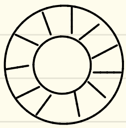
MA3002 General Topology - Revision Questions 1

Compact and locally compact topological spaces

Question 1

Which of the following are compact? Give a reason why or why not. You may refer to any results from the lectures.

a) Annulus, $A_k = \{(x,y) \in \mathbb{R}^2 \mid k \leq ||(x,y)|| \leq 1\}$ equipped with its subspace topology $\mathcal{O}_{A_k}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



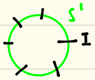
Solution: Compact. At least two possible reasons:

(1) A_k is closed and bounded. A closed and bounded subset of $\mathbb{R}^2$ is compact.

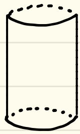
(2) $(A_k, \mathcal{O}_{A_k})$ is homeomorphic to $(S^1 \times I, \mathcal{O}_{S^1 \times I})$. We have that $(S^1 \times I, \mathcal{O}_{S^1 \times I})$ is compact since both $(S^1, \mathcal{O}_{S^1})$ and $(I, \mathcal{O}_I)$ are compact, and a product of compact topological spaces is compact. Moreover a topological space which is homeomorphic to a compact topological space is compact. Hence $(A_k, \mathcal{O}_{A_k})$ is compact.

Two ways to geometrically see that $(A_k, \mathcal{O}_{A_k})$ is homeomorphic to $(S^1 \times I, \mathcal{O}_{S^1 \times I})$ are:

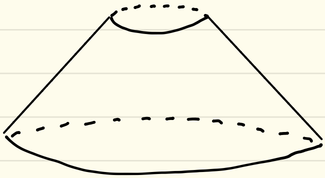
i) Look at $S^1 \times I$ as a 'ring' of copies of I.



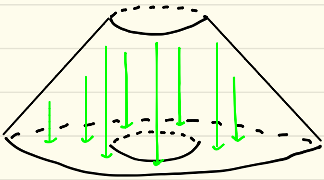
ii) Look at $S^1 \times I$ as a cylinder.



Stretch it a bit to look like a 'lamp shade' as follows.

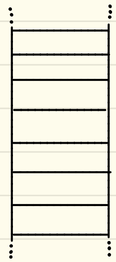


Then project down.



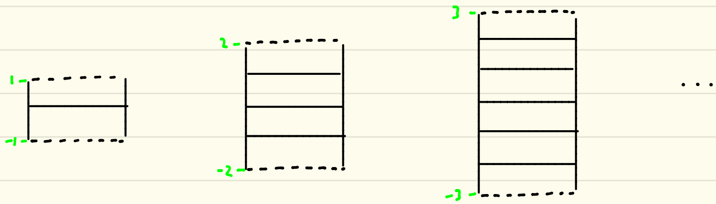
If I were to ask you to rigorously prove that (A_n, ∂_{A_n}) is homeomorphic to $(S^1 \times I, \partial_{S^1 \times I})$, of course these pictures are not enough! Here though this was not the point of the question, and it is enough to assert that $(A_n, \partial A_n)$ is homeomorphic to $(S^1 \times I, \partial_{S^1 \times I})$.

b) $X = \{ (x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1 \}$ equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



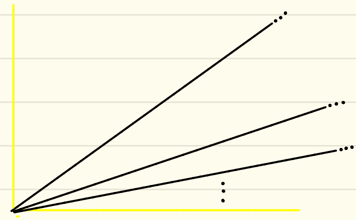
Solution: Not compact. A subset of $\mathbb{R}^2$ is compact if and only if it is bounded. X is not bounded. [Aside: X is closed!]

Alternatively, you could give an example of an open covering of X which does not admit a finite subcovering. For instance $\bigcup_{n \in \mathbb{N}} \{ (x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -n < y < n \}$.



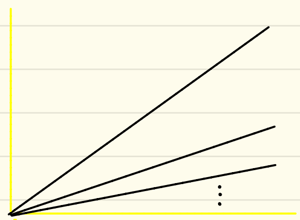
c) $X = \bigcup_{n \in \mathbb{N}} L_n$, where $L_n = \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0$ and $y = \frac{1}{n} x \}$ is the line

of gradient $\frac{1}{n}$ in $\mathbb{R}^2$ which begins at $(0,0)$, equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



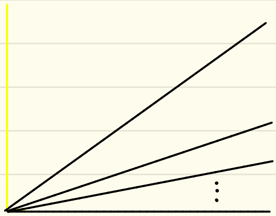
Solution: Not compact. A subset of $\mathbb{R}^2$ is compact if and only if it is closed and bounded. X is neither closed nor bounded.

Instead let $X' := \bigcup_{n \in \mathbb{N}} L_n'$, where $L_n' = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ y = \frac{1}{n} x\}$ is the line of gradient $\frac{1}{n}$ in $\mathbb{R}^2$ of length 1 which begins at $(0,0)$.

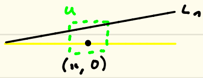


Then X' is still not a compact subset of $\mathbb{R}^2$: it is now bounded, but it is still not closed.

Let $\overline{X'}$ denote the closure of X' in $(\mathbb{R}^2, \mathcal{D}_{\mathbb{R}^2})$. This is $X' \cup \{(x,0) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\}$.



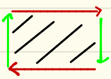
Indeed for every neighbourhood U of a point $(x,0)$ in $\mathbb{R}^2$ with $0 \leq x \leq 1$ we can always find an n such that the line of gradient $1/n$ passes through U.



Thus $(x,0)$ is a limit point of X'.

$\overline{X'}$ is both closed and bounded, and thus is a compact subset of $\mathbb{R}^2$.

d) The projective plane $P^2(\mathbb{R})$. This is the quotient $I^2/\sim$ equipped with the quotient topology $\mathcal{O}_{I^2/\sim}$, where $\sim$ is the equivalence relation on I^2 generated by $(0,t) \sim (1,1-t)$ and $(t,0) \sim (1-t,1)$ for all $t \in I$.



Solution: Compact. We have that $(I, \mathcal{O}_I)$ is compact. Hence $(I^2, \mathcal{O}_{I^2})$ is compact since a product of compact topological spaces is compact. Thus $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ is compact since a quotient of a compact topological space is compact.

e) $I \times [0,\infty)$, equipped with the product topology $\mathcal{O}_{I \times [0,\infty)}$.

Solution: Not compact. There are at least three ways to see this.

1) For any product $(X \times Y, \mathcal{O}_{X \times Y})$ of topological spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, we have that $(X \times Y, \mathcal{O}_{X \times Y})$ is compact if and only if both $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are compact. Since $[0,\infty)$ is not compact, neither is $I \times [0,\infty)$.

More explicitly, we can argue that if $I \times [0,\infty)$ is compact then $[0,\infty)$ is compact (and hence $I \times [0,\infty)$ is not compact) as follows.

Given a compact topological space $(X, \mathcal{O}_X)$, a topological space $(Y, \mathcal{O}_Y)$, and a continuous map $X \xrightarrow{f} Y$, we have that $f(X)$ equipped with its subspace topology with respect to $(Y, \mathcal{O}_Y)$ is compact.

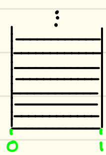
We can use this here as follows: the projection map

$$I \times [0,\infty) \xrightarrow{p_{[0,\infty)}} [0,\infty)$$

given by $(t,x) \mapsto x$ is surjective, i.e., $p_{[0,\infty)}\big(I \times [0,\infty)\big) = [0,\infty)$. Thus if $I \times [0,\infty)$ were compact we would have that $[0,\infty)$ is compact. Contradiction!

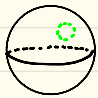
2) Let $\{U_j\}_{j \in J}$ be an open covering of $[0,\infty)$ which does not admit a finite subcovering, such as $\{[0,n)\}_{n \in \mathbb{N}}$. Then $\{I \times [0,n)\}_{n \in \mathbb{N}}$ is an open covering of $I \times [0,\infty)$ which does not admit a finite subcovering.

3) Regard $I \times [0,\infty)$ as a subset of $\mathbb{R}^2$.

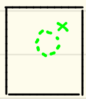


Then $I \times [0, \infty)$ is not bounded, hence not compact.

f) S^2 with an open disc removed.



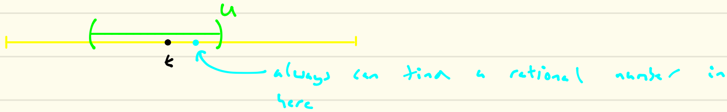
Solution: Compact. As in the lectures, let us view S^2 as $I^2/\sim$, where $\sim$ is the equivalence relation defined by $x \sim y$ if x and y both belong to ∂I^2. Then we may view S^2 with an open disc removed as $(I^2 \setminus X)/\sim$, where X is an open disc in I^2.



We have that $(I, \mathcal{O}_I)$ is compact, hence $(I^2, \mathcal{O}_{I^2})$ is compact since a product of compact topological spaces is compact. Since X is open in I^2, we have that $I^2 \setminus X$ is closed in I^2. A closed subset of a compact topological space is compact. Hence $I^2 \setminus X$ is compact. Since a quotient of a compact topological space is compact, we deduce that S^2 with an open disc removed is compact.

g) $\mathbb{Q} \cap I$, equipped with its subspace topology $\mathcal{O}_{\mathbb{Q} \cap I}$ with respect to $(I, \mathcal{O}_I)$.

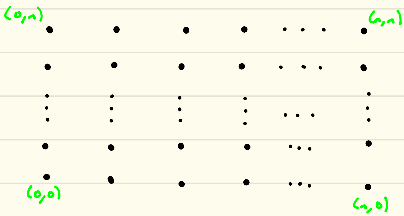
Solution: Not compact. The topology $\mathcal{O}_{\mathbb{Q} \cap I}$ is equal to the subspace topology on $\mathbb{Q} \cap I$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Hence $\mathbb{Q} \cap I$ is compact if and only if it is closed and bounded. It is bounded, but is not closed: $\overline{\mathbb{Q} \cap I} = I$, since for any $t \in I$ and any neighbourhood U of t in $\mathbb{R}$ we can always find a rational number which belongs to U, and thus t is a limit point of $\mathbb{Q}$ in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



h) $T^2 \times k^2 \times M^2$.

Solution: Compact. Each of T^2, k^2, and M^2 is compact since each is a quotient of I^2, which is compact, and any quotient of a compact topological space is compact. Since a product of compact topological spaces is compact we deduce that $T^2 \times k^2 \times M^2$ is compact.

i) $\mathbb{N}_{\leq n} \times \mathbb{N}_{\leq n}$ equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$, where $\mathbb{N}_{\leq n} = \{0, 1, 2, \ldots, n\}$.



Solution: Compact. There are at least two ways to see this.

1) $\mathbb{N}_{\leq n} \times \mathbb{N}_{\leq n}$ is finite. Any topological space $(X, \mathcal{O}_X)$ where X is finite is compact.

2) $\mathbb{N}_{\leq n} \times \mathbb{N}_{\leq n}$ is closed and bounded as a subset of $\mathbb{R}^2$.

The next two parts are harder. The second is non-examinable, and the first would be asked in a different context to the previous examples - it would be worth more marks.

i) $(I, \mathcal{O}')$, where $\mathcal{O}'$ is the subspace topology on I with respect to the topology $\mathcal{O}$ on $\mathbb{R}$ of Example 12.5 in the lecture notes, namely the topology with sub-basis given by $\{(a,b) \mid a, b \in \mathbb{R}\} \cup \{(a,b) \setminus \Sigma \mid a, b \in \mathbb{R}\}$, where $\Sigma = \{1/n \mid n \in \mathbb{N}\}$.

Solution: Not compact. The set

This is the set with just the single element $[0,1] \setminus \Sigma$.

$$\{(1/n, 1]\}_{n \geq 2,\, n \in \mathbb{N}} \cup \{[0,1] \setminus \Sigma\}$$

is an open covering of $(I, \mathcal{O}')$. It does not admit a finite subcovering, since if we have only finitely many of the sets $(1/n, 1]$, say $\{(1/n, 1]\}_{n \in J}$ for a finite set J, then

$$\left(\bigcup_{n \in J} (1/n, 1]\right) \cup \left([0,1] \setminus \Sigma\right)$$

does not contain $1/m$ for any $m \geq \max\{n \mid n \in J\}$.

k) $O(n)$, the set of $n \times n$ real matrices satisfying $A \cdot A^t = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, equipped

with its subspace topology $\mathcal{O}_{O(n)}$ with respect to $(\mathbb{R}^{n^2}, \mathcal{O}_{\mathbb{R}^{n^2}})$. Here we view a

matrix $\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$ as $(a_{11}, \ldots, a_{n1}, a_{12}, a_{22}, \ldots, a_{n2}, \ldots, a_{1n}, a_{2n}, \ldots, a_{nn}) \in \mathbb{R}^{n^2}$.

Also A^t denotes the transpose of A, i.e., given $A = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$

we have that $A^t = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$.

Solution: Compact. We prove that $O(n)$ is closed and bounded in $(\mathbb{R}^{n^2}, \mathcal{O}_{\mathbb{R}^{n^2}})$.

Bounded: Since $A \cdot A^t = I$ we have that $a_{0j}^2 + a_{1j}^2 + \cdots + a_{nj}^2 = 1$ for all $1 \leq j \leq n$. Hence $|a_{ij}| \leq 1$ for all $1 \leq i, j \leq n$.

Closed: The $(i,j)^{th}$ entry of $A \cdot A^t$ is $\sum_{1 \leq k \leq n} a_{ik} a_{jk}$. Let $\mathbb{R}^{n^2} \xrightarrow{f_{i,j}} \mathbb{R}$ be the map given by $(a_{11}, \ldots, a_{n1}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) \mapsto \sum_{1 \leq k \leq n} a_{ik} a_{jk}$.

Then $f_{i,j}$ is a polynomial map. By Question 3 of Exercise Sheet 3 we thus have $f_{i,j}$ is continuous for all $1 \leq i, j \leq n$.

The sets $\{0\}$ and $\{1\}$ are closed in $\mathbb{R}$. Since f is continuous, we deduce by Question 1 (a) on Exercise Sheet 3 that $f_{ij}^{-1}(\{0\})$ and $f_{ij}^{-1}(\{1\})$ are closed in $\mathbb{R}^{n^2}$ for all $1 \leq i, j \leq n$.

We have that $A \in O(n)$ if and only if $f_{i,j}(A) = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$.

In other words $A \in O(n)$ if and only if

$$A \in \Big(\bigcap_{\substack{1 \leq i, j \leq n, \\ i \neq j}} f_{i,j}^{-1}(\{0\}) \Big) \cap \Big(\bigcap_{1 \leq i \leq n} f_{i,i}^{-1}(\{1\}) \Big).$$

Thus $$O(n) = \Big(\bigcap_{\substack{1 \leq i, j \leq n, \\ i \neq j}} f_{i,j}^{-1}(\{0\}) \Big) \cap \Big(\bigcap_{1 \leq i \leq n} f_{i,i}^{-1}(\{1\}) \Big).$$

Since $f_{i,j}^{-1}(\{0\})$ is closed in $\mathbb{R}^{n^2}$ for all $1 \leq i, j \leq n$ with $i \neq j$ and since $f_{i,i}^{-1}(\{1\})$ is closed in $\mathbb{R}^{n^2}$ for all $1 \leq i \leq n$, we deduce that $O(n)$ is closed in $\mathbb{R}^{n^2}$, since a finite intersection of closed sets is closed.

Question 2

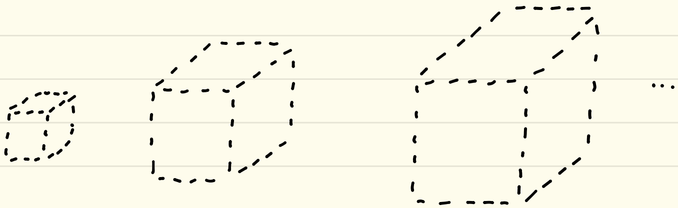
a) Give an example of an open covering $\{U_j\}_{j \in J}$ of $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$ which does not admit a finite subcovering and in which each U_j is bounded.

⚠ You cannot answer this question by appealing to the fact that $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$ is not compact. Remember that for a topological space to be compact <u>every</u> open covering must admit a finite subcovering. Thus if a topological space is not compact, all we know a priori is that there is <u>some</u> open covering which does not admit a finite subcovering. A particular open covering might admit a finite subcovering - for instance it might itself already be finite. For an extreme example, consider the open covering of $\mathbb{R}^3$ consisting just of $\mathbb{R}^3$ itself. An example of an open covering of $\mathbb{R}^3$ consisting of just two sets would be

$$\{(-1, \infty) \times \mathbb{R} \times \mathbb{R},\ (1, -\infty) \times \mathbb{R} \times \mathbb{R}\}.$$

Etc.

Solution: An example is $\{(-n,n) \times (-n,n) \times (-n,n)\}_{n \in \mathbb{N}}$. This is a set of open cubes in $\mathbb{R}^3$ centred at $(0,0,0)$.

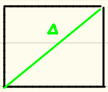


b) Does the open covering $\{(-\infty, -26), (-524, 315)\} \cup \{(0,n)\}_{n \in \mathbb{N}}$ of $\mathbb{R}$ admit a finite subcovering?

⚠ The same warning as in part (a) applies here: even though $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not compact, a particular open covering may admit a finite subcovering. For example if the question asked whether $\{(-\infty,-26), (-524, 315), (101, \infty)\} \cup \{(0,n)\}_{n\in\mathbb{N}}$ admits a finite subcovering, the answer would be yes: we can just throw away all the sets $\{(0,n)\}_{n\in\mathbb{N}}$, for example, leaving us with the subcovering consisting of the three sets $(-\infty,-26)$, $(-524, 315)$, and $(101, \infty)$.

Solution: No. Suppose that we have a set consisting of only finitely many of the sets $(0,n)$, say $\{(0,n)\}_{n\in J}$ for a finite set J. Then $\bigcup_{n\in J}(0,n)$ is $(0,m)$, where $m = \max\{n\in\mathbb{N} \mid n \in J\}$. Thus even if we take the union of $\bigcup_{n\in J}(0,n)$ with both $(-\infty,-26)$ and $(-524, 315)$ we obtain only $(-\infty, m)$, which is not all of $\mathbb{R}$.

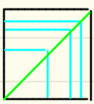
c) Let Δ be the diagonal $\{(t,t)\in I^2 \mid t \in I\}$ in I^2 equipped with its subspace topology $\mathcal{O}_{\Delta}$ with respect to $(I^2, \mathcal{O}_{I^2})$.

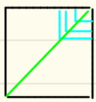


Find a finite subcovering of the open covering

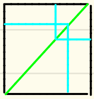
$$\left\{\Delta \cap \left([0, 1-\tfrac{1}{n}) \times [0, 1-\tfrac{1}{n})\right)\right\}_{n\in\mathbb{N}} \cup \left\{\Delta \cap \left((\tfrac{1}{2}+\tfrac{1}{3n}, 1] \times (\tfrac{1}{2}+\tfrac{1}{3n}, 1]\right)\right\}_{n\in\mathbb{N}}$$

of Δ.





Solution: There are many possibilities! One can find a subcovering consisting of just two sets, such as $\left\{ \Delta \cap \left([0, 1-\tfrac{1}{4}) \times [0, 1-\tfrac{1}{4}] \right), \ \Delta \cap \left((\tfrac{1}{2} + \tfrac{1}{3\cdot 4}, 1] \times (\tfrac{1}{2} + \tfrac{1}{3\cdot 4}, 1] \right) \right\}$.

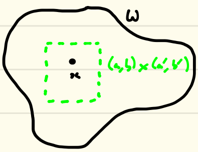


Question 3

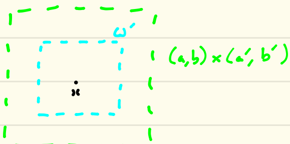
Which of the following topological spaces are locally compact? Give a reason why or why not. You may appeal to any results from the lectures. You may also assume that all the topological spaces are Hausdorff.

a) $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$

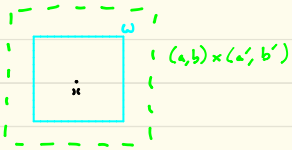
Solution: Locally compact. Let $(x,y) \in \mathbb{R}^2$ and let W be a neighbourhood of (x,y) in $\mathbb{R}^2$. Then there are $a, b, a', b' \in \mathbb{R}$ such that $(a,b) \times (a',b') \subset W$.



Let $W' = (c,d) \times (c',d')$, where $a < c < x$, $x < d < b$, $a' < c' < y$, $y < d' < b'$.



Then $\overline{W'} = [c,d] \times [c',d'] \subset (a,b) \times (a',b') \subset W$.



We have that $[c,d]$ and $[c',d']$ are compact subsets of $\mathbb{R}$. Since a product of compact topological spaces is compact we have that $\overline{W'}$ is compact.

b) K^2.

Solution: Locally compact. Indeed K^2 is compact since it is a quotient of I^2, which is compact. You were allowed to assume that K^2 is Hausdorff. A compact Hausdorff topological space is locally compact.

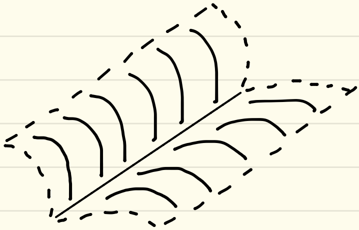
c) $M^2 \times S^1$.

Solution: Locally compact. We have that M^2 is compact since it is a quotient of I^2, which is compact. We have that S^1 is compact since it it is a quotient of I, which is compact. Since a product of compact topological spaces is compact we deduce that $M^2 \times S^1$ is compact. You were allowed to assume that $M^2 \times S^1$ is Hausdorff. A compact Hausdorff topological space is locally compact.

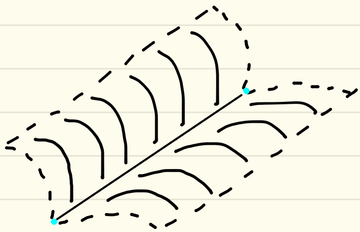
d) $\mathbb{Z}$ equipped with its subspace topology $\mathcal{O}_{\mathbb{Z}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Solution: Locally compact. For example one can observe that $\mathcal{O}_{\mathbb{Z}}$ is the discrete topology on $\mathbb{Z}$. It is proven in the lectures that any set equipped with its discrete topology is locally compact.

e) A Viking ship keel X equipped with its subspace topology $\mathcal{O}_X$ with respect to $\mathbb{R}^3$. The solid line belongs to X, the dashed lines do not.

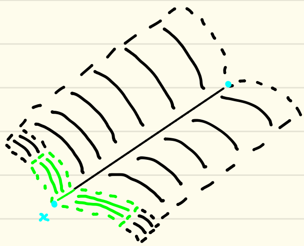


Solution: Not locally compact. Let x be either of the two points on X indicated below.

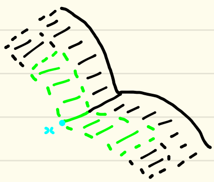


There is no neighbourhood U of x at all such that the closure $\overline{U}$ of U in X is a compact subset of X. Indeed the subspace topology on $\overline{U}$ with respect to $(X, \mathcal{O}_X)$ is equal to the subspace topology on $\overline{U}$ with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$. Hence $\overline{U}$ is a compact subset of $(X, \mathcal{O}_X)$ if and only if it is a compact subset of $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$, and thus if and only if $\overline{U}$ is closed and bounded in $\mathbb{R}^3$. But $\overline{U}$ is not closed in $\mathbb{R}^3$.

Let us see this in pictures. Below is depicted a typical neighbourhood U of x.

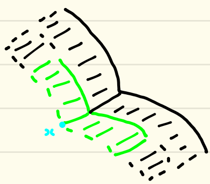


Let us zoom in a little.



U

Then $\overline{U}$ is as follows.



$\overline{U}$

Thus $\overline{U}$ is not closed in $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$.

The two light blue points are the only points of X at which the condition for $(X, \mathcal{O}_X)$ to be locally compact fails to hold.

f) $(\mathbb{Q} \times \mathbb{Q}, \mathcal{O}_{\mathbb{Q} \times \mathbb{Q}})$

Solution: Not locally compact. Let $(q, q') \in \mathbb{Q} \times \mathbb{Q}$. There is no neighbourhood W of (q, q') at all such that the closure $\overline{W}$ of W in $(\mathbb{Q} \times \mathbb{Q}, \mathcal{O}_{\mathbb{Q} \times \mathbb{Q}})$ is compact. Indeed the subspace topology on $\overline{W}$ with respect to $(\mathbb{Q} \times \mathbb{Q}, \mathcal{O}_{\mathbb{Q} \times \mathbb{Q}})$ is the same as the subspace topology on $\overline{W}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Thus $\overline{W}$ is a compact subset of $(\mathbb{Q} \times \mathbb{Q}, \mathcal{O}_{\mathbb{Q} \times \mathbb{Q}})$ if and only if it is a compact subset of $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$, and thus if and only if it is closed in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ and bounded.

There are $a, b, a', b' \in \mathbb{R}$ such that $q \in (a,b)$, $q' \in (a', b')$, and $(\mathbb{Q} \cap (a,b)) \times (\mathbb{Q} \cap (a', b')) \subset W$. The closure of $(\mathbb{Q} \cap (a,b)) \times (\mathbb{Q} \cap (a', b'))$ in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is $[a,b] \times [a', b']$.

If $\overline{W}$ were closed in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ we would thus have that $[a,b] \times [a', b'] \subset \overline{W}$. Since $[a,b] \times [a', b']$ contains pairs (x, x') such that x, x', or both are irrational, this would contradict the fact that $\overline{W} \subset \mathbb{Q}$.

Thus $\overline{W}$ is not closed in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$, and hence not compact.

Question 4

This question is harder. It is examinable, but stronger hints would be given in an actual exam. The phrasing of the third part is closest to the likely phrasing. The important thing is to understand the method, which is the same for all three parts.

1) We wish to prove that a compact topological space $(X, \mathcal{O}_X)$ is homeomorphic to a Hausdorff topological space $(Y, \mathcal{O}_Y)$. By a result from the lectures it is sufficient under these hypotheses to construct

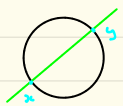
a continuous bijection $X \xrightarrow{f} Y$.

2) In all of the examples below $(X, \mathcal{O}_X)$ is a quotient of a topological space $(X', \mathcal{O}_{X'})$ by an equivalence relation $\sim$ on X'. To construct $X \xrightarrow{f} Y$ we first construct a continuous map $X' \xrightarrow{f'} Y$ such that if $x \sim x'$ for $x, x' \in X$ then $f'(x) = f'(y)$. Then $[x] \longmapsto f'(x)$ gives a well-defined map $X \longrightarrow Y$, which we take to be f.

3) By Question 11(a) of Exercise Sheet 4, since f is continuous we have that f' is continuous.

4) We check that f' is bijective.

a) Let $\sim$ be the equivalence relation on S^1 which identifies antipodal points, namely pairs of points that lie on a common line through the origin.



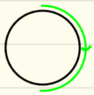
$x \sim y$

Prove that $(S^1, \mathcal{O}_{S^1})$ is homeomorphic to $(S^1/\sim, \mathcal{O}_{S^1/\sim})$. You may assume that $(S^1/\sim, \mathcal{O}_{S^1/\sim})$ is Hausdorff.

Hint: begin by defining a continuous map from I to a semi-circle.

Solution: There are different ways to carry out the proof along the lines of 1) – 4) above. I will view S^1 as $I/\cong$, where $\cong$ is the equivalence relation on I generated by $0 \cong 1$. From this point of view we may define $\sim$ to the equivalence relation on S^1 generated by $[t] \sim [t']$ if $t' = t + 1/2$.

We begin with the map $I \xrightarrow{g} I$ given by $t \mapsto t/2$. Let $I \xrightarrow{\pi} I/\sim$ be the quotient map. Both g and π are continuous, and hence $I \xrightarrow{\pi \circ g} I/\simeq \, = S^1$ is continuous. This map can be viewed as follows.



Let $S^1 \xrightarrow{\pi'} S^1/\sim$ denote the quotient map. It is continuous, and hence $I \xrightarrow{\pi \circ \pi' \circ g} S^1/\sim$ is continuous. Let us denote this map by f'.

We have that $f'(0) = \big[[g(0)]\big] = \big[[0]\big] = \big[[1/2]\big] = \big[[g(1)]\big] = f'(1)$. As in (2) and (3) of the approach outlined above, we thus have that $[t] \mapsto f'(t)$ defines a continuous map $S^1 \longrightarrow S^1/\sim$. We denote it by f.

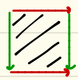
We observe that f is bijective. It is enough to assert this.

Since S^1 is compact and we may assume that $S^1/\sim$ is Hausdorff, we conclude that f is a homeomorphism.

b) Prove that T^2 is homeomorphic to $S^1 \times S^1$.

Solution: Let $\sim$ be the equivalence relation on I^2 which defines T^2, generated by

$$\begin{cases} (0,t) \sim (1,t) & \text{for all } t \in I \\ (t,0) \sim (t,1) & \text{for all } t \in I. \end{cases}$$



Let $\simeq$ be the equivalence relation on I which defines S^1, generated by $0 \simeq 1$.

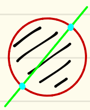
Let $I \xrightarrow{\pi} I/\sim = S^1$ be the quotient map. We have that π is continuous. Hence (see Question 4 c) of Exercise Sheet 3) we have that the map $I \times I \xrightarrow{\pi \times \pi} S^1 \times S^1$ given by $(t, t') \mapsto (\pi(t), \pi(t'))$ is continuous.

We have that $\pi \times \pi(0, t) = ([0], [t]) = ([1], [t]) = \pi \times \pi(1, t)$ and that $\pi \times \pi(t, 0) = ([t], [0]) = ([t], [1]) = \pi \times \pi(t, 1)$.

By (2) and (3) we deduce that $[(t, t')] \mapsto \pi \times \pi(t, t')$ defines a continuous map $T^2 \longrightarrow S^1 \times S^1$. Let us denote it by f.

We observe that f is bijective. Since T^2 is compact and $S^1 \times S^1$ is Hausdorff, we deduce that f is a homeomorphism.

c) Let $\mathbb{P}^2(\mathbb{R})$ be the topological space $(D^2/\sim, \mathcal{O}_{D^2/\sim})$, where $\sim$ is the equivalence relation on D^2 defined by identifying antipodal points on $\partial_{\mathbb{R}^2} D^2 = S^1$ as in a).

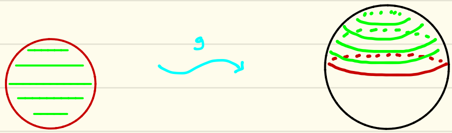


$(D^2/\sim, \mathcal{O}_{D^2/\sim})$ is homeomorphic to the topological space of Question 1 d). You may assume this here. But for further practise, can you see why?

Let $\simeq$ be the equivalence relation on S^2 which identifies antipodal points, namely pairs of points that lie on a line through the origin in $\mathbb{R}^3$.

Prove that $\mathbb{P}^2(\mathbb{R})$ is homeomorphic to $(S^2/\simeq, \mathcal{O}_{S^2/\simeq})$.

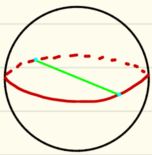
Hint: You may assume the existence of a continuous map $D^2 \xrightarrow{g} S^2$ which maps D^2 injectively onto the upper hemisphere of S^2, and maps $\partial_{\mathbb{R}^2} D^2$ onto the equator.



You may also assume that $S^2/\simeq$ is Hausdorff.

Solution: Let $S^2 \xrightarrow{\pi} S^2/\simeq$ be the quotient map. We have that π is continuous. Thus $D^2 \xrightarrow{\pi \circ g} S^2/\simeq$ is continuous. Let us denote this map by f'.

We have that $f'(x) = f'(y)$ for all $x, y \in D^2$ such that $x \sim y$, since the relation $\simeq$ on S^2 identifies antipodal points on its equator.



As in (2) and (3) we thus have that $[x] \longmapsto f'(x)$ defines a continuous map $D^2/\sim \longrightarrow S^2/\simeq$. Let us denote it by f. We make the following observations:

i) f is bijective

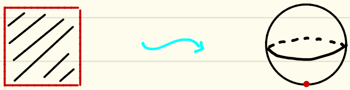
ii) D^2 is compact since it is closed and bounded in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$, and hence $D^2/\sim$ is compact since a quotient of a compact topological space is compact.

3) $S^2/_{\simeq}$ is Hausdorff (you were allowed to assume this).

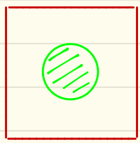
We deduce that f is a homeomorphism.

Aside: I will explain one way to construct a map g as required.

We defined S^2 in the lectures to be $I^2/_{\sim'}$, where $x \sim' y$ if $x, y \in \partial_{\mathbb{R}^2} I^2$.

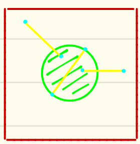


The upper hemisphere of S^2 can be thought of as the disc of radius $\frac{1}{2}$ in I^2 centred at $(\frac{1}{2}, \frac{1}{2})$.



The map $D^2 \xrightarrow{g} S^2$ can be to be the composite of the map $D^2 \longrightarrow I^2$ which shrinks D^2 to radius $\frac{1}{2}$ with the quotient map $I^2 \longrightarrow I^2/_{\sim'} = S^2$.

From this point of view we may define $\simeq$ on S^2 by $(s,s') \simeq (t,t')$ if (s,s') is of distance $\frac{1}{2}$ from (t,t') along a common line through the origin. Three examples of identified points are depicted below.

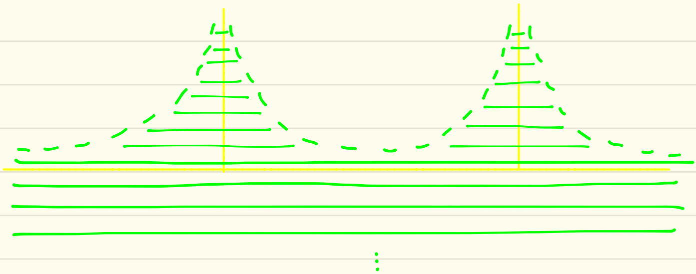


Question 5

a) Find a subset W of $\mathbb{R}^2$ for which there are exactly two real numbers x and x' with $(x,0), (x',0) \in W$ such that the tube lemma fails to hold for x and x'.

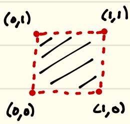
Solution: We may for example take W to be the set

$$\{(x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1,\ x \neq 0,\ y < 1/x\} \cup \{(x,y) \in \mathbb{R}^2 \mid x = 0\}$$
$$\cup \{(x+2,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1,\ x \neq 0,\ y < 1/x\} \cup \{(x,y) \in \mathbb{R}^2 \mid x = 2\}$$



The tube lemma fails to hold exactly for $x = 0$ and $x' = 2$.

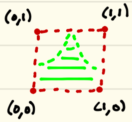
b) Find a subset W of $(0,1) \times (0,1)$ for which the tube lemma fails to hold.



Solution: For example, let $W = \{(x,y) \in (0,1) \times (0,1) \mid 0 < x < 1/2,\ y < x + 1/2\}$

$\cup \{(x,y) \in (0,1) \times (0,1) \mid 1/2 < x < 1,\ y < x - 1/2\}$

$\cup \{(x,y) \in (0,1) \times (0,1) \mid x = 1/2\}$.



The tube lemma fails to hold for $x = 1/2$.

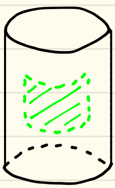
c) Find a subset $X = X' \times X''$ of the cylinder $S^1 \times I$

and a subset W of X for which the tube lemma fails to hold with respect to $(X, \mathcal{O}_{X' \times X''})$, where $\mathcal{O}_{X'}$ is the subspace topology on X' with respect to $(S^1, \mathcal{O}_{S^1})$, and $\mathcal{O}_{X''}$ is the subspace topology on X'' with respect to $(I, \mathcal{O}_I)$.

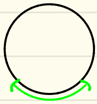
[As an aside, $\mathcal{O}_{X' \times X''}$ is equal to the subspace topology on X with respect to $(S^1 \times I, \mathcal{O}_{S^1 \times I})$.]

A picture of W is sufficient, you do not need to define it explicitly.

Solution: For example, let X be the open subset of the cylinder depicted below.

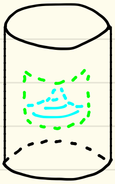


In other words we take X' to be an open arc in S^1



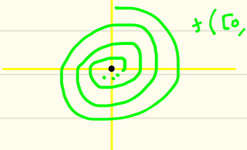
and take X'' to be an open interval in I.

We may take W to be as depicted below.



Question 6

Let $[0,\infty) \xrightarrow{f} \mathbb{R}^2$ be an injective continuous map whose image is a spiral as follows, getting closer and closer to $(0,0)$ without ever reaching it!



Let $f([0,\infty))$ be equipped with its subspace topology $\mathcal{O}_{f([0,\infty))}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Prove that $(f([0,\infty)) \times I, \mathcal{O}_{f([0,\infty)) \times I})$ is not homeomorphic to $(I^2, \mathcal{O}_{I^2})$.

My artistic skills are not good enough to draw $f([0,\infty)) \times I$! It can be thought of as a kind of 'spiralling sheet' around the segment of the z-axis from 0 to 1 in $\mathbb{R}^3$.

Solution: We have that $(f([0,\infty)), \mathcal{O}_{f([0,\infty))})$ is homeomorphic to $([0,\infty), \mathcal{O}_{[0,\infty)})$. [You may assert this without proof - in an exam I might ask you to prove this for any injective continuous map in an earlier part.]

Thus the question is equivalent to proving that $([0,\infty) \times I, \mathcal{O}_{[0,\infty) \times I})$ is not homeomorphic to $(I^2, \mathcal{O}_{I^2})$.

We have that $(I^2, \mathcal{O}_{I^2})$ is compact since it is a product of compact topological spaces. Thus if $[0,\infty) \times I$ were homeomorphic to I^2 we would have that $[0,\infty) \times I$ is compact. Since $[0,\infty)$ is not compact, this is impossible.