Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact person during the exam:
Richard Williamson, (735) 90154

MA3002 General Topology — Mock exam

Spring 2013

Answer four of the five problems. You may answer all five problems, in which case your four best answers will count.

Each problem is worth 25 marks. The apporoximate marks for each part are indicated in square brackets. Your grade will be determined by your mark and by a qualitative overall assessment of your answers.

Problem 1

a) Let $\mathcal{O}$ be the topology on $[-1,1] \times [-1,1]$ with basis

$$\Big\{ \big([-1,1] \times [-1,1]\big) \cap \big((a,b) \times (a',b')\big) \mid a, a', b, b' \in \mathbb{R},\ a < 0,\ a' < 0,\ b > 0,\ b' > 0 \Big\}.$$

Is the set $[-1,-\frac{1}{2}] \times [\frac{1}{2},1]$ is closed in I^2? Justify your answer. [5]

b) Is $(I^2, \mathcal{O})$ Hausdorff? Give a reason. [5]

c) Is $(I^2, \mathcal{O})$ connected? Justify your answer. [5]

d) Are the following statements true or false?

(i) The identity map

$$I^2 \longrightarrow I^2$$

is continuous if the copy of I^2 in the source is equipped with the topology $\mathcal{O}$ and the copy of I^2 in the target is equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

(ii) The identity map

$$I^2 \longrightarrow I^2$$

is continuous if the copy of I^2 in the source is equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ and the copy of I^2 in the target is equipped with the topology $\mathcal{O}$.

Justify your answer in each case. [5]

e) Is $(I^2, \mathcal{O})$ compact? Justify your answer.

You may quote without proof any results from the course. [5]

Problem 2

a) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces.

Let

$$X \xrightarrow{f} Y$$

be a constant map. Recall that this means that there is a $y \in Y$ such that $f(x) = y$ for all $x \in X$.

Prove that f is continuous. [5]

b) Can there exist a continuous map

$$\mathbb{R} \xrightarrow{f} \{0, 1\}$$

which is not constant, where $\{0, 1\}$ is equipped with the discrete topology? Give a reason.

You may quote without proof any results from the course. [5]

c) Find an example of a pair of finite spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ and a map

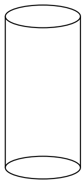
$$X \longrightarrow Y$$

which is a continuous bijection but not a homeomorphism. [5]

d) Let

$$S^1 \times I = \left\{(x, y, z) \in \mathbb{R}^3 \mid \|(x, y)\| = 1 \text{ and } 0 \leq z \leq 1\right\}$$

be equipped with its subspace topology $\mathcal{O}_{S^1 \times I}$ with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$.



Prove that $(S^1 \times I, \mathcal{O}_{S^1 \times I})$ is Hausdorff.

You may not assume without proof any results from the lectures. [5]

e) Let $\sim$ be the equivalence relation on I^2 defined by $(0,t) \sim (1,t)$. Let $I^2/\sim$ be equipped with its quotient topology $\mathcal{O}_{I^2/\sim}$.

Can there exist a continuous bijection

$$I^2/\sim \longrightarrow S^1 \times I$$

which is not a homeomorphism?

You may quote without proof any results from the course. [5]

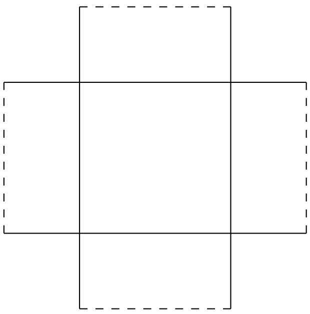
Problem 3

a) Let X be the union of

$$\{(x,y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2 \text{ and } -1 < y < 1\}$$

and

$$\{(x,y) \in \mathbb{R}^2 \mid -1 < x < 1 \text{ and } -2 \leq y \leq 2\}$$



Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $\mathcal{O}_X$.

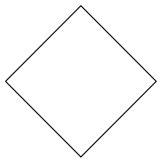
Find an open covering of $(X, \mathcal{O}_X)$ which does not admit a finite subcovering. [4]

b) Let Y be the union of the sets

$$\{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 0 \text{ and } -(x+1) \leq y \leq x+1\}$$

and

$$\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ and } x-1 \leq y \leq -(x-1)\}$$



Let $\mathcal{O}_Y$ denote the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Is $(X, \mathcal{O}_X)$ homeomorphic to $(Y, \mathcal{O}_Y)$? Justify your answer.

You may quote without proof any results from the course. [7]

c) Let $(X, \mathcal{O}_X)$ be a topological space. Let $\sim$ be an equivalence relation on $(X, \mathcal{O}_X)$.

Prove that the number of connected components of $(X/\sim, \mathcal{O}_{X/\sim})$ is less than or equal to the number of connected components of $(X, \mathcal{O}_X)$.

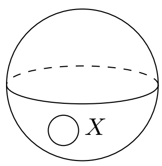
Hint: consider $\pi(A)$ for a connected subset A of X, where

$$X \xrightarrow{\pi} X/\sim$$

is the quotient map $x \mapsto [x]$.

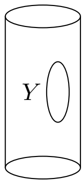
You may quote without proof any results from the course. [8]

d) Let X be a subset of S^2, and let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(S^2, \mathcal{O}_{S^2})$.



The Jordan curve theorem states that if X is a closed subset of S^2 which is homeomorphic to $(S^1, \mathcal{O}_{S^1})$ then $S^2 \setminus X$ equipped with its subspace topology with respect to $(S^2, \mathcal{O}_{S^2})$ has exactly two connected components.

Let Y be a closed subset of the cylinder $S^1 \times I$ which is homeomorphic to $(S^1, \mathcal{O}_{S^1})$.



Outline an argument to deduce from part (d) and the Jordan curve theorem that $(S^1 \times I) \setminus Y$ equipped with its subspace topology with respect to $\mathcal{O}_{S^1 \times I}$ has at least two connected components.

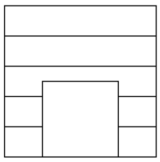
You do not need to give a detailed proof of any steps in your argument. [6]

Problem 4

a) Let

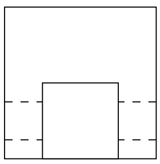
$$X = I^2 \setminus \left\{(x, y) \mid \frac{1}{4} < x < \frac{3}{4} \text{ and } 0 < y < \frac{1}{2}\right\}$$

be equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Let

$$U = \left\{(x, y) \in X \mid 0 \leq x \leq \frac{1}{4} \text{ and } \frac{1}{8} < y < \frac{3}{8}\right\}.$$



Explain why U is open in $(X, \mathcal{O}_X)$. [6]

b) What is the boundary of X in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$? [5]

c) Let $\sim$ be the equivalence relation on X defined by $(x, 0) \sim (x, 1)$ for all $x \in I$ and $(0, y) \sim (1, y)$ for all $y \in I$. Let $X/\sim$ be equipped with its quotient topology $\mathcal{O}_{X/\sim}$.

Draw a picture of a subset Y of $\mathbb{R}^3$ for which $(Y, \mathcal{O}_Y)$ is homeomorphic to $(X/\sim, \mathcal{O}_{X/\sim})$, where $\mathcal{O}_Y$ is the subspace topology on Y with respect to $\mathbb{R}^3$.

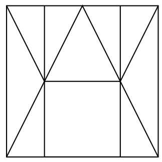
No justification is necessary.

Add a caption to your picture if necessary to indicate any important aspects. [2]

d) Prove that $(X/\sim, \mathcal{O}_{X/\sim})$ is connected.

You may quote without proof any results from the course which you require. [7]

e) A Δ-complex structure on $(X/\sim, \mathcal{O}_{X/\sim})$ is indicated below.



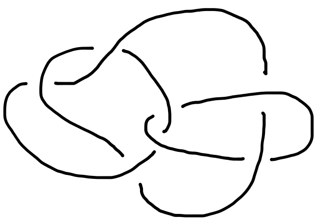
In this picture every line is to be understood to be a 1-simplex and every triangle is to be understood to be the boundary of a 2-simplex.

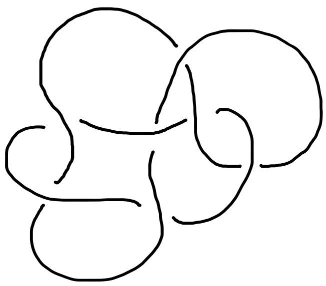
Calculate the Euler characteristic of $(X/\sim, \mathcal{O}_{X/\sim})$. [5]

Problem 5

a) Define a link. [3]

b) Calculate the linking numbers of the two links below. [6]

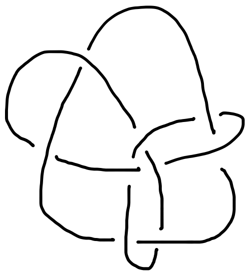




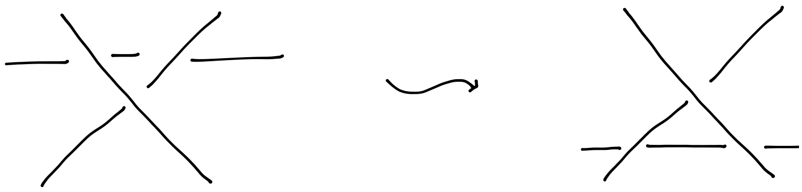
c) Are the two links above isotopic?

State precisely any result from the course which you require. [3]

d) Find a 5-colouring of the following knot. [8]



e) An (R3) Reidemeister move is pictured below.



Let $(L, \mathcal{O}_L)$ be a link, and let $(L', \mathcal{O}_{L'})$ be a link which is obtained from $(L, \mathcal{O}_L)$ by an (R3) Reidemeister move.

Prove that $(L, \mathcal{O}_L)$ is m-colourable for an integer m if and only if $(L', \mathcal{O}_{L'})$ is m-colourable. [5]