7.8 we conclude that every $y \in \mathbb{R}$ such that $f(x) \leq y \leq f(x'')$ belongs to $f(X)$. □

Remark 7.11. Taking $(X, \mathcal{O}_X)$ to be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, Corollary 7.10 is the 'intermediate value theorem' that you met in real analysis/calculus!

7.2 Examples of connected topological spaces

Lemma 7.12. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. For every x in X, let $\{x\} \times Y$ be equipped with the subspace topology $\mathcal{O}_{\{x\} \times Y}$ with respect to $(X \times Y, \mathcal{O}_{X \times Y})$. Then $(\{x\} \times Y, \mathcal{O}_{\{x\} \times Y})$ is homeomorphic to $(Y, \mathcal{O}_Y)$.

For every y in Y, let $X \times \{y\}$ be equipped with the subspace topology $\mathcal{O}_{X \times \{y\}}$ with respect to $(X \times Y, \mathcal{O}_{X \times Y})$. Then $(X \times \{y\}, \mathcal{O}_{X \times \{y\}})$ is homeomorphic to $(X, \mathcal{O}_X)$.

Proof. Exercise. □

Proposition 7.13. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then $(X \times Y, \mathcal{O}_{X \times Y})$ is connected if and only if both $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are connected.

Proof. Suppose that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are connected. Let $X \times Y$ be equipped with the product topology $\mathcal{O}_{X\times Y}$, and let $\{0,1\}$ be equipped with the discrete topology. Let

$$X \times Y \xrightarrow{f} \{0,1\},$$

be a continuous map.

Let $x, x' \in X$, and let $y, y' \in Y$. Let $\{x\} \times Y$ with the subspace topology with respect to $(X \times Y, \mathcal{O}_{X\times Y})$, and let i_x denote the inclusion

$$\{x\} \times Y \hookrightarrow X \times Y.$$

By Proposition 2.15, i_x is continuous. By Proposition 2.16, we deduce that the map

$$\{x\} \times Y \xrightarrow{f \circ i_x} \{0,1\}$$

is continuous.

By Lemma 7.12, $\{x\} \times Y$ is homeomorphic to Y. Thus, since Y is connected, Corollary 7.3 implies that $\{x\} \times Y$ is connected.

We now have that $f \circ i_x$ is continuous and $\{x\} \times Y$ is connected. We deduce by Proposition 6.5 that $f \circ i_x$ cannot be surjective. Since $\{0,1\}$ has only two elements, we conclude that $f \circ i_x$ is constant, and in particular that $f(x, y) = f(x, y')$.

Let $X \times \{y'\}$ be equipped with the subspace topology with respect to $(X \times Y, \mathcal{O}_{X\times Y})$. Let $i_{y'}$ denote the inclusion

$$X \times \{y'\} \hookrightarrow X \times Y.$$

By Proposition 2.15, $i_{y'}$ is continuous. Hence, by Proposition 2.16, the map

$$X \times \{y'\} \xrightarrow{f \circ i_y} \{0,1\}$$

is continuous.

By Lemma 7.12, $X \times \{y'\}$ is homeomorphic to X. Since X is connected, we deduce by Corollary 7.3 that $X \times \{y'\}$ is connected.

We now have that $f \circ i_{y'}$ is continuous and that $X \times \{y'\}$ is connected. We deduce by Proposition 6.5 that $f \circ i_{y'}$ cannot be surjective. Since $\{0,1\}$ has only two elements, we conclude that $f \circ i_{y'}$ is constant, and in particular that $f(x, y') = f(x', y')$.

Putting everything together, we have that $f(x, y) = f(x', y')$. Since $x, x' \in X$ and $y, y' \in Y$ were arbitrary, we conclude that f is constant. In particular, f is not surjective.

We have proven that there does not exist a continuous surjection

$$X \times Y \longrightarrow \{0,1\}.$$

By Proposition 6.5, we conclude that $(X \times Y, \mathcal{O}_{X\times Y})$ is connected.

Conversely, suppose that $(X \times Y, \mathcal{O}_{X \times Y})$ is connected. By Proposition 3.2, we have that the map

$$X \times Y \xrightarrow{p_X} X$$

is continuous. We have that $p_X(X \times Y) = X$. By Proposition 7.1, we conclude that X is connected.

Similarly, by Proposition 3.2, the map

$$X \times Y \xrightarrow{p_Y} X$$

is continuous. We have that $p_Y(X \times Y) = Y$. By Proposition 7.1, we conclude that Y is connected.

$\square$

Examples 7.14.

(1) By Proposition 6.9, we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected. By Proposition 7.13, we deduce $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$ is connected. By Proposition 7.13 and induction, we moreover have that $(\mathbb{R}^n, \mathcal{O}_{\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}})$ is connected for any $n \in \mathbb{N}$.

(2) By Proposition 7.9, I is connected. By Proposition 7.13, we deduce that $(I^2, \mathcal{O}_{I \times I})$ is connected. By Proposition 7.13 and induction, we moreover have that

$$(I^n, \mathcal{O}_{\underbrace{I \times \cdots \times I}})$$

is connected for any $n \in \mathbb{N}$.

Proposition 7.15. Let $(X, \mathcal{O}_X)$ be a connected topological space, and let $\sim$ be an equivalence relation on X. Then $(X/\sim, \mathcal{O}_{X/\sim})$ is connected.

Proof. Let

$$X \xrightarrow{\pi} X/\sim$$

denote the quotient map. By Proposition 3.7, we have that π is continuous. Moreover π is surjective, namely $\pi(X) = X/\sim$. By Proposition 7.1, we deduce that $X/\sim$ is connected. $\square$

Example 7.16. All the topological spaces of Examples 3.9 (1) – (5) are connected. Indeed, by Examples 7.14 (2) we have that I and I^2 are connected. Thus, by Proposition 7.15, a quotient of I or I^2 is connected.

On Exercise Sheet 4 we will prove that $D^2 \cong I^2$. Since I^2 is connected by Examples 7.14 (2), we deduce by Corollary 7.3 that D^2 is connected. By Proposition 7.15, we conclude that S^2, constructed as a quotient of D^2 as in Examples 3.9 (6), is connected.

8 Thursday 7th February

8.1 Using connectedness to distinguish between topological spaces — I

Proposition 8.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that

$$X \xrightarrow{f} Y$$

defines a homeomorphism between $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$.

Let A be a subset of X, equipped with the subspace topology $\mathcal{O}_A$ with respect to X. Let $f(A)$ be equipped with the subspace topology $\mathcal{O}_{f(A)}$ with respect to Y. Let $X \setminus A$ be equipped with the subspace topology $\mathcal{O}_{X \setminus A}$ with respect to X, and let $Y \setminus f(A)$ be equipped with the subspace topology with respect to Y.

Then $(A, \mathcal{O}_A)$ is homeomorphic to $\big(f(A), \mathcal{O}_{f(A)}\big)$, and $(X \setminus A, \mathcal{O}_{X \setminus A})$ is homeomorphic to $\big(Y \setminus f(A), \mathcal{O}_{Y \setminus f(A)}\big)$.

Proof. See Exercise Sheet 4. $\square$

Corollary 8.2. Let $[a, b]$, (a, b), $[a, b)$, and $(a, b]$, for $a, b \in \mathbb{R}$, be equipped with their respective subspace topologies with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then:

(1) $[a, b]$ is not homeomorphic to (a, b), $[a, b)$, or $(a, b]$.

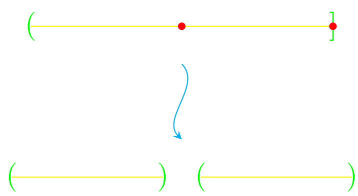
(2) (a, b) is not homeormorphic to $(a, b]$ or $[a, b)$.

(3) $[a, b)$ is homeomorphic to $(a, b]$.

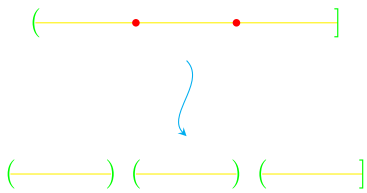
Proof. We have that $[a, b] \setminus \{a, b\} = (a, b)$. By Proposition 7.9, (a, b) equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.

However, by removing any two points from (a, b), $[a, b)$, or $(a, b]$, and equipping the resulting set with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, we will obtain a topological space which is not connected. By Proposition 8.1 and Corollary 7.3, we deduce that $[a, b]$ is not homeomorphic to any of (a, b), $[a, b)$, or $(a, b]$.

An example of removing two points from $(a, b]$ is depicted below. In this case, we obtain a disjoint union of two open intervals.

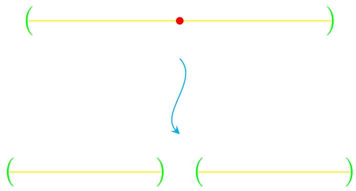


A second example of removing two points from $(a, b]$ is depicted below. In this case, we obtain a disjoint union of two open intervals and a half open interval.



We have that $[a,b) \setminus \{a\} = (a,b)$. Again, by Proposition 7.9, (a,b) equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.

However, removing any one point from (a,b) and equipping the resulting set with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, we will obtain a topological space which is not connected. By Proposition 8.1 and Corollary 7.3, we deduce that $[a,b)$ is not homeomorphic to (a,b).



We also have that $(a,b] \setminus \{b\} = (a,b)$. By the same argument as above, we deduce that $(a,b]$ is not homeomorphic to (a,b).

We will prove that $[a,b) \cong (a,b]$ on Exercise Sheet 4. $\square$

8.2 Connected components

Terminology 8.3. Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Then A is a *connected subset* of X if $(A, \mathcal{O}_A)$ is a connected topological space.

Definition 8.4. Let $(X, \mathcal{O}_X)$ be a topological space, and let $x \in X$. The *connected component* of x in $(X, \mathcal{O}_X)$ is the union of all connected subsets A of X such that $x \in A$.

Proposition 8.5. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\{A_j\}_{j \in J}$ be a set of connected subsets of X such that $\bigcup_{j \in J} A_j = X$. Suppose that $\bigcap_{j \in J} A_j \neq \emptyset$. Then $(X, \mathcal{O}_X)$ is connected.

Proof. Let $\{0,1\}$ be equipped with the discrete topology, and let

$$X \xrightarrow{f} \{0,1\}$$

be a continuous map. Given $j \in J$ let

$$A_j \xhookrightarrow{i_j} X$$

denote the inclusion map. By Proposition 2.15 we have that i_j is continuous. We deduce by Proposition 2.16 that

$$A_j \xrightarrow{f \circ i_j} \{0,1\}$$

is continuous.

By assumption, A_j is a connected subset of X. We deduce by Proposition 6.5 that $f \circ i_j$ is constant. This holds for all $j \in J$. Since $\bigcap_{j \in J} A_j \neq \emptyset$, we deduce that f is constant. $\square$

Corollary 8.6. Let $(X, \mathcal{O}_X)$ be a topological space, and let $x \in X$. The connected component of x in X is a connected subset of X.

Proof. Follows immediately from Proposition 8.5. $\square$

Remark 8.7. Thus the connected component of x in a topological space $(X, \mathcal{O}_X)$ is the largest connected subset of X which contains x.

Terminology 8.8. Let X be a set. A *partition* of X is a set $\{X_j\}_{j \in J}$ of subsets of X such that $X = \bigsqcup_{j \in J} X_j$.

Proposition 8.9. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\{A_x\}_{x \in X}$ denote the set of connected components of X. Then $\{A_x\}_{x \in X}$ defines a partition of X.

Proof. Since $x \in A_x$, we have that $\bigcup_{x \in X} A_x = X$. Suppose that $x, x' \in X$ and that $A_x \cap A_{x'} \neq \emptyset$. We must prove that $A_x = A_{x'}$. By Proposition 8.5 we then have that $A_x \cup A_{x'}$ is connected. By definition of A_x, we deduce that $A_x \cup A_{x'} \subset A_x$. Since we also have that $A_x \subset A_x \cup A_{x'}$, we conclude that $A_x \cup A_{x'} = A_x$ as required. $\square$

Examples 8.10.

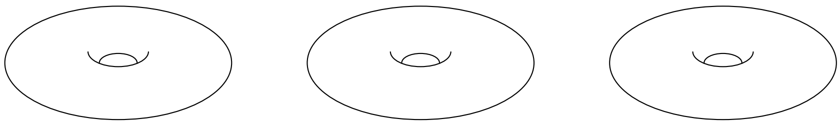
(1) A connected topological space $(X, \mathcal{O})$ has exactly one connected component, namely X itself.

(2) At the other extreme, let X be a set and let $\mathcal{O}^{disc}$ denote the discrete topology on X. The connected components of $(X, \mathcal{O}^{disc})$ are the singleton sets $\{x\}$ for $x \in X$.

Let us prove this. Suppose that $A \subset X$ has more than one element. Let $\mathcal{O}_A$ denote the subspace topology on A with respect to $(X, \mathcal{O}^{disc})$. For any $a \in A$, we have $A = \{a\} \sqcup X \setminus \{a\}$, and both $\{a\}$ and $A \setminus \{a\} = A \cap (X \setminus \{a\})$ are open in $(A, \mathcal{O}_A)$. Thus A is not a connected subset of $(X, \mathcal{O}_X)$.

(3) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be connected topological spaces. Then $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$ has two connected components, namely X and Y. For instance, by Example 7.16 we have that $(T^2, \mathcal{O}_{T^2})$ is connected, and hence that $(T^2 \sqcup T^2, \mathcal{O}_{T^2 \sqcup T^2})$ consists of two connected components.

By induction, we may similarly cook up examples of topological spaces with n connected components for any finite n.



(4) Let the set of rational numbers $\mathbb{Q}$ be equipped with its subspace topology $\mathcal{O}_{\mathbb{Q}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. The connected components of $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ are exactly the singleton sets $\{q\}$ for $q \in \mathbb{Q}$.

Let us prove this. For any $A \subset \mathbb{Q}$ which contains at least two distinct rational numbers q and q', there is an irrational number r with $q < r < q'$. Then

$$A = \Big(A \cap (-\infty, r)\Big) \sqcup \Big(A \cap (r, \infty)\Big).$$

Since both $A \cap (-\infty, r)$ and $A \cap (r, \infty)$ are open in $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$, we deduce that A is not a connected subset of $\mathbb{Q}$.

Terminology 8.11. A topological space $(X, \mathcal{O})$ is *totally disconnected* if the connected components of $(X, \mathcal{O})$ are the singleton sets $\{x\}$ for $x \in X$.

Remark 8.12. By (2) of Examples 8.10, a set equipped with its discrete topology is totally disconnected. However, as (4) of Examples 8.10 demonstrates there are totally disconnected topological spaces $(X, \mathcal{O})$ for which $\mathcal{O}$ is not the discrete topology.

Lemma 8.13. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let

$$X \xrightarrow{f} Y$$

be a continuous map. Let $x \in X$. Let A_x denote the connected component of x in X, and let $B_{f(x)}$ denote the connected component of $f(x)$ in Y. Then $f(A_x) \subset B_{f(x)}$.

Proof. By Corollary 8.6, A_x is a connected subset of $(X, \mathcal{O}_X)$. By Proposition 7.1 we deduce that $f(A)$ is a connected subset of Y. Since $x \in A$, we have that $f(x) \in f(A)$. By definition of $B_{f(x)}$, we conclude that $f(A) \subset B_{f(x)}$. $\square$

Proposition 8.14. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. If $X \cong Y$ then there exists a bijection between the set of connected components of X and the set of connected components of Y.

Proof. Given $x \in X$, let A_x denote the connected component of x in $(X, \mathcal{O}_X)$. Given $y \in Y$, let B_y denote the connected component of y in $(Y, \mathcal{O}_Y)$. Let Γ_X denote the set of connected components of X, and let Γ_Y denote the set of connected components of Y.

Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. By definition of f as a homeomorphism, there is a continuous map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$. For any $x \in X$ we have by Lemma 8.13 that $g(B_{f(x)}) \subset A_{g \circ f(x)} = A_x$. Hence

$$\begin{aligned} B_{f(x)} &= f\big(g(B_{f(x)})\big) \\ &\subset f(A_x). \end{aligned}$$

Moreover, by Lemma 8.13 we have that $f(A_x) \subset B(f_x)$. We deduce that $f(A_x) = B_{f(x)}$. Thus

$$A_x \mapsto f(A_x)$$

defines a map

$$\Gamma_X \xrightarrow{\eta} \Gamma_Y.$$

By an entirely analogous argument we have that $g(B_y) = A_{g(y)}$ for any $y \in Y$. Thus

$$B_y \mapsto g(B_y)$$

defines a map

$$\Gamma_Y \xrightarrow{\zeta} \Gamma_X.$$

We have that $\zeta \circ \eta = id_{\Gamma_X}$ since for any $x \in X$ we have that

$$\begin{aligned} g\big(f(A_x)\big) &= g(B_{f(x)}) \\ &= A_{g \circ f(x)} \\ &= A_x. \end{aligned}$$

Moreover we have that $\eta \circ \zeta = id_{\Gamma_Y}$ since for any $y \in Y$ we have that

$$\begin{aligned} f\big(g(B_y)\big) &= f(A_{g(y)}) \\ &= B_{f \circ g(y)} \\ &= B_y. \end{aligned}$$

□

Observation 8.15. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and suppose that $X \cong Y$. Then every connected component of X, equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$, is homeomorphic to a connected component of Y, equipped with the subspace topology with respect to $(Y, \mathcal{O}_Y)$.

This follows from observations made during the proof of Proposition 8.14. See the Exercise Sheet.

8.3 Using connectedness to distinguish between topological spaces — II

Examples 8.16.

(1) Let us regard the letter T as a subset of $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. For example, we can let

$$\mathsf{T} = \{(x, 1) \in \mathbb{R}^2 \mid x \in [-1, 1]\} \cup \{(0, y) \in \mathbb{R}^2 \mid y \in [0, 1]\}.$$

We equip T with the subspace topology $\mathcal{O}_{\mathsf{T}}$.

Let us also regard the letter I as a subset $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. For example, we can let

$$\mathsf{I} = \{(0, y) \in \mathbb{R}^2 \mid y \in [0, 1]\}.$$

We equip I with the subspace topology $\mathcal{O}_{\mathsf{I}}$.

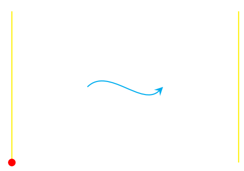
By Examples 4.10 (5) we have that $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ is homeomorphic to the unit interval $(I, \mathcal{O}_I)$.

Let us prove that $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ is not homeomorphic to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$. Let x be the point $(0, 1)$ of T.

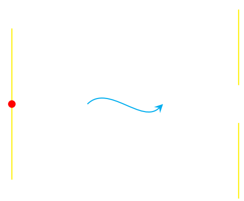
Then $\mathsf{T} \setminus \{x\}$ equipped with the subspace topology with respect to $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ has three connected components.

However, the topological space obtained by removing a single point of I and equipping the resulting set with the subspace topology with respect to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ has either one connected component or two connected components.

We obtain one connected component if we remove one of the two end points of I. Removing the lower end point $(0,0)$ of I is depicted below.



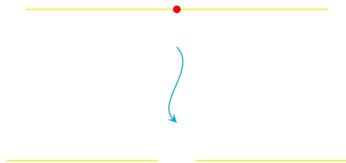
We obtain two connected components if we remove any point of I which is not an end point.



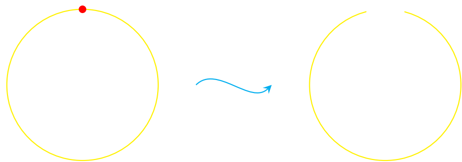
In particular, it is not possible to remove a single point from $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ and obtain a topological space with three connected components.

We deduce by Proposition 8.14 that $\mathsf{T} \setminus \{x\}$ is not homeomorphic to $\mathsf{I} \setminus \{y\}$ for any $y \in \mathsf{I}$. We conclude that T is not homeomorphic to I by Proposition 8.1

(2) The circle S^1 is not homeomorphic to I. Indeed, equipping $I \setminus \{t\}$ for $0 < t < 1$ with the subspace topology with respect to $(I, \mathcal{O}_I)$ gives a topological space with two connected components.



Removing any point from S^1 and equipping the resulting set with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$ gives a topological space with exactly one connected component.



In particular, it is not possible to remove a single point from $(I, \mathcal{O}_I)$ and obtain a topological space with two connected components.

We deduce by Proposition 8.14 that $\mathsf{I} \setminus \{t\}$ is not homeomorphic to $S^1 \setminus \{x\}$ for any $x \in S^1$. We conclude that S^1 is not homeomorphic to I by Proposition 8.1.

(3) Let us regard the letters Å and A as subsets of $\mathbb{R}^2$, equipped with their respective subspace topologies with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. By Proposition 8.14 we have that Å is not homeomorphic to A, since Å has two connected components, whilst A has one.

9 Tuesday 12th February

9.1 Using connectedness to distinguish between topological spaces —- II, continued

Examples 9.1.

(1) Let us regard the letter K as a subset of $\mathbb{R}^2$, equipped with its subspace topology $\mathcal{O}_{\mathsf{K}}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ be as in Examples 8.16 (1). Let us prove that $(\mathsf{K}, \mathcal{O}_{\mathsf{K}})$ is not homeomorphic to $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$. Let x be the point of K indicated below.

Then $\mathsf{K} \setminus \{x\}$ equipped with the subspace topology with respect to $(\mathsf{K}, \mathcal{O}_{\mathsf{K}})$ has four connected components.

However, the topological space obtained by removing a point from $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ and equipping the resulting set with the subspace topology with respect to $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ has at most three connected components.

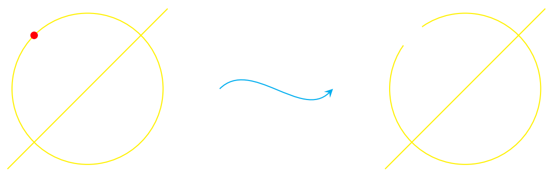
We deduce by Proposition 8.14 that $\mathsf{K} \setminus \{x\}$ is not homeomorphic to $\mathsf{T} \setminus \{y\}$ for any $y \in \mathsf{T}$. We conclude that $(\mathsf{K}, \mathcal{O}_{\mathsf{K}})$ is not homeomorphic to $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ by Proposition 8.1.

(2) Let us regard the letter $\varnothing$ as a subset of $\mathbb{R}^2$, equipped with its subspace topology $\mathcal{O}_{\varnothing}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

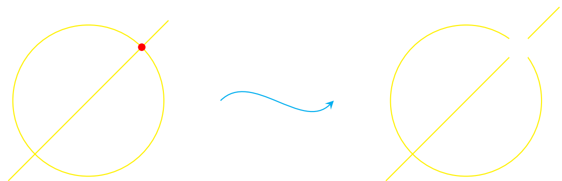
Let I and $\mathcal{O}_{\mathsf{I}}$ be as in Examples 8.16 (1). We cannot distinguish $(\varnothing, \mathcal{O}_{\varnothing})$ from $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ by remvoing one point from $\varnothing$.

Let us see why. Removing one point from $\varnothing$ and equipping the resulting set with the subspace topology with respect to $(\varnothing, \mathcal{O}_{\varnothing})$ we obtain a topological space with either one or two connected components.

For instance, we obtain one connected component by removing a point as shown below.

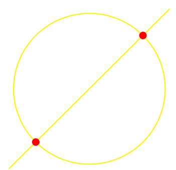


We obtain two connected components by removing a point as shown below, for example.



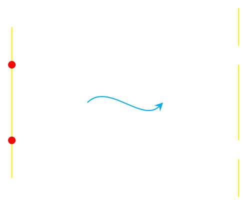
Since we may also obtain a topological space with either one or two connected components by removing a point from $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$, we cannot conclude that $(\varnothing, \mathcal{O}_{\varnothing})$ is not homeomorphic to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$.

However, let x and y be the two points of $\varnothing$ indicated below.



Then $\emptyset \setminus \{x, y\}$ equipped with the subspace topology with respect to $(\emptyset, \mathcal{O}_{\emptyset})$ has five connected components.

Removing two points from I and equipping the resulting set with the subspace topology with respect to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ gives a topological space with at most three connected components.



We deduce by Proposition 8.14 that $\emptyset \setminus \{x, y\}$ is not homeomorphic to $\mathsf{I} \setminus \{x', y'\}$ for any $x', y' \in \mathsf{I}$. We conclude that $(\emptyset, \mathcal{O}_{\emptyset})$ is not homeomorphic to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ by Proposition 8.1.

Lemma 9.2. For any $n > 1$ and any $x \in \mathbb{R}^n$ we have that $\mathbb{R}^n \setminus \{x\}$ equipped with the subspace topology $\mathcal{O}_{\mathbb{R}^n \setminus \{x\}}$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is connected.

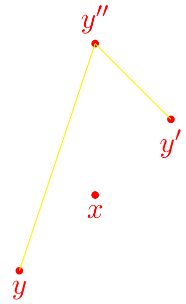
Proof. Let $y, y' \in \mathbb{R}^n \setminus \{x\}$. Since $n > 1$, there exists a line L through y and a line L' through y' such that $L \subset \mathbb{R}^n \setminus \{x\}$ $L' \subset \mathbb{R}^n \setminus \{x\}$, and $L \cap L' \neq \emptyset$. For instance, let y'' denote the point $x + (0, \ldots, 0, 1)$ of $\mathbb{R}^n$. We can take L to be

$$\{y + ty'' \mid t \in [0, 1]\}$$

equipped with the subspace topology $\mathcal{O}_L$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$, and take L' to be

$$\{y' + ty'' \mid t \in [0, 1]\}$$

equipped with the subspace topology with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.



Since $(L, \mathcal{O}_L)$ is homeomorphic to $(I, \mathcal{O}_I)$, and since $(I, \mathcal{O}_I)$ is connected by Proposition 7.9, we deduce by Corollary 7.3 that $(L, \mathcal{O}_L)$ is connected. By exactly the same argument, we also have that $(L', \mathcal{O}_{L'})$ is connected. We deduce by Proposition 8.5 that $L \cup L'$ is connected.

Thus y and y' belong to the same connected component of $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n \setminus \{x\}})$. Since y and y' were arbitrary, we deduce that this connected component is $\mathbb{R}^n \setminus \{x\}$ itself. We conclude by Corollary 8.6 that $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n \setminus \{x\}})$ is connected. $\square$

Proposition 9.3. The topological space $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not homeomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ for any $n > 1$.

Proof. Let $n > 1$, and suppose that

$$\mathbb{R} \xrightarrow{f} \mathbb{R}^n$$

defines a homeomorphism between $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ and $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. Let $x \in \mathbb{R}$. By Lemma 9.2 we have that $(\mathbb{R}^n \setminus \{f(x)\}, \mathcal{O}_{\mathbb{R}^n \setminus \{f(x)\}})$ is connected.

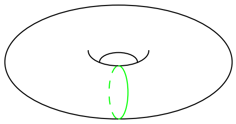
But $(\mathbb{R} \setminus \{x\}, \mathcal{O}_{\mathbb{R} \setminus \{x\}})$ is not connected.

By Proposition 8.1 and Corollary 7.3 we conclude that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not homeomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.

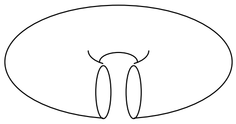
$\square$

Question 9.4. In all our examples of distinguishing topological spaces by means of connectedness at least one of the topological spaces has been 'one dimensional', built out of lines and circles. Can we apply our technique to distinguish between higher dimensional topological spaces?

Remark 9.5. For example, let us try to formulate an argument to distinguish T^2 from S^2. Let X denote the circle on T^2 depicted below.



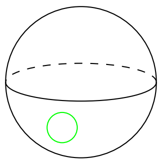
Then $T^2 \setminus X$ is as depicted below.



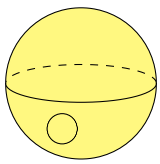
Equipped with the subspace topology with respect to $(T^2, \mathcal{O}_{T^2})$, it is homeomorphic to a cylinder.

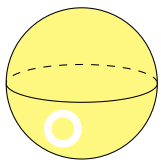
In particular, $T^2 \setminus X$ is connected.

Let us now consider a subset Y of S^2 which, equipped with the subspace topology with respect to $(S^2, \mathcal{O}_{S^2})$, is homeomorphic to a circle.



Then $S^2 \setminus Y$ intuitively appears to have two connected components: the interior of Y and $S^2 \setminus Y$.





If our intuition is correct, by Proposition 8.14 we deduce that $T^2 \setminus X$ is not homeomorphic to $S^2 \setminus Y$ for any subset Y of S^2 which, equipped with the subspace topology with respect to $(S^2, \mathcal{O}_{S^2})$, is homeomorphic to $(S^1, \mathcal{O}_{S^1})$. We conclude that $(T^2, \mathcal{O}_{T^2})$ is not homeomorphic to $(S^2, \mathcal{O}_{S^2})$ by Proposition 8.1.

We have to be very careful! Homeomorphism is a very flexible notion, and Y could be very wild, much more complicated than the circle on S^2 drawn above.

We need to be sure that the requirement that we have a homeomorphism, as opposed to only a continuous surjection, excludes examples which are as wild as the Peano curve that we will meet on a later Exercise Sheet.

In other words, in order to carry out the argument of Remark 9.5 we have to rigorously prove that $S^2 \setminus Y$ has two connected components for any possible Y. This is subtle!

Answer 9.6. Nevertheless, it is true! This is known as the *Jordan curve theorem*, which we will prove towards the end of the course. Thus the argument of Remark 9.5 does after further work prove that $(T^2, \mathcal{O}_{T^2})$ is not homeomorphic to $(S^2, \mathcal{O}_{S^2})$.

Towards the end of the course we will also be able to prove by our technique, using a generalisation of the Jordan curve theorem to higher dimensions, that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ for any $m, n > 0$.

More sophisticated tools, which you will meet if you take Algebraic Topology I in the future, give a simple — after some foundational work! — proof of the Jordan curve theorem and its generalisation to higher dimensions.

Example 9.7. Whilst we do not yet have the tools to explore very wild phenomena such as the Peano curve that we will meet on a later Exercise Sheet, let us give an example of the kind of wildness that topology allows.

We will construct a pair of spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ such that there exists a continuous bijection from X to Y and a continuous bijection from Y to X, but such that X is not homeomorphic to Y.

Let us define

$$X = (0,1) \cup \{2\} \cup (3,4) \cup \{5\} \cup (6,7) \cup \{8\} \cdots .$$

In other words

$$X = \bigcup_{n \in \mathbb{Z},\ n \geq 0} (3n, 3n+1) \cup \{3n+2\}.$$

Here as usual $(3n, 3n+1)$ denotes the open interval from $3n$ to $3n+1$ in $\mathbb{R}$. We equip X with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let us define

$$Y = (0,1] \cup (3,4) \cup \{5\} \cup (6,7) \cup \{8\} \cup \cdots .$$

In other words,

$$Y = (0,1] \cup \Big(\bigcup_{n \in \mathbb{Z},\ n \geq 1} (3n, 3n+1) \cup \{3n+2\} \Big).$$

We equip Y with the subspace topology $\mathcal{O}_Y$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

We have that $(0,1]$ is a connected component of Y. By Corollary 8.2, $(0,1]$ is not homeomorphic to an open or closed interval. We deduce by Observation 8.15 that X is not homeomorphic to Y.

Let

$$X \xrightarrow{f} Y$$

be given by

$$f(x) = \begin{cases} x & \text{if } x \neq 2, \\ 1 & x = 2. \end{cases}$$

Let

$$Y \xrightarrow{g} X$$

be given by

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \in (0, 1], \\ \frac{x-2}{2} & x \in (3, 4), \\ x - 3 & \text{otherwise.} \end{cases}$$

Then both f and g are continuous, by Question 3 (f) of Exercise Sheet 3 and Question 1 of Exercise Sheet 5. Morever, f and g are bijective.

Do not be confused — the bijections f and g are not inverse to one another! If they were, we would have that X is homeomorphic to Y.

9.2 Locally connected topological spaces

Proposition 9.8. Let $(X, \mathcal{O})$ be a topological space. Given $x \in X$, let A_x denote the connected component of x in X. Then A_x is a closed subset of X.

Proof. By Corollary 8.6, A_x is a connected subset of X. We deduce by Corollary 7.6 that $\overline{A_x}$ is a connected subset of X. Hence by definition of A_x we have that $\overline{A_x} \subset A_x$. Since $A_x \subset \overline{A_x}$, we deduce that $\overline{A_x} = A_x$. We conclude by Proposition 5.7 that A_x is closed. $\square$

Remark 9.9. By Examples 8.10 (4) a connected component need not be open.

Definition 9.10. A topological space $(X, \mathcal{O})$ is *locally connected* if for every $x \in X$ and every neighbourhood U of x in $(X, \mathcal{O}_X)$ there is a neighbourhood U' of x in $(X, \mathcal{O}_X)$ such that U' is a connected subset of X and $U' \subset U$.

Proposition 9.11. A topological space $(X, \mathcal{O}_X)$ is locally connected if and only if it admits a basis consisting of connected subsets.

Proof. This is an immediate consequence of Question 3 (a) and Question 3 (b) on Exercise Sheet 2. $\square$

Lemma 9.12. Let $(X, \mathcal{O}_X)$ be a topological space, let $x \in X$, and let U be a neighbourhood of x in $(X, \mathcal{O}_X)$. Equip U with its subspace topology $\mathcal{O}_U$ with respect to $(X, \mathcal{O}_X)$. Let A be a connected subset of X with $A \subset U$. Then A is a connected subset of $(U, \mathcal{O}_U)$.

Proof. See Exercise Sheet 5. $\square$

Proposition 9.13. A topological space $(X, \mathcal{O})$ is locally connected if and only if for every open subset U of X the connected components of $(U, \mathcal{O}_U)$ are open subsets of X, where $\mathcal{O}_U$ denotes the subspace topology on U with respect to $(X, \mathcal{O}_X)$.

Proof. Suppose that $(X, \mathcal{O})$ is locally connected. Let U be an open subset of X, equipped with the subspace topology $\mathcal{O}_U$ with respect to $(X, \mathcal{O}_X)$. Let $x \in U$, and for any $y \in U$, let A_y denote the connected component of y in $(U, \mathcal{O}_U)$. We have that $A_y = A_x$ for all $y \in A_x$.

By Proposition 9.11, $(X, \mathcal{O})$ admits a basis $\{U_j\}_{j \in J}$ such that U_j is a connected subset of $(X, \mathcal{O}_X)$ for every $j \in J$. Thus by Question 3 (a) of Exercise Sheet 2, there is a $j \in J$ such that $y \in U_j$ and $U_j \subset U$. Since U_j is a connected subset of $(X, \mathcal{O}_X)$, we deduce by Lemma 9.12 that U_j is a connected subset of $(U, \mathcal{O}_U)$. Hence $U_j \subset A_y = A_x$. By Question 3 (b) of Exercise Sheet 2, we conclude that A_x is an open subset of X.

Conversely, suppose that for every open subset U of X the connected components of $(U, \mathcal{O}_U)$ are open subsets of X, where $\mathcal{O}_U$ denotes the subspace topology on U with respect to $(X, \mathcal{O}_X)$. For $x \in U$, let A_x^U denote the connected component of x in U.

Then $\{A_x^U\}_{U \in \mathcal{O},\ x \in U}$ defines a basis for $(X, \mathcal{O})$. Indeed, for any $U \in \mathcal{O}$ we have by Proposition 8.9 that $U = \bigcup_{x \in U} A_x^U$.

□

Examples 9.14.

(1) $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally connected. Indeed by definition of $\mathcal{O}_{\mathbb{R}}$ we have that

$$\{(a, b) \mid a, b \in \mathbb{R}\}$$

is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Proposition 7.9, (a, b) is a connected subset of $\mathbb{R}$ for every $a, b \in \mathbb{R}$.

(2) Products and quotients of locally connected topological spaces are locally connected. We will prove this on the Exercise Sheet 5. We deduce that all of the topological spaces of Examples 1.38 and Examples 3.9 are locally connected.

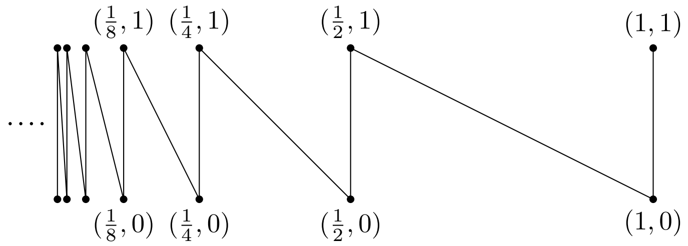
(3) The subset $X = (0, 1) \sqcup (2, 3)$ of $\mathbb{R}$ equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally connected, since $(0, 1)$ and $(2, 3)$ are connected by Proposition 7.9. However X is evidently not connected.

(4) By Examples 8.10 (4), $\mathbb{Q}$ equipped with its subspace topology $\mathcal{O}_{\mathbb{Q}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not locally connected, since its connected components are the singleton sets $\{q\}_{q \in \mathbb{Q}}$, which are not open in $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$.

10 Thursday 14th February

10.1 An example of a topological space which is connected but not locally connected

Example 10.1. Let $X \subset \mathbb{R}^2$ be the set depicted below.



To be explicit X is the union of the sets

$$\bigcup_{n \geq 0} \{ (\frac{1}{2^n}, y) \mid y \in [0,1] \}$$

and

$$\bigcup_{n \geq 0} \{ (x, -2^{n+1}x + 2) \mid x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}] \}.$$

Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. We have that $(X, \mathcal{O}_X)$ is connected. This may be proven as follows.

(1) For every $n \geq 0$, let A_n be the line segment

$$\{ (\frac{1}{2^n}, y) \mid y \in [0,1] \}.$$

Let $\mathcal{O}_{A_n}$ denote the subspace topology on A_n with respect to X, or equivalently with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

The projection map

$$A_n \longrightarrow [0,1]$$

given by $(x,y) \mapsto y$ defines a homeomorphism between $(A_n, \mathcal{O}_{A_n})$ and $([0,1], \mathcal{O}_{[0,1]})$, where $\mathcal{O}_{[0,1]}$ denotes the subspace topology on $[0,1]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

By Proposition 7.9 we have that $([0,1], \mathcal{O}_{[0,1]})$ is connected. By Corollary 7.3 we deduce that $(A_n, \mathcal{O}_{A_n})$ is connected.

(2) For every $n \geq 0$, let A'_n be the line segment

$$\left\{\left(x, -2^{n+1}x + 2\right) \mid y \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]\right\}.$$

Let $\mathcal{O}_{A'_n}$ denote the subspace topology on A'_n with respect to X, or equivalently with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

The projection map

$$A'_n \longrightarrow \left[\tfrac{1}{2^{n+1}}, \tfrac{1}{2^n}\right]$$

given by $(x, y) \mapsto x$ defines a homeomorphism between $(A_n, \mathcal{O}_{A_n})$ and

$$\left(\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \mathcal{O}_{\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]}\right),$$

where $\mathcal{O}_{\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]}$ denotes the subspace topology on $\left[\frac{1}{2^{n+1}}, \frac{1}{2^m}\right]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

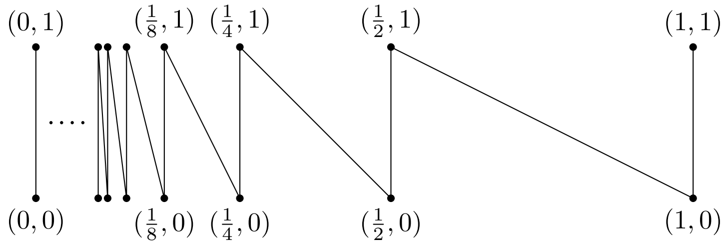
By Proposition 7.9 we have that $\left(\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \mathcal{O}_{\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]}\right)$ is connected. By Corollary 7.3 we deduce that $(A'_n, \mathcal{O}_{A'_n})$ is connected.

(3) We conclude that $(X, \mathcal{O}_X)$ is connected by Proposition 8.5.

The closure $\overline{X}$ of X in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is

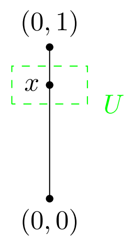
$$X \bigcup \left(\left\{(0, y) \mid y \in [0, 1]\right\}\right).$$

See Exercise Sheet 4.

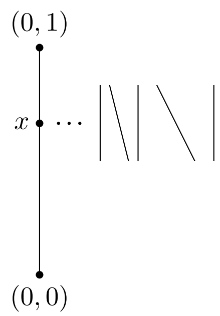


Let $\mathcal{O}_{\overline{X}}$ denote the subspace topology on $\overline{X}$ in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Since X is connected we have by Corollary 7.6 that $(\overline{X}, \mathcal{O}_{\overline{X}})$ is connected.

However $(\overline{X}, \mathcal{O}_{\overline{X}})$ is not locally connected. Let us prove this. Let $x \in \left\{(0, y) \mid y \in [0, 1]\right\}$. There is an open rectangle $U \subset \mathbb{R}^2$ such that $x \in U$ with the property that if $(x', y') \in U$ then $0 < y' < 1$.



Then $U \cap \overline{X}$ is a disjoint union of infinitely many open intervals.



In particular $U \cap \overline{X}$ is not a connected subset of $(\overline{X}, \mathcal{O}_{\overline{X}})$.

Remark 10.2. The topological space $(\overline{X}, \mathcal{O}_{\overline{X}})$ is a variant of the *topologist's sine curve*. If you wish to look up this kind of example in another reference, this is the phrase that you will need!

10.2 Path connected topological spaces

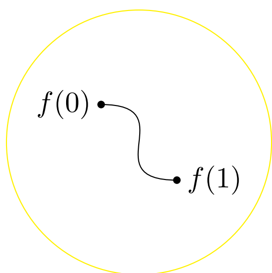
Definition 10.3. Let $(X, \mathcal{O}_X)$ be a topological space. A *path* in X is a continuous map

$$I \longrightarrow X.$$

Terminology 10.4. Let $(X, \mathcal{O})$ be a topological space, and let

$$I \xrightarrow{f} X$$

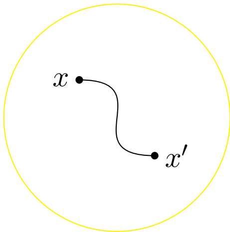
be a path in X. The picture shows a path in D^2.



Terminology 10.5. Let $(X, \mathcal{O}_X)$ be a topological space. Let (x, x') be a pair of elements of X. A path *from* x *to* x' in X is a path

$$I \xrightarrow{f} X$$

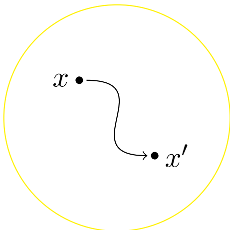
in X such that $f(0) = x$ and $f(1) = x'$.



Notation 10.6. When drawing a path

$$I \xrightarrow{f} X$$

from x to x' in a topological space $(X, \mathcal{O}_X)$ we often use an arrow to indicate that $x = f(0)$ and $x' = f(1)$.



Example 10.7. Look back at Examples 2.13 (3) for several examples of paths in S^1.

Proposition 10.8. Let $(X, \mathcal{O})$ be a topological space. Let $x, x' \in X$, and let

$$I \xrightarrow{f} X$$

be a path from x to x' in X.

Let

$$I \xrightarrow{v} I$$

be the map given by $t \mapsto 1 - t$.

The map

$$I \xrightarrow{f \circ v} X$$

defines a path from x' to x in X.

Proof. By Question 3 (f) of Exercise Sheet 3 we have that v is continuous. By Proposition 2.16 we deduce that $f \circ v$ is continuous.

We have that

$$\begin{aligned}(f \circ v)(0) &= f\big(v(0)\big) \\ &= f(1) \\ &= x'\end{aligned}$$

and that

$$\begin{aligned}(f \circ v)(0) &= f\big(v(1)\big) \\ &= f(0) \\ &= x.\end{aligned}$$

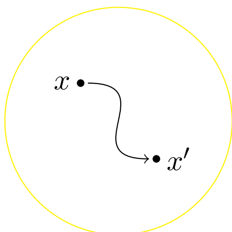
$\square$

Remark 10.9. We met the map v in Examples 2.13 (4).

Remark 10.10. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$I \xrightarrow{f} X$$

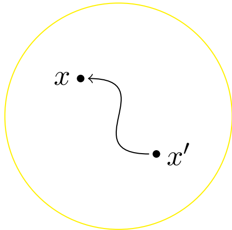
be a path in X.



We think of the path

$$I \xrightarrow{f \circ v} X$$

as obtained by travelling along f in reverse.



Proposition 10.11. Let $(X, \mathcal{O})$ be a topological space. Let $x, x', x'' \in X$. Let

$$I \xrightarrow{f} X$$

be a path from x to x' in X, and let

$$I \xrightarrow{f'} X$$

be a path from x' to x'' in X.

Let

$$I \xrightarrow{g} X$$

be the map given by

$$t \mapsto \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ f'(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then g defines a path from x to x'' in X.

Proof. Note that g is well-defined, since

$$\begin{aligned} f(2 \cdot \frac{1}{2}) &= f(1) \\ &= f'(0) \\ &= f'(2 \cdot \frac{1}{2} - 1). \end{aligned}$$

Moreover by Question 7 (b) of Exercise Sheet 3 we have that g is continuous.

We have that

$$\begin{aligned} g(0) &= f(2 \cdot 0) \\ &= f(0) \\ &= x \end{aligned}$$

and that

$$\begin{aligned} g(1) &= f'(2 \cdot 1 - 1) \\ &= f'(1) \\ &= x'. \end{aligned}$$

□

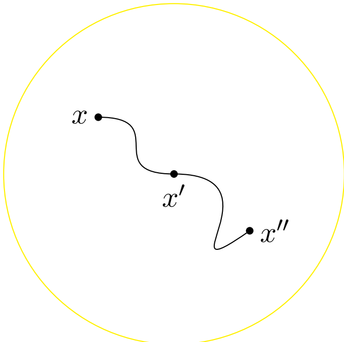
Remark 10.12. Let $(X, \mathcal{O})$ be a topological space. Let $x, x', x'' \in X$. Let

$$I \xrightarrow{f} X$$

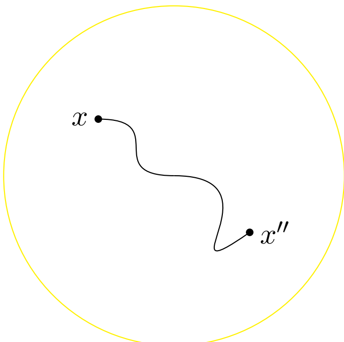
be a path from x to x' in X, and let

$$I \xrightarrow{f'} X$$

be a path from x' to x'' in X.



The corresponding path g from x to x'' of Proposition 10.11 can be thought of as first travelling at double speed from x to x' along f, and then travelling at double speed from x' to x'' along f'.



Proposition 10.13. Let $(X, \mathcal{O})$ be a topological space. Let $x \in X$. The constant map

$$I \xrightarrow{f} X$$

given by $t \mapsto x$ for all $t \in I$ defines a path from x to x in X.

Proof. Easy exercise! $\square$

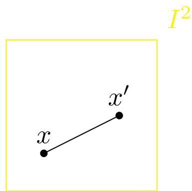
Definition 10.14. A topological space $(X, \mathcal{O})$ is *path connected* if for every pair (x, x') of elements of X there is a path from x to x'.

Examples 10.15.

(1) The topological spaces $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ and $(I^n, \mathcal{O}_{I^n})$ are path connected for every n. Indeed for any x and x' in $\mathbb{R}^n$ or I^n the straight line

$$I \xrightarrow{f} \mathbb{R}^n$$

given by $t \mapsto x + t(x' - x)$ defines a path from x to x'.



(2) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homeomorphic topological spaces. Then X is path connected if and only if Y is path connected. Moreover quotients and products of path connected spaces are path connected. Thus all of the topological spaces of Examples 3.9 (1) – (5) are path connected.

On Exercise Sheet 5 you will be asked to prove these assertions.

(3) Let $X = \{a, b\}$ be equipped with the topology $\mathcal{O} = \big\{\emptyset, \{b\}, X\big\}$. In other words $(X, \mathcal{O})$ is the Sierpiński interval. We have that $(X, \mathcal{O})$ is path connected.

Let us prove this. By virtue of Proposition 10.13 and Proposition 10.8 it suffices to prove that there is a path in X from a to b.

The map

$$I \xrightarrow{f} X$$

given by

$$t \mapsto \begin{cases} a & \text{if } t = 0, \\ b & \text{if } 0 < t \leq 1 \end{cases}$$

is continuous. Indeed $f^{-1}(b) = (0, 1]$, which is an open subset of I.

Moreover $f(0) = a$ and $f(1) = b$. Thus f defines a path from a to b in X.

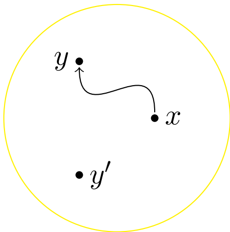
Proposition 10.16. Let $(X, \mathcal{O})$ be a topological space. Let $x \in X$. Then $(X, \mathcal{O})$ is path connected if and only if for every $x' \in X$ there is a path from x to x' in X.

Proof. Suppose that for every $x' \in X$ there is a path from x to x' in X. Let $y, y' \in X$. We must prove that there is a path from y to y' in X.

By assumption there is a path

$$I \xrightarrow{f_{x,y}} X$$

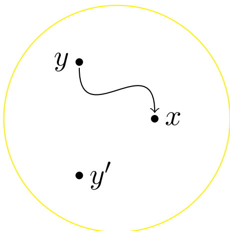
from x to y in X.



By Proposition 10.8 we deduce that there is a path

$$I \xrightarrow{f_{y,x}} X$$

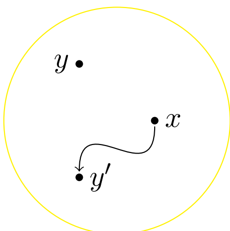
from y to x in X.



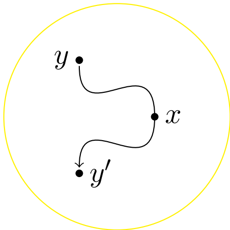
By assumption there is also a path

$$I \xrightarrow{f_{x,y'}} X$$

from x to y' in X,



By Proposition 10.11 applied to the paths $f_{y,x}$ and $f_{x,y'}$ in X we conclude that there is a path from y to y' in X, as required.



Conversely if $(X, \mathcal{O})$ is path connected then by definition there is a path from x to x' for every $x' \in X$.

$\square$

Proposition 10.17. Let $(X, \mathcal{O})$ be a path connected topological space. Then $(X, \mathcal{O})$ is connected.

Proof. Let $x \in X$. Since $(X, \mathcal{O})$ is path connected we have that for every $x' \in X$ there is a path

$$I \xrightarrow{f_{x,x'}} X$$

from x to x' in X.

Since $x' \in f_{x,x'}(I)$ we have that

$$X = \bigcup_{(x,x') \in X \times X} f_{x,x'}(I).$$

By Proposition 7.9 we have that $(I, \mathcal{O}_I)$ is connected. We deduce by Proposition 7.1 that $f_{x,x'}(I)$ is connected for all $x, x' \in X$.

We conclude by Proposition 8.5 that $(X, \mathcal{O})$ is connected. $\square$

Remark 10.18. A connected topological space is not necessarily path connected. For instance the topological space $(\overline{X}, \mathcal{O}_{\overline{X}})$ of Example 10.1 is connected but not path connected. You will be asked to prove this on Exercise Sheet 5.

10.3 Locally path connected topological spaces

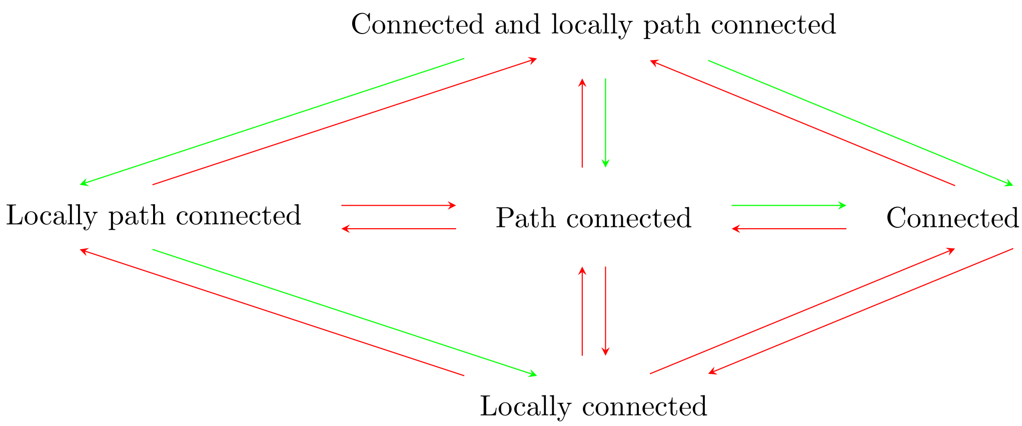
Definition 10.19. A topological space $(X, \mathcal{O}_X)$ is *locally path connected* if for every $x \in X$ and every neighbourhood U of x in $(X, \mathcal{O}_X)$ there is a neighbourhood U' of x in $(X, \mathcal{O}_X)$ such that U' is path connected and $U' \subset U$.

Remark 10.20. We will explore locally path connected topological spaces on Exercise Sheet 5. We will prove that a connected and locally path connected topological space is path connected.

By Proposition 10.17 we have that a locally path connected topological space is locally connected.

By contrast, we will see on Exercise Sheet 5 that a locally connected topological space need not be locally path connected. Moreover we will see that a path connected topological space need not be locally path connected and need not be locally connected.

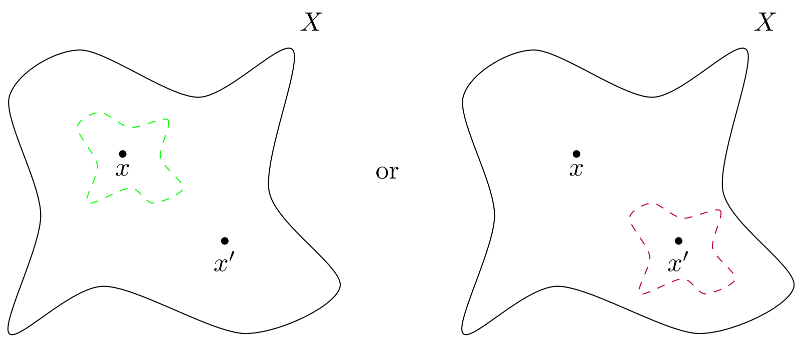
Synopsis 10.21. Let us summarise the relationship between connected, path connected, locally connected, and locally path connected topological spaces. A green arrow indicates an implication. A red arrow indicates that there is no implication.



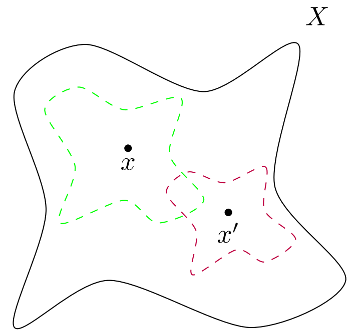
10.4 Separation axioms

Definition 10.22. Let $(X, \mathcal{O})$ be a topological space. The following are axioms which $(X, \mathcal{O})$ may satisfy.

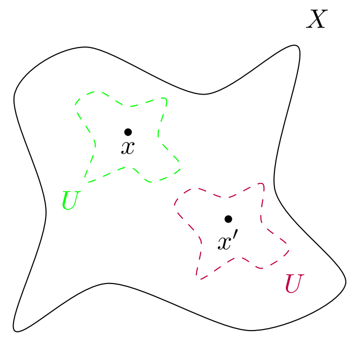
(T0) For every $x, x' \in X$ such that $x \neq x'$ there is a neighbourhood of either x or x' which does not contain both x and x'.



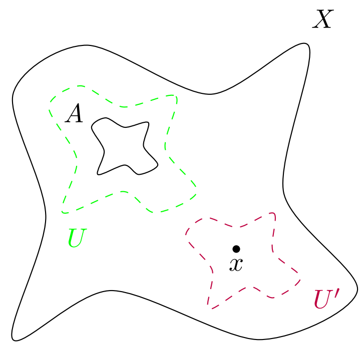
(T1) For every ordered pair (x, x') of elements of X such that $x \neq x'$ there is a neighbourhood of x which does not contain x'.



(T2) For every $x, x' \in X$ such that $x \neq x'$ there is a neighbourhood U of x and a neighbourhood U' of x' such that $U \cap U' = \emptyset$.



(T3) For every non-empty closed subset A of X and every $x \in X \setminus A$ there is an open subset U of X such that $A \subset U$ and a neighbourhood U' of x such that $U \cap U' = \emptyset$.

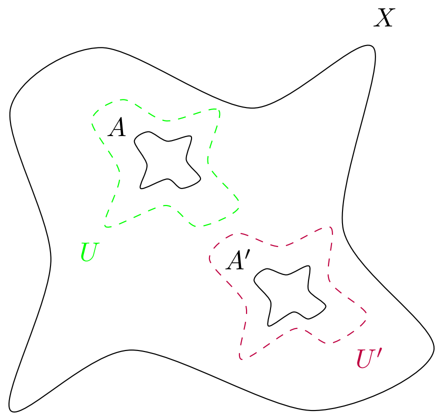


(T3$\frac{1}{2}$) For every non-empty closed subset A of X and every $x \in X \setminus A$ there is a continuous map

$$X \longrightarrow I$$

such that $f(x) = 1$ and $f(A) = \{0\}$.

(T4) For every pair A and A' of closed subsets of X such that $A \cap A' = \emptyset$ there is an open subset U of X with $A \subset U$ and an open subset U' of X with $A' \subset U'$ such that $U \cap U' = \emptyset$.



(T6) For every pair of non-empty closed subsets A and A' of X such that $A \cap A' = \emptyset$ there is a continuous map

$$X \longrightarrow I$$

such that $f^{-1}(0) = A$ and $f^{-1}(1) = A'$.

Remark 10.23. There is an axiom along the above lines which is denoted (T5), but we will not need it. Be careful if you look up these axioms in another source, as there are differing conventions.

11 Tuesday 19th Febuary

11.1 Separation axioms, continued

Definition 11.1. Let $(X, \mathcal{O})$ be a topological space.

(0) If $(X, \mathcal{O})$ satisfies (T0) we refer to $(X, \mathcal{O})$ as a *T0* topological space.

(1) If $(X, \mathcal{O})$ satisfies (T1) we refer to $(X, \mathcal{O})$ as a *T1* topological space.

(2) If $(X, \mathcal{O})$ satisfies (T2) we refer to $(X, \mathcal{O})$ as a *Hausdorff* topological space.

(3) If $(X, \mathcal{O})$ satisfies both (T1) and (T3) we refer to $(X, \mathcal{O})$ as a *regular* topological space.

($3\frac{1}{2}$) If $(X, \mathcal{O})$ satisfies both (T1) and (T$3\frac{1}{2}$) we refer to $(X, \mathcal{O})$ as a *completely regular* topological space.

(4) If $(X, \mathcal{O})$ satisfies both (T1) and (T4) we refer to $(X, \mathcal{O})$ as a *normal* topological space.

(6) If $(X, \mathcal{O})$ satisfies both (T1) and (T6) we refer to $(X, \mathcal{O})$ as a *perfectly normal* topological space.

11.2 T0 and T1 topological spaces

Examples 11.2.

(1) Let $<$ be a pre-order on a set X. Let $\mathcal{O}$ denote the corresponding topology on X defined in Question 8 of Exercise Sheet 1.

A *partial order* is a pre-order $<$ such that if both $x < x'$ and $x' < x$ then $x = x'$. In other words there is at most one arrow between every pair of elements of X.

For instance the the pre-orders

$$\begin{tikzcd} 0 \arrow[r] & 1 \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd}$$

are partial orders. The pre-orders

$$\begin{tikzcd} 0 \arrow[r, shift left] & 1 \arrow[l, shift left] \end{tikzcd}$$

and

$$\begin{tikzcd}
0 & 1 \arrow[l] \arrow[d, shift right] \\
& 2 \arrow[ul] \arrow[u, shift right]
\end{tikzcd}$$

are not partial orders.

The topological space $(X, \mathcal{O})$ is a T0 topological space if and only if $<$ is a partial order. Let us prove this.

By definition a subset U of X belongs to $\mathcal{O}$ if and only if for all $x, x' \in X$ we have that if $x \in X$ and $x < x'$ then $x' \in X$.

Suppose that $<$ is not a partial order. Then for some $x, x' \in X$ with $x \neq x'$ we have that $x < x'$ and $x' < x$. By the previous paragraph, every neighbourhood of x in $(X, \mathcal{O})$ will contain x', and every neighbourhood of x' in $(X, \mathcal{O})$ will contain x. Hence $(X, \mathcal{O})$ is not a T0 topological space.

Suppose instead that $<$ is a partial order. Then for every $x, x' \in X$ with $x \neq x'$ we have either:

(i) $x < x'$ and $x' \not< x$

(ii) $x' < x$ and $x \not< x'$.

Without loss of generality — we may simply relabel x and x' — let us assume that (i) holds. Then the neighbourhood $U^x = \{x'' \in X \mid x < x''\}$ of x in $(X, \mathcal{O})$ contains x but not x'. We conclude that $(X, \mathcal{O})$ is T0.

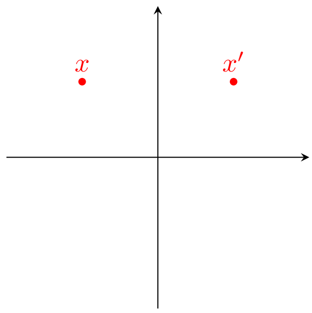
In Question 10 of Exercise Sheet 2 we saw that Alexandroff topological spaces correspond exactly to pre-orders. In this way we obtain a characterisation of T0 Alexandroff topological spaces.

(2) Let X be a set, and let $\mathcal{O}^{\text{indis}}$ denote the indiscrete topology on X. Then $(X, \mathcal{O})$ is not T0. Let us prove this.

Let $x, x' \in X$. The only neighbourhood of x in $(X, \mathcal{O}^{\text{indisc}})$ is X itself, which contains x'. Similarly the only neighbourhood of x' in $(X, \mathcal{O}^{\text{indisc}})$ is X itself, which contains x.

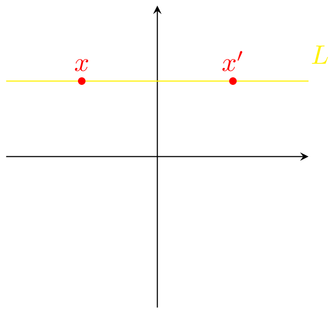
(3) Let $\mathcal{O}$ denote the topology on $\mathbb{R}^2$ generated in the sense of Question 5 of Exercise Sheet 2 by straight lines parallel to the x-axis. Then $(\mathbb{R}^2, \mathcal{O})$ is not T0.

Let us prove this. Let $(x, y) \in \mathbb{R}^2$ and $(x', y) \in \mathbb{R}^2$ be such that $x < x'$.



Every neighbourhood of (x, y) in $(\mathbb{R}^2, \mathcal{O})$ contains the straight line

$$L = \{(x'', y \mid x'' \in \mathbb{R}\}.$$



Moreover L is contained in every neighbourhood of (x', y) in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Thus there is no neighbourhood of (x, y) in $(\mathbb{R}^2, \mathcal{O})$ which does not contain (x', y), and no neighbourhood of (x', y) in $(\mathbb{R}^2, \mathcal{O})$ which does not contain (x, y).

Observation 11.3. Every T1 topological space is a T0 topological space.

Examples 11.4.

(1) Let X be a set, and let $<$ be a pre-order on X. Let $\mathcal{O}$ denote the corresponding topology on X defined in Question 8 of Exercise Sheet 1.

The topological space $(X, \mathcal{O})$ is T1 if and only $<$ is equality, by which we mean that $x < x'$ if and only if $x = x'$. Let us prove this.

Suppose that $<$ is not equality. Then there is a pair (x, x') of elements of X such that $x < x'$ but $x \neq x'$. By definition of $\mathcal{O}$, which we recalled in Examples 11.2 (1), every neighbourhood of x contains x'. Thus $(X, \mathcal{O})$ is not T1.

Suppose instead that $<$ is equality. Then $\mathcal{O}$ is the discrete topology on X. In particular, for any pair (x, x') of elements of X, the singleton set $\{x\}$ is a neighbourhood of x in $(X, \mathcal{O})$ which does not contain x'. Thus $(X, \mathcal{O})$ is T1.

Let us summarise.

(a) If $<$ is not a partial order, then $(X, \mathcal{O})$ is not a T0 topological space.

(b) If $<$ is a partial order which is not equality then $(X, \mathcal{O})$ is a T0 topological space but is not a T1 topological space.

(c) If $<$ is equality, then $(X, \mathcal{O})$ is a T1 topological space — and in particular a T0 topological space.

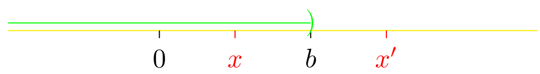
(2) Let $\mathcal{O}$ be the topology on $\mathbb{R}$ generated by

$$\mathcal{O}' = \{(a, \infty) \mid a < 0\} \cup \{(-\infty, b) \mid b > 0\}.$$

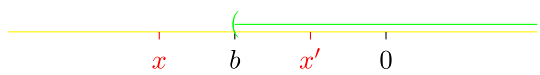
Then $(\mathbb{R}, \mathcal{O})$ is a T0 topological space. Let us prove this.

Suppose that $x, x' \in X$ and that $x \neq x'$. Without loss of generality — we may relabel x and x' if necessary — suppose that $x < x'$. We have the following two cases.

(a) If $x' > 0$ then for any $b \in \mathbb{R}$ such that $x < b$ and $0 < b < x'$ we have that $(-\infty, b)$ is a neighbourhood of x in $(\mathbb{R}, \mathcal{O})$ which does not contain x'.



(b) If $x' \leq 0$ then for any $x < a < x'$ we have that (a, ∞) is a neighbourhood of x' in $(\mathbb{R}, \mathcal{O})$ which does not contain x.



In each case we have a neighbourhood of either x or x' in $(\mathbb{R}, \mathcal{O})$ which does not contain both x and x'. Thus $(\mathbb{R}, \mathcal{O})$ is T0.

However $(\mathbb{R}, \mathcal{O})$ is not T1. For any $x \in \mathbb{R}$ we have that every neighbourhood of x in $(\mathbb{R}, \mathcal{O})$ contains 0. Thus for any $x \in \mathbb{R}$ with $x \neq 0$ we cannot find a neighbourhood of x in $(\mathbb{R}, \mathcal{O})$ which does not contain 0.

Proposition 11.5. Let $(X, \mathcal{O})$ be a topological space. The following are equivalent.

(1) $(X, \mathcal{O})$ is T1.

(2) The singleton set $\{x\}$ is closed in X for every $x \in X$.

(3) Every finite subset A of X is a closed subset of X.

(4) For every subset A of X we have that

$$A = \bigcap_{U \in \mathcal{O} \text{ and } A \subset U} U.$$

Proof. It suffices to prove the following implications.

(1) $\Rightarrow$ (2) Suppose that $(X, \mathcal{O})$ is T1. Let $x' \in X$ be such that $x \neq x'$. Since $(X, \mathcal{O})$ is T1 there is a neighbourhood U of x' in $(X, \mathcal{O})$ which does not contain x. Thus x' is not a limit point of $\{x\}$ in $(X, \mathcal{O})$.

(2) $\Rightarrow$ (3) Suppose that (2) holds. Let A be a finite subset of X. Let $x \in X \setminus A$. For each $a \in A$ there is by (2) a neighbourhood U_a of x in $(X, \mathcal{O})$ such that $a \notin U_x$.

Let $U = \bigcap_{a \in A} U_a$. Since A is finite we have U is open in X. Moreover we have that $x \in U$ and that $U \cap A = \emptyset$. Thus x is not a limit point of A in $(X, \mathcal{O})$.

(3) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (4) Suppose that (2) holds. Let A be a subset of X. Let

$$A' = \bigcap_{U \in \mathcal{O} \text{ and } A \subset U} U.$$

Let $x \in X \setminus A$. By (2) we have that $\{x\}$ is closed in X. Thus $X \setminus \{x\}$ is open in X. Hence

$$A' \subset \bigcap_{x \in X \setminus A} X \setminus \{x\}.$$

We have that

$$\begin{aligned} \bigcap_{x \in X \setminus A} X \setminus \{x\} &= X \setminus \bigcup_{x \in X \setminus A} \{x\} \\ &= X \setminus \big(X \setminus A\big) \\ &= A. \end{aligned}$$

Thus $A' \subset A$. It is moreover clear that $A \subset A'$. We conclude that $A = A'$.

(4) $\Rightarrow$ (1) Suppose that (4) holds. Let (x, x') be an ordered pair of elements of X be such that $x \neq x'$.

By (4) we have that

$$\{x\} = \bigcap_{U \in \mathcal{O} \text{ and } x \in U} U.$$

In particular we have that

$$x' \notin \bigcap_{U \in \mathcal{O} \text{ and } x \in U} U.$$

Thus there is a neighbourhood of x in $(X, \mathcal{O})$ which does not contain x'.

$\square$

11.3 Hausdorff topological spaces

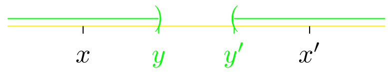
Observation 11.6. Every Hausdorff topological space is a T1 topological space. In particular every Hausdorff topological space is a T0 topological space.

Examples 11.7.

(1) $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is a Hausdorff topological space. Let us prove this.

Let $x, x' \in \mathbb{R}$ be such that $x \neq x'$. Without loss of generality — we may relabel x and x' — let us assume that $x < x'$.

Let $y, y' \in \mathbb{R}$ be such that $x < y \leq y' < x'$. We then have that $x \in (-\infty, y)$, that $x' \in (y', \infty)$, and that $(-\infty, y) \cap (y', \infty) = \emptyset$.



(3) Let $\mathsf{Spec}(\mathbb{Z})$ denote the set of primes. Let $\mathcal{O}$ denote the topology on $\mathsf{Spec}(\mathbb{Z})$ defined in Question 9 of Exercise Sheet 1. Recall that

$$\mathcal{O} = \left\{ \mathsf{Spec}(\mathbb{Z}) \setminus V(n) \mid n \in \mathbb{Z} \right\}$$

where

$$V(n) = \{ p \in \mathsf{Spec}(\mathbb{Z}) \mid p \mid n \}.$$

The topological space $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$ is T1. Let us prove this.

For any $n \in \mathbb{Z}$ we have that $V(n)$ is closed in $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$. In particular for any $p \in \mathsf{Spec}(\mathbb{Z})$ we have that $V(p)$ is closed in $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$. Thus since $V(p) = \{p\}$ we have that $\{p\}$ is a closed subset of $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$ for every $p \in \mathsf{Spec}(\mathbb{Z})$.

By Proposition 11.5 we conclude that $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$ is T1. However $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$ is not Hausdorff. Let us prove this.

Let $p, p' \in \mathsf{Spec}(\mathbb{Z})$. The neighbourhoods of p in $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$ are the sets

$$\mathsf{Spec}(\mathbb{Z}) \setminus V(n)$$

for which $p \nmid n$. The neighbourhoods of p' in $\big(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\big)$ are the sets

$$\mathsf{Spec}(\mathbb{Z}) \setminus V(n')$$

for which $p' \nmid n'$.

Let $n, n' \in \mathbb{Z}$ be such that $p \nmid n$ and $p' \nmid n'$. Let $p'' \in \mathsf{Spec}(\mathbb{Z})$ be such that $p'' \nmid n$ and $p'' \nmid n'$. For example, we may take p'' to be any prime larger than n and n'.

Then $p'' \in \mathsf{Spec}(\mathbb{Z}) \setminus V(n)$ and $p'' \in \mathsf{Spec}(\mathbb{Z}) \setminus V(n')$. Hence

$$\big(\mathsf{Spec}(\mathbb{Z}) \setminus V(n)\big) \cap \big(\mathsf{Spec}(\mathbb{Z}) \setminus V(n')\big) \neq \emptyset.$$

We have shown that for every neighbourhood U of p in $\big(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\big)$ and every neighbourhood U' of p' in $\big(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\big)$ we have that $U \cap U' \neq \emptyset$. Thus $\big(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\big)$ is not Hausdorff.

Notation 11.8. Let X be a set. We denote by $\Delta(X)$ the subset

$$\{(x, x) \in X \times X \mid x \in X\}$$

of $X \times X$.

Proposition 11.9. Let $(X, \mathcal{O}_X)$ be a topological space. Then $(X, \mathcal{O}_X)$ is Hausdorff if and only if $\Delta(X)$ is closed in $(X \times X, \mathcal{O}_{X \times X})$.

Proof. Suppose that $(X, \mathcal{O}_X)$ is Hausdorff. Let $(x, x') \in (X \times X) \setminus \Delta(X)$. By definition of $\Delta(X)$, we have that $x \neq x'$. Since $(X, \mathcal{O})$ is Hausdorff there is a neighbourhood U of x in $(X, \mathcal{O})$ and a neighbourhood U' of x' in $(X, \mathcal{O})$ such that $U \cap U' = \emptyset$.

We have that $\Delta(X) \cap (U \times U') = \Delta(U \cap U') = \emptyset$. Hence (x, x') is not a limit point of $\Delta(X)$. We deduce that $\Delta(X) = \overline{\Delta(X)}$. By Proposition 5.7 we conclude that $\Delta(X)$ is closed in $(X \times X, \mathcal{O}_{X \times X})$.

Suppose instead that $\Delta(X)$ is closed in $(X \times X, \mathcal{O}_{X \times X})$. For any $x, x' \in X$ with $x \neq x'$ we have by Proposition 5.7 that (x, x') is not a limit point of $\Delta(X)$. Hence there is a neighbourhood U of x in $(X, \mathcal{O}_X)$ and a neighbourhood U' of x' in $(X, \mathcal{O}_X)$ such that $\Delta(X) \cap (U \times U') = \emptyset$.

Appealing again to the fact that $\Delta(X) \cap (U \times U') = \Delta(U \cap U')$ we deduce that $\Delta(U \cap U') = \emptyset$. Hence $U \cap U' = \emptyset$. We conclude that $(X, \mathcal{O})$ is Hausdorff.

$\square$

Proposition 11.10. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let A be a subset of X, and let $\mathcal{O}_A$ denote the subspace topology $\mathcal{O}_A$ on A with respect to $(X, \mathcal{O}_X)$. Then $(A, \mathcal{O}_A)$ is a Hausdorff topological space.

Proof. Exercise. $\square$

Proposition 11.11. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then

$$(X \times Y, \mathcal{O}_{X \times Y})$$

is Hausdorff if and only if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are both Hausdorff.

Proof. Exercise. □

Examples 11.12. By Examples 11.7 (1) we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. By Proposition 11.10 and Proposition 11.11 we deduce that all the topological spaces of Examples 1.38 are Hausdorff.

Proposition 11.13. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a bijection which is an open map. If $(X, \mathcal{O}_X)$ is Hausdorff then $(Y, \mathcal{O}_Y)$ is Hausdorff.

Proof. Exercise. □

Corollary 11.14. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. If $(X, \mathcal{O}_X)$ is Hausdorff then $(Y, \mathcal{O}_Y)$ is Hausdorff.

Proof. Follows immediately from Proposition 11.13 since a homeomorphism is in particular a bijection which is an open map. □

Proposition 11.15. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$ is Hausdorff if and only if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are both Hausdorff.

Proof. Exercise. □

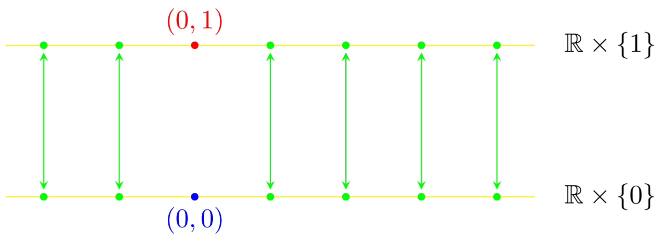
12 Thursday 21st February

12.1 Quotients of Hausdorff topological spaces

Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space, and let $\sim$ be an equivalence relation on X. Then $(X/\sim, \mathcal{O}_{X/\sim})$ is not necessarily Hausdorff.

Example 12.1. Recall from Recollection 5.17 that $\mathbb{R} \sqcup \mathbb{R}$ is the set $\big(\mathbb{R} \times \{0\}\big) \cup \big(\mathbb{R} \times \{1\}\big)$. By Examples 11.7 (1) we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. Thus by Proposition 11.15 we have that $(\mathbb{R} \sqcup \mathbb{R}, \mathcal{O}_{\mathbb{R} \sqcup \mathbb{R}})$ is Hausdorff.

Let $\sim$ be the equivalence relation on $\mathbb{R} \sqcup \mathbb{R}$ defined by $(x, 0) \sim (x, 1)$ for all $x \neq 0$. To put it slightly less formally, we have two copies of $\mathbb{R}$ and identify every real number except zero in the first copy of $\mathbb{R}$ with the same real number in the second copy of $\mathbb{R}$.



The topological space $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R} \sqcup \mathbb{R})/\sim}\big)$ is known as the *real line with two origins*. It is not Hausdorff — indeed is not even T0. Let us prove this.

To avoid confusion let us for the remainder of this example adopt the notation $]a, b[$ for the open interval from a real number a to a real number b, rather than our usual (a, b). Let

$$\mathbb{R} \sqcup \mathbb{R} \xrightarrow{\pi} (\mathbb{R} \sqcup \mathbb{R})/\sim$$

denote the quotient map.

Let U be a neighbourhood of $\pi\big((0,0)\big)$ in $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R} \sqcup \mathbb{R})/\sim}\big)$. We claim that $\pi\big((0,1)\big) \in U$. First let us make two observations.

(1) By definition of $\mathcal{O}_{(\mathbb{R} \sqcup \mathbb{R})/\sim}$ we have that $\pi^{-1}(U)$ is open in $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R} \sqcup \mathbb{R})/\sim}\big)$.

(2) By definition of $\mathcal{O}_{\mathbb{R}}$ we have that

$$\big\{\,]a, b[\mid a, b \in \mathbb{R} \big\}$$

is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Hence — see the Exercise Sheet — we have that

$$\big\{\,]a, b[\times \{0\} \mid a, b \in \mathbb{R} \big\} \cup \big\{\,]a, b[\times \{1\} \mid a, b \in \mathbb{R} \big\}$$

is a basis for $(\mathbb{R} \sqcup \mathbb{R}, \mathcal{O}_{\mathbb{R} \sqcup \mathbb{R}})$.

(3) We have that $(0,0) \in \pi^{-1}(U)$.

By (1) – (3) together with Question 3 (a) on Exercise Sheet 2 we deduce there are $a, b \in \mathbb{R}$ such that $0 \in \,]a,b[$ and $]a,b[\, \times \{0\} \subset \pi^{-1}(U)$.

From the latter we deduce that

$$\pi\big(\,]a,b[\, \times \{0\}\big) \subset \pi\big(\pi^{-1}(U)\big) = U.$$

Thus

$$\pi^{-1}\Big(\pi\big(\,]a,b[\, \times \{0\}\big)\Big) \subset \pi^{-1}(U).$$

Moreover

$$\pi^{-1}\Big(\pi\big(\,]a,b[\, \times \{0\}\big)\Big) = \big(\,]a,b[\, \times \{0\}\big) \sqcup \big(\,]a,b[\, \times \{1\}\big).$$

Since $0 \in \,]a,b[$ we have that $(0,1) \in \,]a,b[\, \times \{1\}$. We deduce that $(0,1) \in \pi^{-1}(U)$, and hence that $\pi\big((0,1)\big) \in U$ as claimed.

We have now proven that every neighbourhood of $\pi\big((0,0)\big)$ in $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R} \sqcup \mathbb{R})/\sim}\big)$ contains $\pi\big((0,1)\big)$. Thus $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R} \sqcup \mathbb{R})/\sim}\big)$ is not T1. An entirely analogous argument demonstrates that every neighbourhood of $\pi\big((0,1)\big)$ in $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R} \sqcup \mathbb{R})/\sim}\big)$ contains $\pi\big((0,0)\big)$. We conclude that $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R} \sqcup \mathbb{R})/\sim}\big)$ is not T0.

Notation 12.2. Let X be a set and let $\sim$ be a relation on X. Let

$$R_\sim := \{(x, x') \in X \times X \mid x \sim x'\}.$$

Proposition 12.3. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let $\sim$ be an equivalence relation on X. If $(X/\sim, \mathcal{O}_{X/\sim})$ is a Hausdorff topological space then $R_\sim$ is a closed subset of $(X \times X, \mathcal{O}_{X \times X})$.

Proof. Let

$$X \xrightarrow{\pi} X/\sim$$

be the quotient map.

Let

$$X \times X \xrightarrow{\pi \times \pi} (X/\sim) \times (X/\sim)$$

be the map given by $(x, x') \mapsto \big(\pi(x), \pi(x')\big)$. By Observation 3.7 we have that π is continuous. By Question 4 (c) on Exercise Sheet 3 we deduce that $\pi \times \pi$ is continuous.

If $X/\sim$ is Hausdorff then by Proposition 11.9 we have that $\Delta(X/\sim)$ is closed in

$$(X/\sim) \times (X/\sim).$$

By Question 1 (a) on Exercise Sheet 3 we deduce that $(\pi \times \pi)^{-1}\Big(\Delta(X/\sim)\Big)$ is closed in $X \times X$. Note that $R_\sim = (\pi \times \pi)^{-1}\Big(\Delta(X/\sim)\Big)$. We conclude that $R_\sim$ is closed in $X \times X$. □

Remark 12.4. We will shortly introduce compact topological spaces. If $(X, \mathcal{O}_X)$ is compact we will see in a later lecture that the requirement that $R_\sim$ be closed in $(X \times X, \mathcal{O}_{X \times X})$ is not only necessary but sufficient to ensure that if $(X, \mathcal{O}_X)$ is Hausdorff then $(X/\sim, \mathcal{O}_{X/\sim})$ is Hausdorff.

That $R_\sim$ be closed in $(X \times X, \mathcal{O}_{X \times X})$ is not sufficient in general to ensure that $(X/\sim, \mathcal{O}_{X/\sim})$ is Hausdorff, as the following example demonstrates.

Example 12.5. For the purposes of this example let $\mathbb{N}$ be the set $\{1, 2, \ldots\}$, namely the set of natural numbers without 0. Let $\Sigma = \left\{\frac{1}{n}\right\}_{n \in \mathbb{N}}$. Let $\mathcal{O}'$ be the set

$$\left\{(a,b) \mid a, b \in \mathbb{R}\right\} \cup \left\{(a,b) \setminus \big((a,b) \cap \Sigma\big) \mid a, b \in \mathbb{R}\right\}.$$

Then $\mathcal{O}'$ satisfies the conditions of Proposition 2.2. Let $\mathcal{O}$ denote the corresponding topology on $\mathbb{R}$ with basis $\mathcal{O}'$. Note that $\mathcal{O}_{\mathbb{R}} \subset \mathcal{O}$. Since $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff we deduce that $(\mathbb{R}, \mathcal{O})$ is Hausdorff.

Let $\sim$ be the equivalence relation on $\mathbb{R}$ generated by $1 \sim \frac{1}{n}$ for all $n \in \mathbb{N}$. Let $\mathcal{O}_\sim$ denote the quotient topology on $\mathbb{R}/\sim$ with respect to $(\mathbb{R}, \mathcal{O})$ and $\sim$. Then:

(1) $(\mathbb{R}/\sim, \mathcal{O}_\sim)$ is not Hausdorff.

(2) $R_\sim$ is closed in $(\mathbb{R}^2, \mathcal{O}^2)$, where $\mathcal{O}^2$ denotes the product topology on $\mathbb{R}^2$ with respect to $(\mathbb{R}, \mathcal{O})$ and $(\mathbb{R}, \mathcal{O})$.

We shall first prove (1). Let

$$\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\sim$$

be the quotient map. Let U be a neighbourhood of $\pi(1)$ in $(\mathbb{R}/\sim, \mathcal{O}_\sim)$ and let U' be a neighbourhood of $\pi(0)$ in $(\mathbb{R}, \mathcal{O}_\sim)$. We claim that $U \cap U' \neq \emptyset$. Let us prove this.

Since π is continuous we have that $\pi^{-1}(U)$ is open in $(\mathbb{R}, \mathcal{O})$. Moreover we have that

$$\left\{\frac{1}{n}\right\}_{n \in \mathbb{N}} = \pi^{-1}\big(\pi(1)\big) \subset \pi^{-1}(U).$$

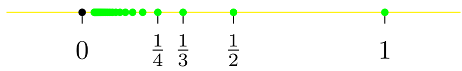
Let $n \in \mathbb{N}$. By Question 3 (a) of Exercise Sheet 2 there is a $U_n \in \mathcal{O}'$ such that $\frac{1}{n}$ in $(\mathbb{R}, \mathcal{O})$ and such that $U_n \subset \pi^{-1}(U)$. We make the following observations.

(i) Since $U_n \subset \pi^{-1}(U)$ for all $n \in \mathbb{N}$ we have that $\bigcup_{n \in \mathbb{N}} U_n \subset \pi^{-1}(U)$.

(ii) Since $\frac{1}{n}$ does not belong to $(a,b) \setminus \Sigma$ for any $a, b \in \mathbb{R}$ we have that $U_n \in \mathcal{O}_{\mathbb{R}}$.

By (ii) we have that $U_n = (a_n, b_n)$ for some $a_n, b_n \in \mathbb{R}$. Moreover we have that that

$$\inf\Big(\bigcup_{n \in \mathbb{N}} U_n\Big) \leq \inf\left\{\frac{1}{n}\right\}_{n \in \mathbb{N}} = 0.$$



Since π is continuous we have that $\pi^{-1}(U')$ is open in $(\mathbb{R}, \mathcal{O})$ and that $0 \in \pi^{-1}(U')$. By Question 3 (a) of Exercise Sheet 2 there is a $W \in \mathcal{O}'$ such that $0 \in W$ and such that $W \subset \pi^{-1}(U')$.

By definition of $\mathcal{O}'$ there are $a, b \in \mathbb{R}$ that $W = (a, b)$ or $W = (a, b) \setminus \Sigma$. Either way, since $0 \in W$ we must have that $a < 0$ and $b > 0$. We also have that

$$\inf\left(\pi^{-1}(U)\right) \leq \inf\left(\bigcup_{n \in \mathbb{N}} U_n\right) \leq 0.$$

Any $x \in \mathbb{R}$ such that $x \notin \Sigma$ and $0 < x < b$ belongs to $W \cap \pi^{-1}(U')$. Thus

$$W \cap \pi^{-1}(U') \neq \emptyset.$$

Hence $\pi^{-1}(U) \cap \pi^{-1}(U') \neq \emptyset$, Since $\pi^{-1}(U \cap U') = \pi^{-1}(U) \cap \pi^{-1}(U')$ we deduce that $\pi^{-1}(U \cap U') \neq \emptyset$. Thus $U \cap U' \neq \emptyset$ as claimed.

We have now proven that for any neighbourhood U of $\pi(0)$ in $(\mathbb{R}/\sim, \mathcal{O}_\sim)$ and any neighbourhood U' of $\pi(1)$ in $(\mathbb{R}/\sim, \mathcal{O}_\sim)$ we have that $U \cap U \neq \emptyset$. Thus $(\mathbb{R}/\sim, \mathcal{O}_\sim)$ is not Hausdorff.

Let us now prove (2). We claim that Σ is closed in $(\mathbb{R}, \mathcal{O})$. Let us prove this.

(i) If $x \in \mathbb{R}$ is a limit point of Σ in $(\mathbb{R}, \mathcal{O})$ then x is a limit point of Σ in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. The only limit point of Σ in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ which does not belong to Σ is 0.

(ii) Let $a < 0$ and $b > 0$ be real numbers. Then $(a, b) \setminus \Sigma$ is a neighbourhood of 0 in $(\mathbb{R}, \mathcal{O})$. We have that $\left((a, b) \setminus \Sigma\right) \cap \Sigma = \emptyset$. Thus 0 is not a limit point of Σ in $(\mathbb{R}, \mathcal{O})$.

Thus every limit point of Σ in $(\mathbb{R}, \mathcal{O})$ belongs to Σ. By Proposition 5.7 we deduce that Σ is closed in $(\mathbb{R}, \mathcal{O})$ as claimed. By Question 5 of Exercise Sheet 3 we thus have that $\Sigma \times \Sigma$ is closed in $(\mathbb{R}^2, \mathcal{O}^2)$. Moreover note that $R_\sim = \Sigma \times \Sigma$. We conclude that $R_\sim$ is closed in $(\mathbb{R}^2, \mathcal{O}^2)$.

12.2 Compact topological spaces

Terminology 12.6. Let $(X, \mathcal{O})$ be a topological space. An *open covering* of X is a set $\{U_j\}_{j \in J}$ of open subsets of X such that $X = \bigcup_{j \in J} U_j$.

Definition 12.7. A topological space $(X, \mathcal{O})$ is *compact* if for every open covering $\{U_j\}_{j \in J}$ of X there is a finite subset J' of J such that $X = \bigcup_{j' \in J'} U_{j'}$.

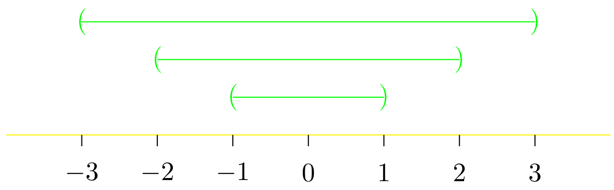
Terminology 12.8. Let $(X, \mathcal{O})$ be a topological space, and let $\{U_j\}_{j \in J}$ be an open covering of X. Suppose that there is a finite subset J' of J such that $X = \bigcup_{j' \in J'} U_{j'}$. We write that $\{U_{j'}\}_{j' \in J'}$ is a *finite subcovering* of $\{U_j\}_{j \in J}$.

Examples 12.9.

(1) Let $(X, \mathcal{O})$ be a topological space. If $\mathcal{O}$ is finite then $(X, \mathcal{O})$ is compact. For if $\mathcal{O}$ is finite then every set $\{U_j\}_{j \in J}$ of open subsets of X is finite.

In particular if X is finite then $(X, \mathcal{O})$ is compact. For if X is finite there are only finitely many subsets of X, and thus $\mathcal{O}$ is finite.

(2) $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not compact. The open covering $\{(-n, n)\}_{n \in \mathbb{N}}$ of $\mathbb{R}$ has no finite subcovering for instance.

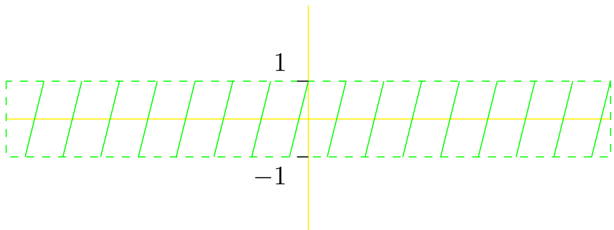


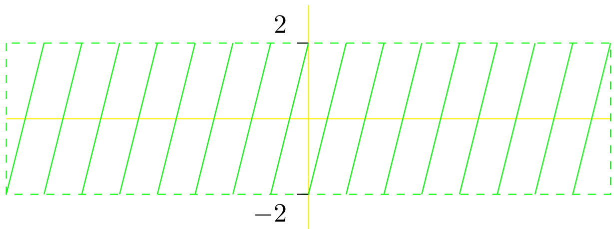
(3) $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is not compact. The open covering of $\mathbb{R}^2$ given by

$$\left\{\mathbb{R} \times (-n, n)\right\}_{n \in \mathbb{N}}$$

has no finite subcovering for instance.

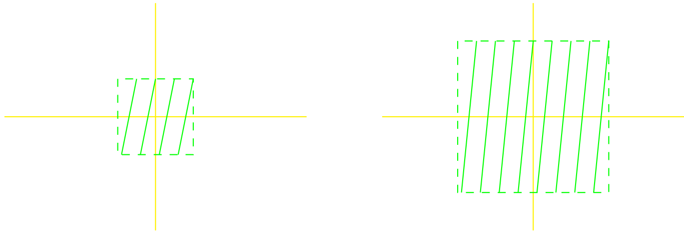
This open covering consists of horizontal strips of increasing height.





A different open covering of $\mathbb{R}^2$ which has no finite subcovering is given by

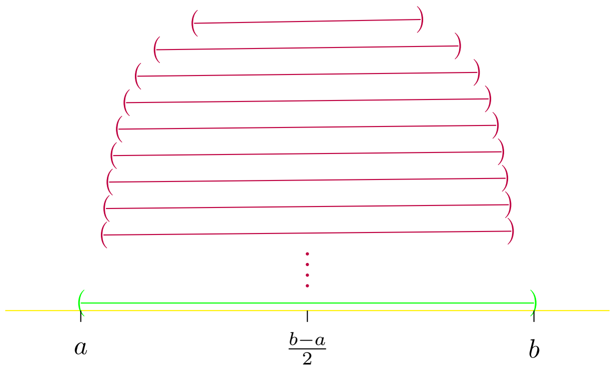
$$\left\{(-n, n) \times (-n, n)\right\}_{n \in \mathbb{N}}.$$



(4) An open interval $\big((a,b),\mathcal{O}_{(a,b)}\big)$, where $\mathcal{O}_{(a,b)}$ denotes the subspace topology with respect to $(\mathbb{R},\mathcal{O}_{\mathbb{R}})$, is not compact for any $a,b\in\mathbb{R}$. The open covering of (a,b) given by

$$\big\{(a+\frac{1}{n},b-\frac{1}{n})\big\}_{n\in\mathbb{N}\text{ and }\frac{1}{n}<\frac{b-a}{2}}$$

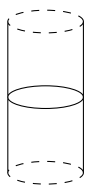
has no finite subcovering for instance.



(5) Generalising (2) let $(X,\mathcal{O}_X)$ and $(Y,\mathcal{O}_Y)$ be topological spaces, one of which is not compact. Then $(X\times Y,\mathcal{O}_{X\times Y})$ is not compact.

Suppose for example that $(X,\mathcal{O}_X)$ is not compact. Let $\{U_j\}_{j\in J}$ be an open covering of X which does not admit a finite subcovering. Then $\big\{U_j\times Y\big\}_{j\in J}$ is an open covering of $X\times Y$ which does not admit a finite subcovering.

For instance let $(S^1\times(0,1),\mathcal{O}_{S^1\times(0,1)})$ be a cylinder with the two circles at its ends cut out.

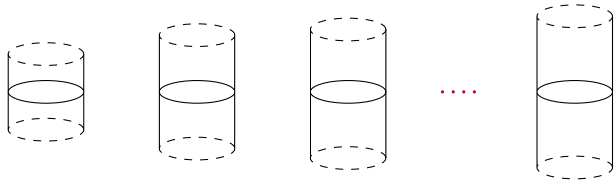


Since $(0,1)$ is not compact by (4) we have that $(S^1\times(0,1),\mathcal{O}_{S^1\times(0,1)})$ is not compact.

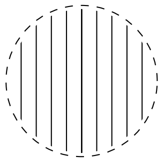
The open covering

$$\left\{ S^1 \times \left(\frac{1}{n}, 1 - \frac{1}{n} \right) \right\}_{n \in \mathbb{N} \text{ and } n \geq 2}$$

of $S^1 \times (0,1)$ is pictured below. It does not admit a finite subcovering.



(6) Let $D^2 \setminus S^1$ be the disc D^2 with its boundary circle removed.



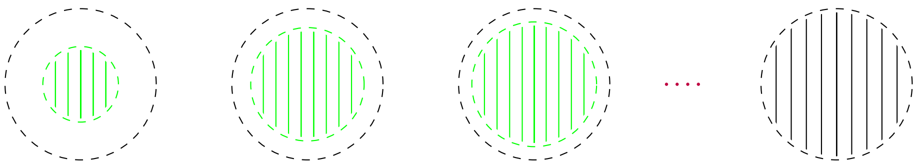
In other words $D^2 \setminus S^1$ is

$$\left\{ (x,y) \in \mathbb{R}^2 \mid \|(x,y)\| < 1 \right\}$$

equipped with the subspace topology $\mathcal{O}_{D^2 \setminus S^1}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Then $(D^2 \setminus S^1, \mathcal{O}_{D^2 \setminus S^1})$ is not compact. The open covering

$$\left\{ (x,y) \in \mathbb{R}^2 \mid \|(x,y)\| < 1 - \frac{1}{n} \right\}_{n \in \mathbb{N}}$$

of $D^2 \setminus S^1$ does not admit a finite subcovering for instance.



Proposition 12.10. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a surjective continuous map. If $(X, \mathcal{O}_X)$ is compact then $(Y, \mathcal{O}_Y)$ is compact.

Proof. Let $\{U_j\}_{j \in J}$ be an open covering of Y. Since f is continuous we have that $f^{-1}(U_j) \in \mathcal{O}_X$ for all $j \in J$. Moreover

$$\begin{aligned} \bigcup_{j \in J} f^{-1}(U_j) &= f^{-1}(\bigcup_{j \in J} U_j) \\ &= f^{-1}(Y) \\ &= X. \end{aligned}$$

Thus $\{f^{-1}(U_j)\}_{j \in J}$ is an open covering of X.

Since $(X, \mathcal{O}_X)$ is compact there is a finite subset J' of J such that $\{f^{-1}(U_{j'})\}_{j' \in J'}$ is an open covering of X. We have that

$$\begin{aligned} \bigcup_{j' \in J'} U_{j'} &= \bigcup_{j' \in J'} f\big(f^{-1}(U_{j'})\big) \\ &= f\Big(\bigcup_{j' \in J'} f^{-1}(U_{j'})\Big) \\ &= f(X) \\ &= Y. \end{aligned}$$

Thus $\{U_{j'}\}_{j' \in J'}$ is an open covering of Y. $\square$

Corollary 12.11. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. If $(X, \mathcal{O}_X)$ is compact then $(Y, \mathcal{O}_Y)$ is compact.

Proof. Follows immediately from Proposition 12.10, since by Proposition 3.15 a homeomorphism is in particular surjective and continuous. $\square$

Remark 12.12. Let the open interval (a, b) for $a, b \in \mathbb{R}$ be equipped with its subspace topology $\mathcal{O}_{(a,b)}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Examples 4.7 (6) we have that $\big((a, b), \mathcal{O}_{(a,b)}\big)$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Once we know by Examples 12.9 (2) that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not compact we could appeal to Corollary 12.11 to deduce that $\big((a, b), \mathcal{O}_{(a,b)}\big)$ is not compact. We observed this directly in Examples 12.9 (4).

13 Tuesday 26th February

13.1 $(I, \mathcal{O}_I)$ is compact

Lemma 13.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{U_j\}_{j \in J}$ and $\{W_k\}_{k \in K}$ be open coverings of X. Suppose that for every $k \in K$ there is a $j_k \in J$ such that $W_k \subset U_{j_k}$. If $\{W_k\}_{k \in K}$ has a finite subcovering then $\{U_j\}_{j \in J}$ has a finite subcovering.

Proof. Let K' be a finite subset of K such that $\{W_{k'}\}_{k' \in K}$ is an open covering of X. We have that

$$\begin{aligned} X &= \bigcup_{k' \in K'} W_{k'} \\ &\subset \bigcup_{k' \in K'} U_{j_{k'}}. \end{aligned}$$

Then $X = \bigcup_{k' \in K'} U_{j_{k'}}$. Thus $\{U_{j_{k'}}\}_{k' \in K'}$ is a finite subcovering of $\{U_j\}_{j \in J}$. $\square$

Proposition 13.2. The unit interval $(I, \mathcal{O}_I)$ is compact.

Proof. By definition of $\mathcal{O}_{\mathbb{R}}$ we have that $\big\{(a,b) \mid a, b \in \mathbb{R}\big\}$ is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Question 2 of Exercise Sheet 2 we deduce that

$$\mathcal{O}' = \big\{ I \cap (a,b) \mid a, b \in \mathbb{R} \big\}$$

is a basis for $(I, \mathcal{O}_I)$.

Let $\{U_j\}_{j \in J}$ be an open covering of I. By Question 3 (a) of Exercise Sheet 2 with respect to $\mathcal{O}'$, for every $t \in I$ and every $j \in J$ there is an open interval (a_t, b_t) such that $I \cap (a_t, b_t) \subset U_j$ and $t \in I \cap (a_t, b_t)$. Let $A_t = I \cap (a_t, b_t)$. Then A_t is either an open interval, a half open interval, or a closed interval. For brevity we shall simply refer to A_t as an interval.

We have that

$$\begin{aligned} I &= \bigcup_{t \in I} \{t\} \\ &\subset \bigcup_{t \in I} A_t. \end{aligned}$$

Thus $\{A_t\}_{t \in I}$ is an open covering of I. By Lemma 13.1 it suffices to prove that $\{A_t\}_{t \in I}$ admits a finite subcovering.

Let $\{0, 1\}$ be equipped with its discrete topology. Define

$$I \xrightarrow{f} \{0,1\}$$

to be the map given by $s \mapsto 0$ if there is a finite subset J of I such that

$$[0, s] \subset \bigcup_{t' \in J} A_{t'}$$

and by $s \mapsto 1$ otherwise. Let us first prove that f is continuous.

Let $t \in I$. Let $s \in A_t$ and suppose that $f(s) = 0$. Then by definition of f there is a finite subset J of I such that

$$[0, s] \subset \bigcup_{t' \in J} A_{t'}.$$

Let $s' \in A_t$, and let $s'' \in I$ be such that $s'' \in [0, s']$. If $s'' \leq s$ then since

$$[0, s] \subset \bigcup_{t' \in J} A_{t'}$$

we have that $s'' \in A_{t'}$ for some $t' \in J$. If $s \leq s''$ then since A_t is an interval and both s and s' belong to A_t we have by Lemma 7.8 that $s'' \in A_t$. Thus for any $s'' \in [0, s']$ we have that

$$s'' \in \bigcup_{t'' \in J \cup \{t\}} A_{t''}.$$

We deduce that

$$[0, s'] \subset \bigcup_{t'' \in J \cup \{t\}} A_{t''}.$$

Since J is finite we have that $J \cup \{t\}$ is finite. Hence $f(s') = 0$.

We have now proven that for any $t \in I$, if $f(s) = 0$ for some $s \in A_t$ then $f(A_t) = \{0\}$. We draw the following conclusions.

(1) We have that

$$f^{-1}\big(\{0\}\big) = \bigcup_{t \in I \text{ such that } f(s) = 0 \text{ for some } s \in A_t} A_t.$$

We deduce that $f^{-1}\big(\{0\}\big) \in \mathcal{O}_I$ since $A_t \in \mathcal{O}_I$ for all $t \in I$.

(2) We have that

$$f^{-1}\big(\{1\}\big) = \bigcup_{t \in I \text{ such that } f(s) = 1 \text{ for all } s \in A_t} A_t.$$

We deduce that $f^{-1}\big(\{1\}\big) \in \mathcal{O}_I$ since $A_t \in \mathcal{O}_I$ for all $t \in I$.

This completes our proof that f is continuous.

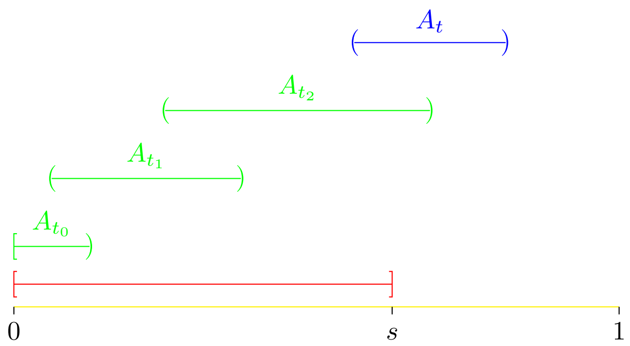
By Proposition 7.9 we have that $(I, \mathcal{O}_I)$ is connected. Since f is continuous we deduce by Proposition 6.5 that f is constant. Moreover $f(0) = 0$ since $[0, 0] = \{0\} \subset A_0$.

We deduce that $f(s) = 0$ for all $s \in I$. In particular we have that $f(1) = 0$. Thus by definition of f there is a finite subset J of I such that $I = \bigcup_{t' \in J} A_{t'}$. We conclude that $\{A_{t'}\}_{t' \in J}$ is a finite subcovering of $\{A_t\}_{t \in I}$.

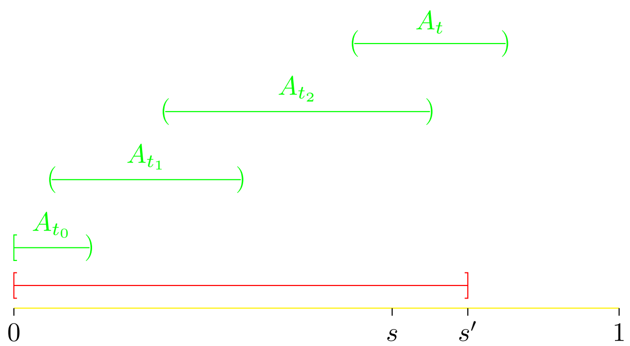
$\square$

Scholium 13.3. The proof of Proposition 13.2 is perhaps the most difficult in the course. The idea is to carry out a kind of inductive argument. We begin by noting that we need only the singleton set $\{A_0\}$ to ensure that $0 \in \bigcup_{t\in I}\{A_t\}$.

Next we suppose that we know that for some $t \in I$ and some $s \in A_t$ we need only a finite number of the sets $A_{t'}$ to ensure that $[0, s] \subset \bigcup_{t'\in I} A_{t'}$. In the picture below we suppose that we need only three sets $A_0 = A_{t_0}$, A_{t_1}, and A_{t_2}.



We then observe that for any $s' \in A_t$ we require only the finite number of sets $A_{t'}$ which we needed to cover $[0, s]$ together with the set A_t to ensure that $[0, s'] \subset \bigcup_{t'\in I} A_{t'}$.



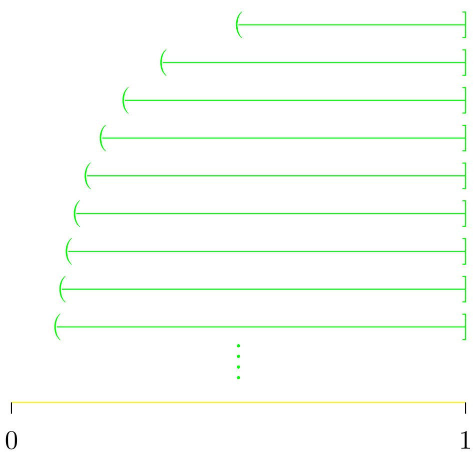
The tricky part is to show that by 'creeping along' in this manner we can arrive at 1 after a finite number of steps. Ultimately this is a consequence of the completeness of $\mathbb{R}$ — indeed it is possible to give a proof in which one appeals to the completeness of $\mathbb{R}$ directly.

We instead gave a proof which builds upon the hard work we already carried out to prove that $(I, \mathcal{O}_I)$ is connected. The role of the completeness of $\mathbb{R}$ lay in the proof of Lemma 7.8.

Remark 13.4. To help us to appreciate why $(I, \mathcal{O}_I)$ is compact, let us compare it to $(0,1]$. We equip $(0,1]$ with its subspace topology $\mathcal{O}_{(0,1]}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. The open covering

$$\left\{ \left(\frac{1}{n}, 1\right] \right\}_{n \in \mathbb{N}}$$

of $(0,1]$ has no finite subcovering.



Thus $(0,1]$ is not compact.

To obtain an open covering of $[0,1]$ we have to add an open set containing 0. Let us take this open set to be $[0,t)$ for some $0 < t < 1$. Thus our open covering of $[0,1]$ is

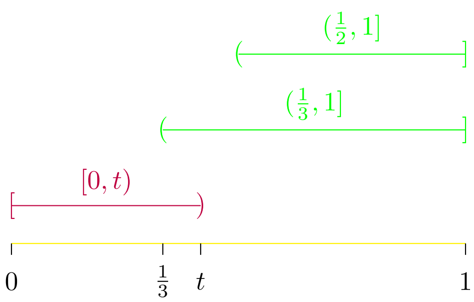
$$\left\{ \left(\frac{1}{n}, 1\right] \right\}_{n \in \mathbb{N}} \cup \{[0,t)\}$$

where $\{[0,t)\}$ is the singleton set containing $[0,t)$.

This open covering has a finite subcovering! For instance

$$\left\{ \left(\frac{1}{n}, 1\right] \right\}_{n \,\in\, \mathbb{N} \text{ such that } n \leq m} \cup \{[0,t)\}$$

where $m \in \mathbb{N}$ is such that $\frac{1}{m} < t$.



Corollary 13.5. Let the closed interval $[a, b]$ be equipped with its subspace topology $\mathcal{O}_{[a,b]}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then $\big([a, b], \mathcal{O}_{[a,b]}\big)$ is compact.

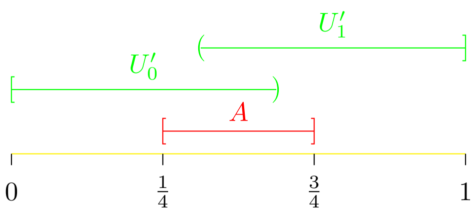
Proof. Follows immediately by Corollary 12.11 from Proposition 13.2, since by Examples 4.7 (4) we have that $\big([a, b], \mathcal{O}_{[a,b]}\big)$ is homeomorphic to $(I, \mathcal{O}_I)$. $\square$

Remark 13.6. Corollary 13.5 is one of the cornerstones of mathematics. Analysis relies indispensably upon it, and it is at the heart of many constructions in topology.

13.2 Compact vs Hausdorff vs closed

Proposition 13.7. Let $(X, \mathcal{O}_X)$ be a compact topological space, and let A be a closed subset of X. Then $(A, \mathcal{O}_A)$ is compact. Here $\mathcal{O}_A$ is the subspace topology on A with respect to $(X, \mathcal{O}_X)$.

Proof. Let $\{U_j\}_{j \in J}$ be an open covering of $(A, \mathcal{O}_A)$. By definition of $\mathcal{O}_A$ we have that $U_j = A \cap U'_j$ for some $U'_j \in \mathcal{O}_X$. Suppose first that $\{U'_j\}_{j \in J}$ is an open covering of $(X, \mathcal{O}_X)$. An example is pictured below.

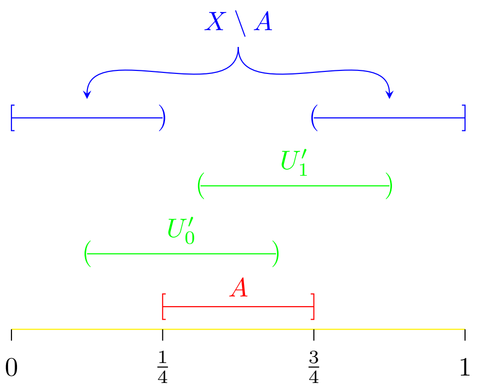


Since $(X, \mathcal{O}_X)$ is compact there is a finite subset J' of J such that $\{U'_{j'}\}_{j' \in J'}$ is an open covering of $(X, \mathcal{O}_X)$. Then:

$$\begin{aligned} A &= A \cap X \\ &= A \cap \Big(\bigcup_{j' \in J'} U'_{j'} \Big) \\ &= \bigcup_{j' \in J'} A \cap U'_{j'} \\ &= \bigcup_{j' \in J'} U_{j'}. \end{aligned}$$

Thus $\{U_{j'}\}_{j' \in J'}$ is a finite subcovering of $\{U_j\}_{j \in J}$.

Suppose now that $X \setminus \bigcup_{j \in J} U'_j \neq \emptyset$. Since A is closed in $(X, \mathcal{O}_X)$ we have that $X \setminus A$ is open in $(X, \mathcal{O}_X)$. Hence $\{U'_j\}_{j \in J} \cup \{X \setminus A\}$ is an open covering of $(X, \mathcal{O}_X)$. Here $\{X \setminus A\}$ denotes the set with the single element $X \setminus A$. An example is pictured below.



Since $(X, \mathcal{O}_X)$ is compact this open covering admits a finite subcovering. Moreover $X \setminus A$ must belong to this finite subcovering by our assumption that $X \setminus \bigcup_{j \in J} U_j' \neq \emptyset$. Thus there is a finite subset J' of J such that

$$\{U_{j'}'\}_{j' \in J'} \cup \{X \setminus A\}$$

is an open covering of X. We now observe that:

$$\begin{aligned}
A &= A \cap X \\
&= A \cap \left(\left(\bigcup_{j' \in J'} U_{j'}' \right) \cup \{X \setminus A\} \right) \\
&= \left(A \cap \bigcup_{j' \in J'} U_{j'}' \right) \cup \left\{ A \cap (X \setminus A) \right\} \\
&= \left(\bigcup_{j' \in J'} A \cap U_{j'}' \right) \cup \emptyset \\
&= \bigcup_{j' \in J'} U_{j'}.
\end{aligned}$$

Thus $\{U_{j'}\}_{j' \in J'}$ is a finite subcovering of $\{U_j\}_{j \in J}$. $\square$

Terminology 13.8. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X equipped with its subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Then A is a *compact subset* of X if $(A, \mathcal{O}_A)$ is compact.

Lemma 13.9. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let A be a compact subset of X. Suppose that $x \in X \setminus A$. There is a pair of open subsets U and U' of X such that:

(1) $A \subset U$,

(2) U' is a neighbourhood of x,

(3) $U \cap U' = \emptyset$.

Proof. Let $a \in A$. Since $(X, \mathcal{O}_X)$ is Hausdorff there is a neighbourhood U_a of a in $(X, \mathcal{O}_X)$ and a neighbourhood U'_a of x in $(X, \mathcal{O}_X)$ such that $U_a \cap U'_a = \emptyset$.

Let A be equipped with its subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Then $\{A \cap U_a\}_{a \in A}$ defines an open covering of $(A, \mathcal{O}_A)$, since

$$\begin{aligned} A &= \bigcup_{a \in A} \{a\} \\ &\subset \bigcup_{a \in A} A \cap U_a \end{aligned}$$

and thus $A = \bigcup_{a \in A} A \cap U_a$.

Since $(A, \mathcal{O}_A)$ is compact there is a finite subset J of A such that $\{A \cap U_a\}_{a \in J}$ is an open covering of A. Let $U = \bigcup_{a \in J} U_a$. Let $U' = \bigcap_{a \in J} U'_a$. Then:

(1) We have that $A \subset U$, since

$$\begin{aligned} A &= \bigcup_{a \in A} A \cap U_a \\ &= \bigcup_{a \in J} A \cap U_a \\ &\subset \bigcup_{a \in J} U_a \\ &= U. \end{aligned}$$

Moreover we have that U_a is open in $(X, \mathcal{O}_X)$ for all $a \in A$, and in particular for all $a \in J$. Thus we have that U is open in $(X, \mathcal{O}_X)$.

(2) Since $x \in U'_a$ for all $a \in A$ we have that

$$x \in \bigcap_{a \in A} U'_a \subset \bigcap_{a \in J} U'_a.$$

Moreover we have that U'_a is open in $(X, \mathcal{O}_X)$ for all $a \in A$, and in particular for all $a \in J$. Since J is finite we thus have that $\bigcap_{a \in J} U'_a$ is open in $(X, \mathcal{O}_X)$.

(3) Since $U_a \cap U_{a'} = \emptyset$ for all $a \in A$ we have that

$$\begin{aligned} U \cap U' &= \Big(\bigcup_{a \in J} U_a \Big) \cap \Big(\bigcap_{a' \in J} U'_{a'} \Big) \\ &= \bigcup_{a \in J} \Big(U_a \cap \bigcap_{a' \in J} U'_{a'} \Big) \\ &\subset \bigcup_{a \in J} (U_a \cap U_a) \\ &= \bigcup_{a \in J} \emptyset \\ &= \emptyset. \end{aligned}$$

$\square$

Remark 13.10. The proof of Lemma 13.9 is a very typical example of an appeal to compactness in practise. The key step is (2). Our conclusion that $\bigcap_{a \in J} U'_a$ is open in $(X, \mathcal{O}_X)$ relies on the fact that J is finite — as we know, an arbitrary intersection of open sets in a topological space need not be open.

Proposition 13.11. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let A be a compact subset of X. Then A is closed in $(X, \mathcal{O}_X)$.

Proof. Let $x \in X \setminus A$. By Lemma 13.9 there is a pair of open subsets U and U' of X such that:

(1) $A \subset U$,

(2) U' is a neighbourhood of x,

(3) $U \cap U' = \emptyset$.

In particular $U' \cap A \subset U' \cap U = \emptyset$. Thus x is not a limit point of A in $(X, \mathcal{O}_X)$.

We deduce that $\overline{A} = A$. By Proposition 5.7 we conclude that A is closed in X. $\square$

Remark 13.12. Proposition 13.11 does not necessarily hold if $(X, \mathcal{O}_X)$ is not Hausdorff. Here are two examples.

(1) Let $X = \{0, 1\}$ be equipped with the topology $\mathcal{O} = \{\emptyset, \{1\}, X\}$. In other words, $(X, \mathcal{O})$ is the Sierpiński interval.

By Examples 11.4 (1) a finite topological space is T1 if and only if its topology is the discrete topology. Since $\mathcal{O}$ is not the discrete topology on X, the Sierpiński interval is not T1 and hence not Hausdorff.

Since X is finite every subset of X is compact. In particular $\{1\}$ is a compact subset of $(X, \mathcal{O})$. But as we observed in Examples 5.6 (1) the set $\{1\}$ is not closed in $(X, \mathcal{O})$.

(2) Let $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R} \sqcup \mathbb{R})/\sim}\big)$ be the real line with two origins of Example 12.1. As in Example 12.1 to avoid confusion we adopt the notation $]a, b[$ for the open interval from a to b. Similarly we denote by $]a, b]$ the half open interval from a to b.

Let

$$\mathbb{R} \sqcup \mathbb{R} \xrightarrow{\pi} (\mathbb{R} \sqcup \mathbb{R})/\sim$$

denote the quotient map.

Let

$$I \xrightarrow{i} \mathbb{R} \sqcup \mathbb{R}$$

denote the map $t \mapsto (t, 0)$. Here as usual we think of $\mathbb{R} \sqcup \mathbb{R}$ as

$$\big(\mathbb{R} \times \{0\}\big) \cup \big(\mathbb{R} \times \{1\}\big).$$

We have that i is the composition of the following two maps.

(1) The inclusion map

$$I \hookrightarrow \mathbb{R},$$

which is continuous by Proposition 2.15

(2) The map

$$\mathbb{R} \longrightarrow \mathbb{R} \sqcup \mathbb{R}$$

given by $x \mapsto (x, 0)$, which is continuous by Observation 5.21.

We deduce by Proposition 2.16 that i is continuous. Thus by Proposition 13.2 and Proposition 12.10 we have that $\pi\big(i(I)\big)$ is a compact subset of

$$\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big).$$

We claim that $\pi\big(i(I)\big)$ is not a closed subset of $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$.

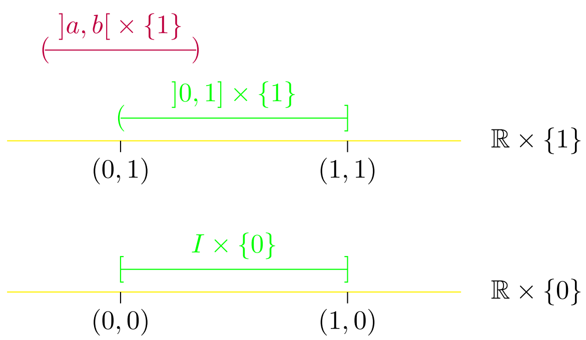
Let us prove that $\pi\big((0,1)\big)$ is a limit point of $\pi\big(i(I)\big)$ in $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$. Let U be a neighbourhood of $\pi\big((0,1)\big)$ in $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$. Then $\pi^{-1}(U)$ is a neighbourhood of $(0,1)$ in $\mathbb{R} \sqcup \mathbb{R}$.

As in Example 12.1 we have that

$$\big\{\,]a, b[\times \{0\} \mid a, b \in \mathbb{R} \big\} \cup \big\{\,]a, b[\times \{1\} \mid a, b \in \mathbb{R} \big\}$$

is a basis for $(\mathbb{R}\sqcup\mathbb{R}, \mathcal{O}_{\mathbb{R}\sqcup\mathbb{R}})$. By Question 3 (a) of Exercise Sheet 2 there are $a, b \in \mathbb{R}$ such that $0 \in\,]a, b[$ and $]a, b[\times \{1\} \subset \pi^{-1}(U)$.

We have that $\pi^{-1}\Big(\pi\big(i(I)\big)\Big) = \big(I \times \{0\}\big) \times \big(]0, 1] \times \{1\}\big)$.



Then

$$]a, b[\cap]0, 1] =]0, b[\neq \emptyset.$$

Hence

$$\big( ]a, b[\times \{1\} \big) \cap \big( ]0, 1] \times \{1\} \big) \neq \emptyset.$$

Thus

$$\pi^{-1}\Big(U \cap \pi\big(i(I)\big) \Big) = \pi^{-1}(U) \cap \pi^{-1}\Big(\pi\big(i(I)\big) \Big) \neq \emptyset.$$

We deduce that

$$U \cap \pi\big(i(I)\big) \neq \emptyset.$$

This completes our proof that $\pi\big((0,1)\big)$ is a limit point of $\pi\big(i(I)\big)$ in

$$\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R} \sqcup \mathbb{R})/\sim} \big).$$

But $\pi\big((0,1)\big) \notin \pi\big(i(I)\big)$. Thus $\pi\big(i(I)\big)$ is not closed in $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R} \sqcup \mathbb{R})/\sim} \big)$.

Proposition 13.13. Let $(X, \mathcal{O}_X)$ be a compact topological space. Let $(Y, \mathcal{O}_Y)$ be a Hausdorff topological space. A map

$$X \xrightarrow{f} Y$$

is a homeomorphism if and only if f is continuous and bijective.

Proof. If f is a homeomorphism then by Proposition 3.15 we have that f is continuous and bijective.

Suppose instead that f is continuous and bijective. That f is bijective implies that $x \mapsto f^{-1}(x)$ gives a well defined map

$$Y \xrightarrow{g} X.$$

We have that g is inverse to f. To prove that f is a homeomorphism we shall prove that g is continuous.

By Question 1 (a) of Exercise Sheet 3 we have that g is continuous if and only if $g^{-1}(A)$ is a closed subset of Y for any closed subset A of X. By definition of g we have that $g^{-1}(A) = f(A)$. Thus it suffices to prove that if A is a closed subset of X then $f(A)$ is a closed subset of Y.

Suppose that A ia a closed subset of X. Then since $(X, \mathcal{O}_X)$ is compact we have by Proposition 13.7 that A is a compact subset of X. Thus by Proposition 12.10 we have that $f(A)$ is a compact subset of Y. Since $(Y, \mathcal{O}_Y)$ is Hausdorff we deduce by Proposition 13.11 that $f(A)$ is closed in $(Y, \mathcal{O}_Y)$ as required.

□

Proposition 13.14. Let $(X, \mathcal{O}_X)$ be a compact topological space and let $\sim$ be an equivalence relation on X. Then $(X/\sim, \mathcal{O}_{X/\sim})$ is compact.

Proof. The quotient map

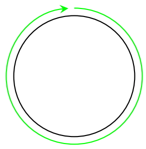
$$X \xrightarrow{\pi} X/\sim$$

is continuous and surjective. We deduce that $(X/\sim, \mathcal{O}_{X/\sim})$ is compact by Proposition 12.10. $\square$

Example 13.15. As in Examples 3.9 (1) let $\sim$ be the equivalence relation on I generated by $0 \sim 1$. Let

$$I \xrightarrow{\phi} S^1$$

be the continuous map constructed in Question 9 of Exercise Sheet 3.



We have that $\phi(0) = \phi(1)$. Thus we obtain a map

$$I/\sim \xrightarrow{f} S^1$$

given by $[t] \mapsto \phi(t)$. By Question 11 (a) of Exercise Sheet 4 we have that f is continuous. Moreover f is a bijection.

(1) By Examples 11.7 we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. Thus by Proposition 11.11 we have that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff. By Proposition 11.10 we deduce that $(S^1, \mathcal{O}_{S^1})$ is Hausdorff.

(2) By Proposition 13.2 we have that $(I, \mathcal{O}_I)$ is compact. Thus by Proposition 13.14 we have that $(I/\sim, \mathcal{O}_{I/\sim})$ is compact.

We conclude by Proposition 13.13 that f is a homeomorphism. This gives a rigorous affirmative answer to Question 3.10.

14 Thursday 28th February

14.1 A product of compact topological spaces is compact

Proposition 14.1. Let $(X, \mathcal{O}_X)$ be a topological space and let $(X', \mathcal{O}_{X'})$ be a compact topological space. Let $x \in X$ and let W be a subset of $X \times X'$ satisfying the following conditions.

(1) $W \in \mathcal{O}_{X \times X'}$.

(2) $\{x\} \times X' \subset W$.

Then there is a neighbourhood U of x in $(X, \mathcal{O}_X)$ such that $U \times X' \subset W$.

Proof. Let $x' \in X'$. By (2) we have that $(x, x') \in W$. By (1) and the definition of $\mathcal{O}_{X \times X'}$ we deduce that there is a neighbourhood $U_{x'}$ of x in $(X, \mathcal{O}_X)$ and a neighbourhood $U'_{x'}$ of x' in $(X', \mathcal{O}_{X'})$ such that $U_{x'} \times U'_{x'} \subset W$. We have that

$$\begin{aligned} X' &= \bigcup_{x' \in X'} \{x'\} \\ &\subset \bigcup_{x' \in X'} U'_{x'}. \end{aligned}$$

Thus $X = \bigcup_{x' \in X'} U'_{x'}$ and we have that $\{U'_{x'}\}_{x' \in X}$ is an open covering of X'.

Since $(X', \mathcal{O}_{X'})$ is compact there is a finite subset J of X' such that $\{U'_{x'}\}_{x' \in J}$ is an open covering of X'. Let $U = \bigcap_{x' \in J} U_{x'}$. We make the following observations.

(1) Since J is finite we have that $U \in \mathcal{O}_X$.

(2) Since $x \in U_{x'}$ for all $x' \in X'$, we in particular have that $x \in U_{x'}$ for all $x' \in J$. Thus $x \in U$.

(3) For any $x' \in J$ we have that $U \times U'_{x'} \subset U_{x'} \times U'_{x'} \subset W$. Thus we have that

$$\begin{aligned} U \times X' &= U \times \Big(\bigcup_{x' \in J} U'_{x'} \Big) \\ &= \bigcup_{x' \in J} \left(U \times U'_{x'} \right) \\ &\subset \bigcup_{x' \in J} W \\ &= W. \end{aligned}$$

$\square$

Remark 14.2. Proposition 14.1 is sometimes known as the *tube lemma*. It does not necessarily hold if X' is not compact. Let us illustrate this by an example.

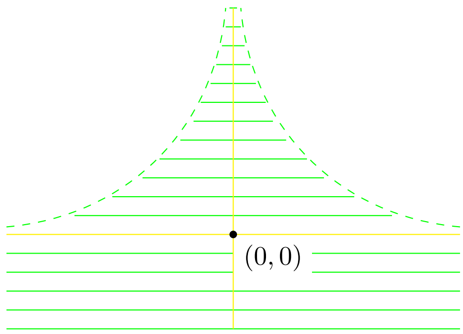
(1) Let $(X, \mathcal{O}_X) = (\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

(2) Let $(X', \mathcal{O}_{X'}) = (\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

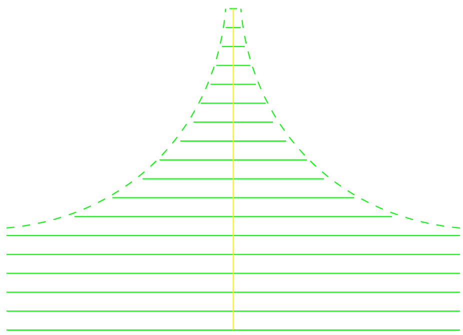
(3) Let $x = 0$.

(4) Let $W = \left\{(x, y) \in \mathbb{R}^2 \mid x \neq 0 \text{ and } y < \left|\frac{1}{x}\right|\right\} \cup \left\{(0, y) \mid y \in \mathbb{R}\right\}$.

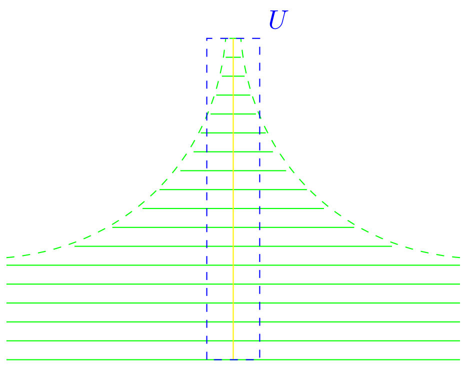
In the picture below W is the shaded green area. The two dashed green curves do not themselves belong to W.



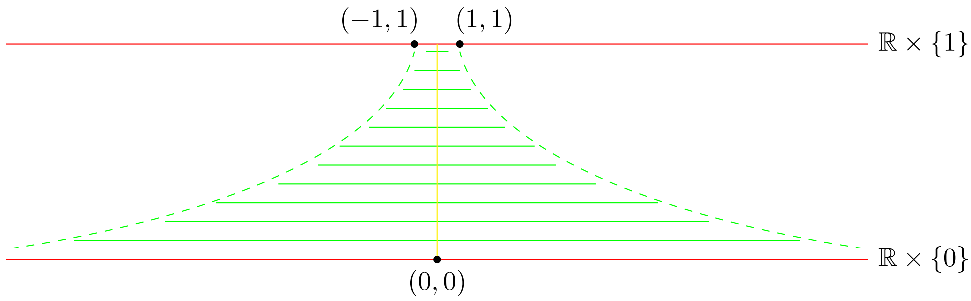
Then W is a neighbourhood of $\{x\} \times \mathbb{R} = \{0\} \times \mathbb{R}$ in $(X \times X', \mathcal{O}_{X \times X'}) = (\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. The set $\{0\} \times \mathbb{R}$, or in other words the y-axis, is depicted in yellow below.



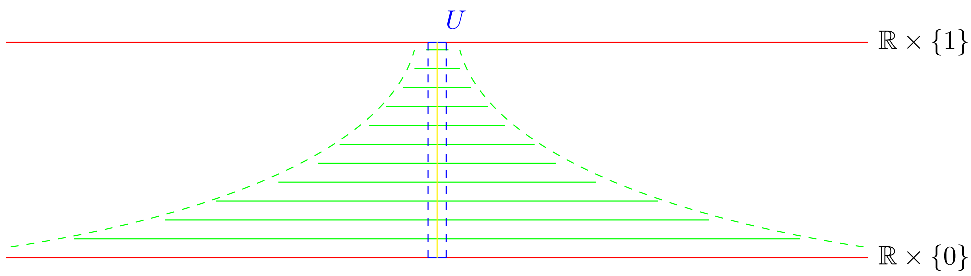
There is no neighbourhood U of x in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ such that $U \times \mathbb{R} \subset W$.



This is due to the 'infinitesimal narrowing' of W. Proposition 14.1 establishes that this kind of behaviour cannot occur if $(X', \mathcal{O}_{X'})$ is compact. For instance, suppose that we instead let $(X', \mathcal{O}_{X'}) = (I, \mathcal{O}_I)$. By Proposition 13.2 we have that $(I, \mathcal{O}_I)$ is compact. The restriction of W to $\mathbb{R} \times I$ is pictured below.



We can find a neighbourhood U of $\{0\} \times \mathbb{R}$ such that $U \subset W$.



Remark 14.3. The role of compactness in the proof of Proposition 14.1 is very similar to its role in the proof of Lemma 13.9, which was discussed in Remark 13.10. The key to the proof is observation (1), that if J is finite then $U \in \mathcal{O}_X$.

The idea of the proof of Proposition 14.1 is that since $(X', \mathcal{O}_{X'})$ is compact we can find a finite set of points $x' \in X'$ and a neighbourhood $U_{x'} \times U'_{x'} \subset W$ of (x, x') for each of these points x' such that $\{x\} \times X'$ is contained in the union of the sets $U_{x'} \times U'_{x'}$.

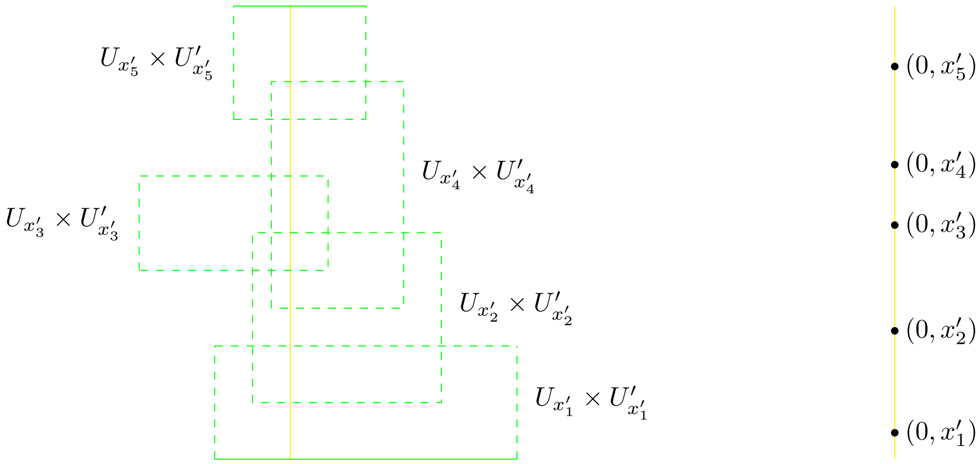
An example is depicted below when we have the following.

(1) $(X, \mathcal{O}_X) = (\mathbb{R}, \mathcal{O}_\mathbb{R})$,

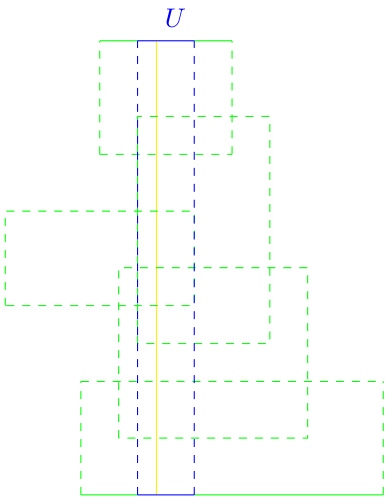
(2) $(X', \mathcal{O}_{X'}) = (I, \mathcal{O}_I)$,

(3) $x = 0$.

A set W is not drawn, but should be thought of as an open subset of $\mathbb{R} \times I$ which contains all the green rectangles. To avoid cluttering the picture, possible locations for the points $(0, x'_1), \ldots, (0, x'_5)$ are indicated seperately to the right.



The open set U in the proof of Proposition 14.1 is the intersection of all the neighbourhoods $U_{x'} \times U'_{x'}$.



Proposition 14.4. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be compact topological spaces. Then the topological space $(X \times Y, \mathcal{O}_{X \times Y})$ is compact.

Proof. Let $\{W_j\}_{j \in J}$ be an open covering of $X \times Y$. For any $x \in X$, let $\{x\} \times Y$ be equipped with its subspace topology $\mathcal{O}_{\{x\} \times Y}$ with respect to $(X \times Y, \mathcal{O}_{X \times Y})$. Then

$$\Big\{ W_j \cap \big(\{x\} \times Y \big) \Big\}_{j \in J}$$

is an open covering of $\{x\} \times Y$.

By Lemma 7.12 we have that $\big(\{x\} \times Y, \mathcal{O}_{\{x\} \times Y}\big)$ is homeomorphic to $(Y, \mathcal{O}_Y)$. Since Y is compact we have by Corollary 12.11 that $\big(\{x\} \times Y, \mathcal{O}_{\{x\} \times Y}\big)$ is compact. We deduce that there is a finite subset J_x of J such that

$$\Big\{ W_j \cap \big(\{x\} \times Y \big) \Big\}_{j \in J_x}$$

is an open covering of $\{x\} \times Y$.

Let $W_x = \bigcup_{j \in J_x} W_j$. Then

$$\begin{aligned}\{x\} \times Y \subset W_x &= \bigcup_{j \in J_x} \Big(W_j \cap \big(\{x\} \times Y \big) \Big) \\ &\subset \bigcup_{j \in J_x} W_j \\ &= W_x.\end{aligned}$$

Since Y is compact we deduce by Proposition 14.1 that there is a neighbourhood U_x of x in $(X, \mathcal{O}_X)$ such that $U_x \times Y \subset W_x$. We have that

$$\begin{aligned}X &= \bigcup_{x \in X} \{x\} \\ &\subset \bigcup_{x \in X} U_x.\end{aligned}$$

Hence $X = \bigcup_{x \in X} U_x$. Thus $\{U_x\}_{x \in X}$ is an open covering of X. Since $(X, \mathcal{O}_X)$ is compact we deduce that there is a finite subset K of X such that $\{U_x\}_{x \in K}$ is an open covering of X.

Let $J' = \bigcup_{x \in K} J_x$. We make the following observations.

(1) We have that J_x is finite for every $x \in X$. In particular, J_x is finite for every $x \in K$. Thus J' is finite.

(2) We have that

$$\begin{aligned}X \times Y &= \Big(\bigcup_{x \in K} U_x \Big) \times Y \\ &= \bigcup_{x \in K} \big(U_x \times Y \big) \\ &\subset \bigcup_{x \in K} W_x.\end{aligned}$$

Hence $X \times Y = \bigcup_{x \in K} W_x$. We deduce that

$$\begin{aligned}X \times Y &= \bigcup_{x \in K} W_x \\ &= \bigcup_{x \in K} \Big(\bigcup_{j \in J_x} W_j \Big) \\ &= \bigcup_{j \in J'} W_j.\end{aligned}$$

We conclude that $\{W_j\}_{j \in J'}$ is a finite subcovering of $\{W_j\}_{j \in J}$.

□

Examples 14.5. By Proposition 14.4 we have that $(I^2, \mathcal{O}_{I^2})$ is compact. Thus by Proposition 13.14 we have that all the topological spaces of Examples 3.9 (1) – (5) are compact.

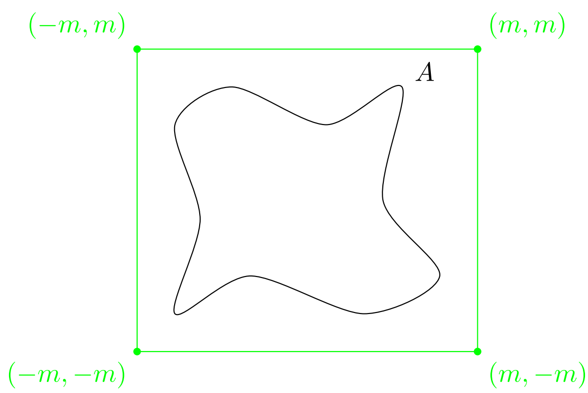
Moreover we have by Examples 4.10 (3) that $D^2 \cong I^2$. By Corollary 12.11 we deduce that $(D^2, \mathcal{O}_{D^2})$ is compact. Hence by Proposition 13.14 we have that the topological space $(S^2, \mathcal{O}_{S^2})$ constructed in Examples 3.9 (6) is compact.

14.2 Characterisation of compact subsets of $\mathbb{R}^n$

Terminology 14.6. Let A be a subset of $\mathbb{R}^n$. Then A is *bounded* if there is an $m \in \mathbb{N}$ such that

$$A \subset \underbrace{[-m, m] \times \ldots \times [-m, m]}_{n}$$

where $[-m, m]$ denotes the closed interval in $\mathbb{R}$ from $-m$ to m.



Remark 14.7. Roughly speaking a subset A of $\mathbb{R}^n$ is bounded if we can enclose it in a box. There are many ways to express this, all of which are equivalent to the definition of Terminology 14.6.

Notation 14.8. Let $m \in \mathbb{N}$. We denote the subset

$$\underbrace{[-m, m] \times \ldots \times [-m, m]}_{n}$$

of $\mathbb{R}^n$ by $[-m, m]^n$.

Let $(-m, m)$ denote the open interval from $-m$ to m in $\mathbb{R}$. We denote the subset

$$\underbrace{(-m, m) \times \ldots \times (-m, m)}_{n}$$

of $\mathbb{R}^n$ by $(-m, m)^n$.

Proposition 14.9. Let A be a subset of $\mathbb{R}^n$ equipped with its subspace topology $\mathcal{O}_A$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. Then $(A, \mathcal{O}_A)$ is compact if and only if A is bounded and closed in $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.

Proof. Suppose that A is bounded and closed in $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. Since A is bounded there is an $m \in \mathbb{N}$ such that $A \subset [-m, m]^n$. Let $\mathcal{O}_{[-m,m]^n}$ denote the subspace topology on $[-m, m]^n$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. We make the following observations.

(1) The subspace topology $\mathcal{O}_A$ on A with respect to $(X, \mathcal{O}_X)$ is equal to the subspace topology on A with respect to $\big([-m, m]^n, \mathcal{O}_{[-m,m]^n}\big)$.

(2) By Corollary 13.5 we have that $[-m, m]$ is a compact subset of $\mathbb{R}$. By Proposition 14.4 and induction we deduce that $[-m, m]^n$ is a compact subset of $\mathbb{R}^n$.

(3) Since A is closed in $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ we have by Question 1 (c) of Exercise Sheet 4 that A is closed in $\big([-m, m]^n, \mathcal{O}_{[-m,m]^n}\big)$.

We deduce by Proposition 13.7 that $(A, \mathcal{O}_A)$ is compact.

Suppose instead now that $(A, \mathcal{O}_A)$ is compact. By Examples 11.7 (1) and Proposition 11.11 we have that $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is Hausdorff. We deduce by Proposition 13.11 that A is closed.

The set $\big\{A \cap (-m, m)^n\big\}_{m \in \mathbb{N}}$ is an open covering of A. Thus since $(A, \mathcal{O}_A)$ is compact there is a finite subset J of $\mathbb{N}$ such that $\big\{A \cap (-m, m)^n\big\}_{m \in J}$ is an open covering of A. Let m' be the largest natural number which belongs to J. Then

$$\begin{aligned} A &= \bigcup_{m \in J} \big(A \cap (-m, m)^n\big) \\ &\subset \bigcup_{m \in J} (-m, m)^n \\ &\subset \bigcup_{m \in J} (-m', m')^n \\ &= (-m', m')^n. \end{aligned}$$

Thus A is bounded. $\square$

Corollary 14.10. Let $(X, \mathcal{O}_X)$ be a compact topological space. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. There is an $x \in X$ such that $f(x) = \inf f(X)$ and an $x' \in X$ such that $f(x') = \sup f(X)$. Equivalently we have that

$$f(x) \leq f(x'') \leq f(x')$$

for all $x'' \in X$.

Proof. Since $(X, \mathcal{O}_X)$ is compact we have by Proposition 12.10 that $f(X)$ is a compact subset of $\mathbb{R}$. By Proposition 14.9 we deduce that $f(X)$ is closed in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ and bounded.

Since $f(X)$ is bounded we have that $\sup f(X) \in \mathbb{R}$ and $\inf f(X) \in \mathbb{R}$. Thus by Question 5 (a) and (b) on Exercise Sheet 4 we have that $\sup f(X)$ and $\inf f(X)$ are limit points of $f(X)$ in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Since $f(X)$ is closed in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ we deduce by Proposition 5.7 that $\sup f(X)$ belongs to $f(X)$ and that $\inf f(X)$ belongs to $f(X)$. □

Remark 14.11. You will have met Corollary 14.10 when X is a closed interval in $\mathbb{R}$ in real analysis/calculus, where it is sometimes known as the 'extreme value theorem'!

14.3 Locally compact topological spaces

Definition 14.12. A topological space $(X, \mathcal{O}_X)$ is *locally compact* if for every $x \in X$ and every neighbourhood U of x in $(X, \mathcal{O}_X)$ there is a neighbourhood U' of x in $(X, \mathcal{O}_X)$ such that the following hold.

(1) The closure $\overline{U'}$ of U in $(X, \mathcal{O}_X)$ is a compact subset of X.

(2) We have that $\overline{U'} \subset U$.

Examples 14.13.

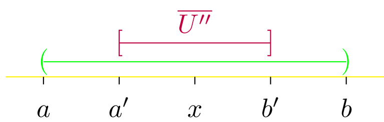
(1) The topological space $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally compact, whereas in Examples 12.9 (2) we saw that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not compact. Let us prove that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally compact.

Let $x \in \mathbb{R}$ and let U be a neighbourhood of x in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By definition of $\mathcal{O}_{\mathbb{R}}$ we have that

$$\{(a, b) \mid a, b \in \mathbb{R}\}$$

is a basis of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Question 3 (a) of Exercise Sheet 2 we deduce that there is an open interval (a, b) in $\mathbb{R}$ such that $x \in (a, b)$ and $(a, b) \subset U$.

Let $a' \in \mathbb{R}$ be such that $a < a' < x$. Let $b' \in \mathbb{R}$ be such that $x < b' < b$. Let $U' = (a', b')$. By Question 5 (c) of Exercise Sheet 4 we have that $\overline{U'} = [a', b']$. Here $\overline{U'}$ denotes the closure of U' in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. In particular we have that $\overline{U} \subset (a, b) \subset U$.



By Corollary 13.5 we have that $[a, b]$ is a compact subset of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Putting everything together we have that:

(i) $U' \in \mathcal{O}_{\mathbb{R}}$

(ii) $x \in U'$

(iii) $\overline{U'} \subset U$

(iv) $\overline{U'}$ is a compact subset of $\mathbb{R}$.

(2) Let X be a set and let $\mathcal{O}^{\text{disc}}$ be the discrete topology on X. Then $(X, \mathcal{O}^{\text{disc}})$ is locally compact. Let us prove this.

Let $x \in X$ and let U be a neighbourhood of x in $(X, \mathcal{O}^{\text{disc}})$. We make the following observations.

(i) $\{x\} \subset U$.

(ii) $\{x\}$ is open in $(X, \mathcal{O}^{\text{disc}})$.

(iii) $\{x\}$ is closed in $(X, \mathcal{O}^{\text{disc}})$ since $X \setminus \{x\}$ is open in $(X, \mathcal{O}^{\text{disc}})$. By Proposition 5.7 we deduce that $\overline{\{x\}} = \{x\}$.

Thus $(X, \mathcal{O}^{\text{disc}})$ is locally compact.

By contrast, if X is infinite then $(X, \mathcal{O}_X)$ is not compact. For

$$\big\{\{x\}\big\}_{x \in X}$$

is an open covering of X which if X is infinite has no finite subcovering.

(3) A product of locally compact topological spaces is locally compact. This is left as an exercise. Thus $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is locally compact.

(4) Let $(X, \mathcal{O}_X)$ be a locally compact topological space. Let U be an open subset of X equipped with its subspace topology $\mathcal{O}_U$ with respect to $(X, \mathcal{O}_X)$. Then $(U, \mathcal{O}_U)$ is locally compact. This is left as an exercise.

By (3) we conclude that any 'open blob' in $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is locally compact.

(5) Let $(X, \mathcal{O}_X)$ be a locally compact topological space. Let A be a closed subset of X equipped with its subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Then $(A, \mathcal{O}_A)$ is locally compact. This is left as an exercise.

Proposition 14.14. Let $(X, \mathcal{O}_X)$ be a compact Hausdorff topological space. Then $(X, \mathcal{O}_X)$ is locally compact.

Proof. Let $x \in X$ and let U be a neighbourhood of x in $(X, \mathcal{O}_X)$. Since U is open in $(X, \mathcal{O}_X)$ we have that $X \setminus U$ is closed in $(X, \mathcal{O}_X)$. Since $(X, \mathcal{O}_X)$ is compact we deduce by Proposition 13.7 that $X \setminus U$ is a compact subset of X.

Since $(X, \mathcal{O}_X)$ is Hausdorff we deduce by Lemma 13.9 that there are open subsets U' and U'' of X with the following properties:

(1) $X \setminus U \subset U'$,

(2) $x \in U''$,

(3) $U' \cap U'' = \emptyset$.

By (3) and Question 1 (f) of Exercise Sheet 4 we have that $U' \cap \overline{U''} = \emptyset$. Here $\overline{U''}$ is the closure of U'' in $(X, \mathcal{O}_X)$. We deduce by appeal to (1) that

$$\begin{aligned} \overline{U''} &\subset X \setminus U' \\ &\subset X \setminus (X \setminus U) \\ &= U. \end{aligned}$$

By Proposition 13.7 we have that $\overline{U''}$ is closed in $(X, \mathcal{O}_X)$. Since $(X, \mathcal{O}_X)$ is compact we deduce by Proposition 13.7 that $\overline{U''}$ is a compact subset of X. Putting everything together we have the following.

(1) $U'' \in \mathcal{O}_X$.

(2) $x \in U''$.

(3) $\overline{U''} \subset U$.

(4) $\overline{U''}$ is a compact subset of X.

Thus $(X, \mathcal{O}_X)$ is locally compact. □

Example 14.15. Let $n \geq 1$. We make the following observations.

(1) By Proposition 13.2, Proposition 14.4, and induction we have that $(I^n, \mathcal{O}_{I^n})$ is compact.

(2) By Examples 11.7 (1) we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. We deduce that $(I, \mathcal{O}_I)$ is Hausdorff by Proposition 11.10. Thus by Proposition 11.11 we have that $(I^n, \mathcal{O}_{I^n})$ is Hausdorff.

We conclude by Proposition 14.14 that $(I^n, \mathcal{O}_{I^n})$ is locally compact. We can also see this by appealing to Examples 14.13 (1), (3), and (5).

14.4 Topological spaces which are not locally compact

Example 14.16. Let $\mathbb{Q}$ be equipped with its subspace topology $\mathcal{O}_{\mathbb{Q}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not locally compact. Let us prove this.

Let $q \in \mathbb{Q}$ and let U be a neighbourhood of q in $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$. By definition of $\mathcal{O}_{\mathbb{Q}}$ there is a $U' \in \mathcal{O}_{\mathbb{R}}$ such that $U = \mathbb{Q} \cap U'$. By definition of $\mathcal{O}_{\mathbb{R}}$ we have that

$$\{(a, b) \mid a, b \in \mathbb{R}\}$$

is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. We deduce by Question 3 (a) of Exercise Sheet 2 that there are $a, b \in \mathbb{R}$ such that $q \in (a, b)$ and $(a, b) \subset U'$.

Suppose that $\overline{U}$ is a compact subset of $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$. The subspace topology on $\overline{U}$ with respect to $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is equal to the subspace topology on $\overline{U}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Thus we have that $\overline{U}$ is a compact subset of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

By Examples 11.7 (1) we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. We deduce by Proposition 13.11 that $\overline{U}$ is closed in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Since $(a, b) \subset U'$ we have that

$$\overline{\mathbb{Q} \cap (a, b)} \subset \overline{\mathbb{Q} \cap U'} = \overline{U},$$

where $\overline{\mathbb{Q} \cap (a, b)}$ denotes the closure of $\mathbb{Q} \cap (a, b)$ in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

By Question 5 (d) of Exercise Sheet 4 we have that $\overline{\mathbb{Q} \cap (a, b)} = [a, b]$. We deduce that $[a, b] \subset \overline{U}$. Since $[a, b]$ contains irrational numbers this contradicts the fact that $\overline{U} \subset \mathbb{Q}$.

We conclude that no neighbourhood of q in $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is compact. Hence $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not locally compact.