MA3002 Generell Topologi — Revision Checklist

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Contents

1 Overview

Here is a synopsis of the structure of the exam.

(1) The exam problems will test your understanding of the ideas introduced in the course primarily by asking you to consider various examples of topological spaces and answer questions about them.

(2) There will be five problems, of which you must answer four. You may answer five, in which case your mark will consist of your best four answers. Your grade will be determined by your mark and a qualitative assessment of your answers overall.

(3) Each problem will be divided into several smaller parts.

(4) The parts of each problem will mix all the topics of the course. It is essential that you do not restrict to revising just some topics.

(5) The only exception to this is that there will be one question which is taken only from the last two topics — knot theory or surfaces or both.

(6) Altogether about $1\frac{2}{3}$ of the questions will be taken from the last two topics of the course, namely knot theory and surfaces.

This revision checklist is structured as follows.

(1) Everything listed as 'Must know' has a high probability of being asked on the exam. If you have a good and thorough understanding of everything here, you should at least be at the level of a B. I can see that you all have the talent to achieve this — just keep working hard until the exam!

The revision classes will focus on this material. If there's anything that you do not understand, you must get in touch with me by email or otherwise. I will be happy to help.

(2) Everything listed as 'Very important' is potentially examinable. There will be one or two parts altogether — at most one per problem — which ask about this material. If you have a good understanding of everything here, you should certainly be at the level of an A.

I strongly advise all of you not to ignore this material. Start out by finding proofs on each topic which you understand, and make an effort to remember the idea well enough to be able to give the proof on the exam — almost all the proofs in the 'Fundamentals' section should be short and simple, for example. Then work up to the longer and more difficult proofs.

This should moreover help you to improve your understanding of the material listed as 'Must know'.

Again just let me know if you are having difficulties, and I will be happy to help.

(3) I will not ask on the exam anything which requires knowledge of something listed as 'Non-examinable'. Nevertheless you may find it very helpful to read over this, to obtain a deeper understanding of the examinable material.

2 Fundamentals

2.1 Must know

The most important thing here is to know the definitions inside out and be able to apply your knowledge in examples. No proofs needed yet.

(1) Definition of a topological space (Definition 1.1). Understand Examples 1.7 and the warning after Remark 1.26.

(2) Definition of subspace topology (Proposition 1.33). Understand Example 1.35.

(3) Definition of product topology (Proposition 1.36). Understand Examples 1.38.

(4) Definition of a basis (Definition 2.1). Definition of the standard topology on $\mathbb{R}$ (Definition 2.5). Understand Examples 2.9.

(5) Definition of a continuous map (Definition 2.9). Understand Examples 2.13.

(6) Definition of a quotient topology (Proposition 3.5). Observation 3.7. Understand Examples 3.9. Remember the equivalence relation on I defining a circle and the equivalence relations on I^2 defining a cylinder, torus, Möbius band, Klein bottle, and S^2.

(7) Definition of a homeomorphism (Definition 3.14) and of homeomorphic topological spaces (Definition 4.5). Understand Examples 4.7 and 4.10. Be aware that a continuous bijection need not be a homeomorphism, and be able to quote an example.

(8) Definition of a neighbourhood (Definition 4.13). Understand 4.13.

(9) Definition of a limit point (Definition 4.14).

(10) Definition of closure (Definition 4.15) and of a dense subset (Definition 5.4). Understand Examples 5.6.

(11) A subset A is closed if and only if $\overline{A} = A$. (Proposition 5.7).

(12) Definition of boundary (Definition 5.12). Understand Examples 5.16.

(13) Definition of coproduct topology (Proposition 5.19). Observation 5.21 and Examples 5.22.

2.2 Very important

Improve your understanding by working through the following until you are able to recall the statements and give the proofs yourself. They are mostly quite simple and short.

(1) Proof of Proposition 1.33.

(2) Proof of Proposition 1.36.

(3) Statement of Proposition 2.2.

(4) Statement and proof of Proposition 2.15.

(5) Statement and proof of Proposition 2.16.

(6) Statement and proof of Proposition 2.18.

(7) Statement and proof of Proposition 3.2.

(8) Proof of Proposition 3.5.

(9) Statement and proof of Proposition 4.4.

(10) Proof of Proposition 5.7. This one is a little harder — make a really good effort on it!

(11) Statement and proof of Proposition 5.9.

2.3 Non-examinable

(1) Proof of Proposition 2.2. It would nevertheless be a good idea to check that you understand the idea of the proof.

(2) All of §1.2 except the warning after Remark 1.26. Since this section discusses a motivation for the axioms in the definition of a topological space, it would be a good idea to read over it.

3 Connectedness

3.1 Must know

No proofs at this stage, except as indicated.

(1) Definition of a connected topological space (Definition 6.2). Understand Examples 6.6.

(2) $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected. (Proposition 6.9).

(3) Characterisation of connected subsets of $\mathbb{R}$. (Proposition 7.9).

(4) Products and quotients of connected topological spaces are connected. Understand Examples 7.14 and 7.16.

(5) Definition of a connected component (Definition 8.4). Understand Examples 8.10.

(6) Distinguishing topological spaces by means of connectedness.

   (i) Corollary 8.2 and its proof. You must be able to state Proposition 8.1 and pinpoint where it is used.

   (ii) Examples 8.16 and 9.1. You must be able to give a full careful argument here, being able to state Proposition 8.1 and Proposition 8.14 and be able to pinpoint where you appeal to them.

   (iii) Statement and proof of Proposition 9.3. You do not need to know the proof of Lemma 9.2, but must then be able to prove that $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n \setminus \{x\}})$ is path connected and hence connected.

(7) Definition of a locally connected topological space. Understand Examples 9.14. Be aware that a connected topological space is not necessarily locally connected (Example 10.1).

(8) Definition of a path connected topological space (Definitions 10.3 and 10.14). Understand Examples 10.15. Be able to prove that topological spaces are path connected, and hence by (9) connected.

(9) A path connected topological space is connected. (Proposition 10.17). Be aware that a connected topological space is not necessarily path connected (Remark 10.18).

3.2 Very important

(1) Equivalent characterisations of connectedness. Propositions 6.3 and 6.5 and their proofs.

(2) Statement that the closure of a connected subset is closed (Corollary 7.6). Proof not needed.

(3) Continuous image of a connected topological space is connected. (Proposition 7.1). Corollary 7.3. Both statements and proofs.

(4) Closure of a connected subset is closed. (Corollary 7.6).

(5) Statement and proof of the generalised intermediate value theorem (Corollary 7.10).

(6) Proof that a quotient of a connected topological space is connected (Proposition 7.15).

(7) Proof of Proposition 8.1.

(8) Statement and proof of Proposition 8.5. Corollary 8.6.

(9) Statement and proof of Lemma 8.13.

(10) A connected component is closed (Proposition 9.8 and its proof). Be able to quote an example to show that a connected component need not be open (Remark 9.9).

(11) Statement of Proposition 9.13.

(12) The details of Example 10.1.

(13) Construction of reverse paths (statement and proof of Proposition 10.8), composite paths (statement and proof of Proposition 10.11), and constant paths (statement and proof of Proposition 10.13).

These proofs and the proof needed in (14) are quite geometric, and you may find them easier than some of the others.

(14) Proposition 10.16 and its proof. Although not particularly significant from a theoretical point of view, it can be very helpful when demonstrating that a particular topological space is path connected.

(15) Proof of Proposition 10.17.

3.3 Non-examinable

(1) Proof that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected. Lemma 6.7 and Proposition 6.9.

(2) Statement and proof of Proposition 7.4. Proof of Corollary 7.6.

(3) Lemma 7.7, Lemma 7.8, and the proof of Proposition 7.9.

(4) Proof of Proposition 7.13.

(5) Proposition 8.9. Be aware of its statement though.

(6) Proof of Proposition 8.14. Observation 8.15.

(7) Example 9.7. A very good idea to keep it in mind though.

(8) Proposition 9.11, Lemma 9.12, and the proof of Proposition 9.13.

(9) Section 10.3.

4 Separation Axioms

4.1 Must know

All the definitions here are from Definition 10.22 and Definition 11.1.

(1) Definition of a T0 topological space. Understand Examples 11.2.

(2) Definition of a T1 topological space. Understand Observation 11.3 and Examples 11.4.

(3) Definition of a Hausdorff topological space. Understand Observation 11.6 and Examples 11.7. $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff.

(4) Be aware that a quotient of a Hausdorff topological space is not necessarily Hausdorff.

4.2 Very important

(1) Statement and proof of Proposition 11.5. The entire proof would not be asked on an exam, but a part of it might be. The most important part is (1) $\Rightarrow$ (2), that singleton sets in a T1 topological space (and hence a Hausdorff topological space) are closed.

(2) Proposition 11.9, that a topological space is Hausdorff if and only if its diagonal is closed, and its proof.

(3) Statements and proofs in 11.10 – 11.14. The proofs here are short and simple.

4.3 Non-examinable

(1) Separation axioms T3, T3$\frac{1}{2}$, T4, and T6. Regular, completely regular, normal, and completely normal topological spaces.

(2) All of section 12.1. Working through the details of Examples 12.1 and 12.5 may nevertheless be very beneficial to you. Understanding the proof of Proposition 12.3 would be a good way to help with the revision of other material you have learnt.

5 Compactness and local compactness

5.1 Must know

No proofs at this stage.

(1) Definition of a compact topological space.

(2) $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not compact. An example of an open covering of $\mathbb{R}$ with no finite subcovering to illustrate this. Examples 12.9.

(3) $(I, \mathcal{O}_I)$ is compact. Remark 13.4. Corollary 13.5.

(4) Products and quotients of compact topological spaces are compact. Examples 14.5.

(5) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Use this to recognise compact subsets of $\mathbb{R}^n$.

(6) Definition of a locally compact topological space.

(7) $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally compact. Examples 14.13 (1) and (2).

5.2 Very important

(1) Remember that a closed subset of a compact topological space is compact. No proof needed at this stage.

(2) Remember that a compact subset of a Hausdorff topological space is closed. Be able to prove it assuming Lemma 13.9.

(3) Remember the statement of and be able to prove Proposition 12.10.

(4) Be able to prove that a quotient of a compact topological space is compact given (3).

(5) Remember the statement of Proposition 13.13. Be able to prove it assuming (1), (2), and (3).

(6) Proof of Corollary 13.5.

(7) Understand Example 13.15 and be able to reproduce the proof.

(8) Understand Remark 14.2. Be able to recognise examples where the tube lemma does not hold.

(9) Proof of Corollary 14.10.

(10) Remember the statement of Proposition 14.14. Be able to prove it assuming Lemma 13.9. Example 14.15.

(11) Be able to prove Lemma 13.9 — perhaps slightly harder, so make a really good effort to be able to do this! Keep in mind Remark 13.10.

(11) Know that $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not locally compact. Be able to prove it — you may assume that the closure of $\mathbb{Q} \cap [a, b]$ in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is $[a, b]$.

5.3 Non-examinable

(1) All of Lectures 15–16 except for an example of a topological space which is not locally compact at the beginning of Lecture 15.

(2) Lemma 13.1 and the proof of Proposition 13.2.

(3) Proof of Proposition 13.7.

(4) Proof of Proposition 14.1.

(5) Proof of Proposition 14.4.

(6) Remark 13.12 (2).

6 Knot theory

6.1 Must know

There are two main themes to this part of the course.

(1) To be able to prove that a given pair of knots or links are not isotopic via knot invariants. For this you must be able to calculate the three knot invariants we have seen — linking number, m-colourability, and the Jones polynomial — in practise.

(2) To be able to show that these are indeed knot invariants by showing that they are unchanged under the Reidemeister moves.

Here is a more detailed list.

(1) Definition of a knot and of a link.

(2) Definition of an isotopy of knots and links.

(3) Be able to use the three Reidemeister moves in practise.

(4) Statement that two knots are isotopic if and only if the link diagram of one can be obtained from the link diagram of the other via the Reidemeister moves.

(5) Definition of linking number. Be able to calculate it.

(6) Proof that linking number is a knot invariant, namely that it is unchanged by the three Reidemeister moves).

(7) Definition of writhe. Be aware of the difference between the writhe and linking number. Be able to calculate it. Proof that it is unchanged under the $(R2)$ and $(R3)$ moves, but not under $(R1)$. Be able to give a counterexample in the latter case.

(8) Definition of m-colouring. Be able to show that a given link is m-colourable. Be able to prove that a link is only m-colourable for certain m, and thus to distinguish links via colourability.

(9) Proof that m-colouring is a knot invariant, namely that it is unchanged by the three Reidemeister moves.

(11) Be able to calculate the Jones polynomial of an oriented link using the skein relations. You do not need to remember the skein relations.

(12) Statement that if a knot is isotopic to its mirror image then its Jones polynomial is palindromic.

6.2 Very important

(1) Definition of the bracket polynomial. Be able to calculate it.

(2) Definition of the Jones polynomial assuming the bracket polynomial.

(3) Be able to calculate the Jones polynomial using the bracket polynomial and writhe.

(4) Proof that the Jones polynomial is a knot invariant, namely that it is unchanged under the three Reidemeister moves.

(5) Proof that the skein relations hold. You do not need to remember the statement, but should be able to prove it given a reminder of the statement.

(6) Statement and proof that the Jones polynomial of a knot does not depend on the choice of orientation.

(7) Proof that if a knot is isotopic to its mirror image then its Jones polynomial is palindromic.

6.3 Non-examinable

(1) Definition of a link diagram, and the definition of the Reidemeister moves using it.

7 Surfaces

No proofs required.

7.1 Must know

(1) Definition of a surface. Be able to recognise surfaces. Especially focus on the 'locally homeomorphic to a disc' condition.

(2) Definition of the Euler characteristic of a Δ-complex. Be able to calculate it. Remember the statement that the Euler characteristic does not depend on the choice of Δ-complex structure. In particular, remember that $\chi(S^2) = 2$.

(3) Be able to decide whether or not a Δ-complex is a simplicial complex.

(4) Statement of the classification of surfaces. Be able to explain the definition of a handlebody H_n and a crosscap C_n.

(5) Be able to calculate Euler characteristic using by its 'motivic' property. Know that glueing on a handle decreases the Euler characteristic by 2, and glueing on a Möbius band decreases the Euler characteristic by 1. Be able to justify this.

(6 Be able to apply successive surgeries to pass from a given surface to S^2. Be able to use this to decide which surface in the statement of the classification a given surface is homeomorphic to. Also be able to use this to calculate the Euler characteristic of a given surface.

7.2 Non-examinable

Everything else.