MA3002 Generell Topologi — Revision Questions

Richard Williamson

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2 Lectures 1–12

Question 2.1. Regard the letters B, C, D, E, F, G, H as subsets of $\mathbb{R}^2$, equipped with their respective subspace topologies with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Which of the letters are homeomorphic to one another, and which are not?

You may quote without proof any results from the course which you require.

Solution 2.1. The letters E and F are homeomorphic. The letters C and G are homeomorphic. There are no other pairs of these letters which are homeomorphic.

Let us prove for example that $(\mathsf{B}, \mathcal{O}_{\mathsf{B}})$ is not homeomorphic to $(\mathsf{H}, \mathcal{O}_{\mathsf{H}})$. It is very important to be able to give a careful presentation of the argument in the manner below, quoting the necessary results from the course.

Suppose that

$$\mathsf{H} \xrightarrow{f} \mathsf{B}$$

is a homeomorphism. Let $x \in \mathsf{H}$. Let $\mathsf{H} \setminus \{x\}$ be equipped with its subspace topology with respect to $(\mathsf{H}, \mathcal{O}_{\mathsf{H}})$, and let $\mathsf{B} \setminus \{f(x)\}$ be equipped with its subspace topology with respect to $(\mathsf{B}, \mathcal{O}_{\mathsf{B}})$.

By Proposition 8.1 in the Lecture Notes we have that the map

$$\mathsf{H} \setminus \{x\} \longrightarrow \mathsf{B} \setminus \{f(x)\}$$

given by restricting f is a homeomorphism. Hence by Proposition 8.14 in the Lecture Notes we have that $\mathsf{H} \setminus \{x\}$ and $\mathsf{B} \setminus \{f(x)\}$ have the same number of connected components.

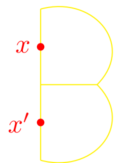
Let x be the junction point of H depicted below.

Then $\mathsf{H} \setminus \{x\}$ has three connected components.

There is no $y \in \mathsf{B}$ such that $\mathsf{B} \setminus \{y\}$ has three connected components. Indeed, removing any point from B yields a topological space with one connected component.

Practise writing out a proof as above for one of the other pairs of letters which are not homeomorphic.

Don't forget that you may need to remove more than one point. For example, to distinguish B from D it is not possible to use only one point. Instead we can observe that it is possible to remove two points from B such as those shown below

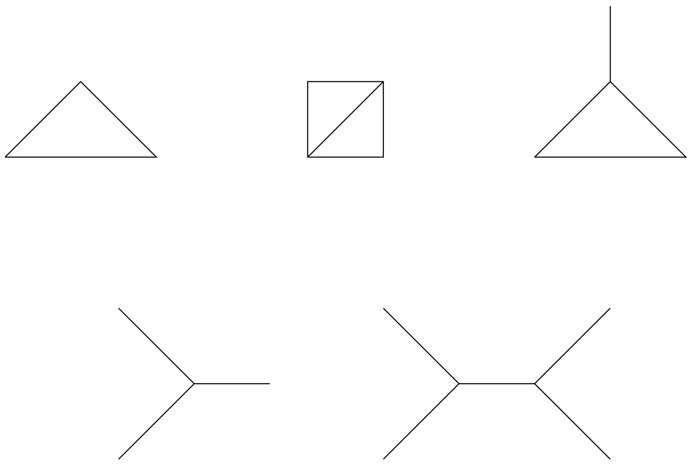


and obtain a topological space with only one connected component.

Removing any two points from D we obtain a topological space with two connected components.

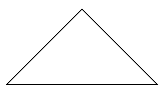
Question 2.2. Show that no two of the following Δ-complexes are homeomorphic.

All consist of 0-simplices and 1-simplices, there are no 2-simplices.



Calculate their Euler characteristics.

Solution 2.2. The argument required here is the same as in the first question. The Δ-complexes



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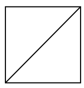
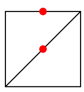
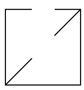


are not homeomorphic since it is possible to remove two points from the latter shape

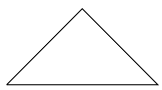


and obtain a topological space with one connected component

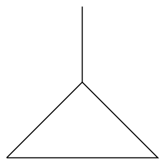


whereas this is impossible for a triangle.

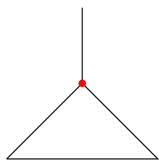
The Δ-complexes



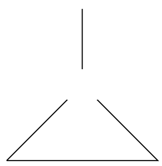
and



are not homeomorphic since it is possible to remove one point from the latter shape



and obtain a topological space with two connected components



whereas this is impossible for a triangle.

Similar arguments can be made for the other shapes. Let me know if there are any which you are unsure about!

Note that on an exam the above sketches are not enough. A careful argument such as that presented in Solution 2.1 is needed.

Clockwise from the top left, the Euler characteristics are:

(1) $3 - 3 = 0$,

(2) $4 - 5 = -1$,

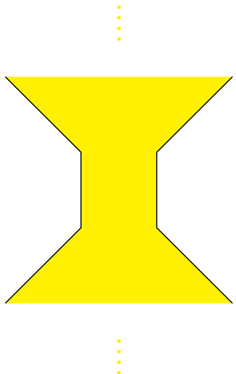
(3) $4 - 4 = 0$,

(4) $6 - 6 = 0$,

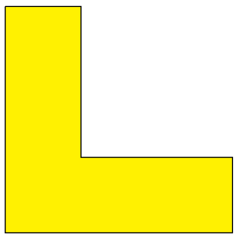
(5) $4 - 3 = 1$.

Question 2.3. Prove that the following topological spaces are connected.

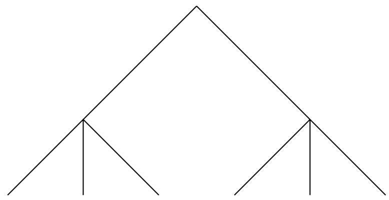
(1) The following subset of $\mathbb{R}^2$, equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



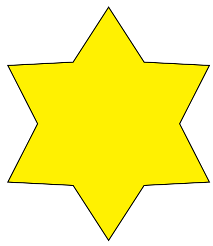
(2) The following subset of $\mathbb{R}^2$, equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



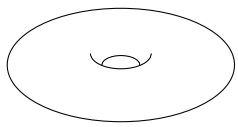
(3) The following subset of $\mathbb{R}^2$, equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



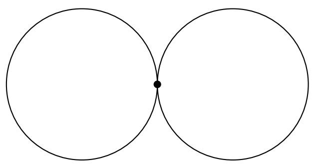
(4) The following subset of $\mathbb{R}^2$, equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



(5) $(T^2, \mathcal{O}_{T^2})$.



(6) The following subset of $\mathbb{R}^2$ consisting of two circles joined at a point, equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Solution 2.3. We have three principal methods to show that a topological space is connected.

(A) Observe that it can be obtained canonically, or is homeomorphic to a topological space which can be obtained canonically, by beginning with a topological space or a collection of topological spaces which we know to be connected, and then taking products and quotients. This relies upon Proposition 7.13, Proposition 7.15, and Corollary 7.3 in the Lecture Notes.

(B) Prove that it is path connected, which is often straightforward in geometric examples, and then appeal to the fact that a path connected topological space is connected, which is Proposition 10.17 in the Lecture Notes.

(C) Apply Proposition 8.5. We can refer to this as: by glueing connected topological spaces which have non-empty intersection we obtain a connected topological space.

It is very important that you are comfortable with all three methods to prove that a topological space is connected, and are able to recognise when each method can be used. Let us look at this for the examples in the question.

(1) All three methods can be used here. Let us discuss the use of methods A and B.

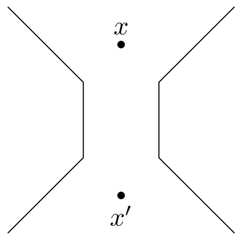
For method A, we can note that this topological space is homeomorphic to

$$(I \times \mathbb{R}, \mathcal{O}_{I\times\mathbb{R}}).$$

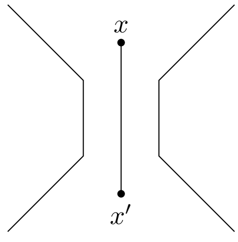
We know by Proposition 7.9 in the Lecture Notes that $(I, \mathcal{O}_I)$ and $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ are connected.

For method B, we make the following observations.

(i) Every point can be joined by a straight line — which is a particular kind of path — from the two points indicated below.



(ii) There is a straight line from x to x',

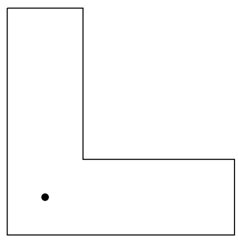


It follows by applying Proposition 10.8 and Proposition 10.11 in the Lecture Notes that we have a path connected topological space. This a slight generalisation of the argument given to prove Proposition 10.16 in the Lecture Notes.

There is no necessity to use only straight lines here, but if we were to write everything out explicitly it would probably be easiest to work with straight lines.

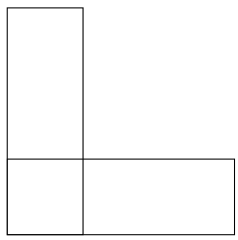
(2) Again all three methods can be used here. Let us discuss methods B and C.

For method B we can observe that every point can be joined by a straight line from the point indicated below.



By Proposition 10.16 in the Lecture Notes, it follows that we have a path connected topological space.

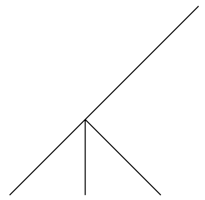
For method C we can observe that we have a union with non-empty intersection of two strips as follows.



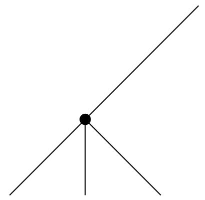
Each strip is homeomorphic to $(I^2, \mathcal{O}_{I^2})$, and is thus connected.

(3) Methods B and C can be applied here. Let us discuss method C.

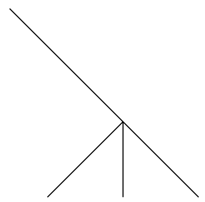
We can first apply method C to see that the topological space



is connected, since we have a union of four lines — each homeomorphic to $(I, \mathcal{O}_I)$ and therefore connected — which intersect at the point indicated below.

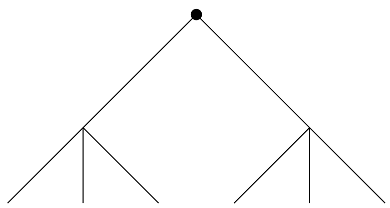


Next we can apply method C in the same way to see that the topological space



is connected.

We can then apply method C a third time to conclude that the topological space we are interested in is connected, since it is a union



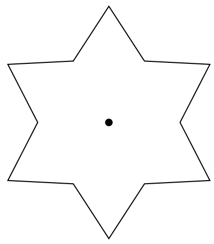
of the two topological spaces we have already observed to be connected, and these two topological spaces intersect at the point indicated.

There are other possible ways of using method C here. It is crucial to note that we must have a non-empty intersection in order to apply method C, which is why we divided our argument into three steps.

(4) All three methods can be applied here. Let us discuss methods A and B.

For method A we can observe for example that our star is homeomorphic to $(D^2, \mathcal{O}_{D^2})$ or $(I^2, \mathcal{O}_{I^2})$, which we know to be connected.

For method B we can observe that every point in our star can be joined by a straight line to a point as indicated below.



By Proposition 10.16 in the Lecture Notes, it follows that we have a path connected topological space.

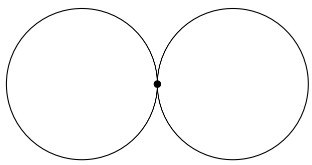
(5) We can apply method A. We have that $(T^2, \mathcal{O}_{T^2})$ is $(I^2/\sim, \mathcal{O}_{I^2/\sim})$, where $\sim$ is the equivalence relation on I^2 defined in Examples 3.9 (3).

We know by Proposition 7.9 in the Lecture Notes that $(I, \mathcal{O}_I)$ is connected. Since a product of connected topological spaces is connected by Proposition 7.13 in the Lecture Notes, we deduce that $(I^2, \mathcal{O}_{I^2})$ is connected. Since a quotient of a connected topological space is connected by Proposition 7.15 in the Lecture Notes, we conclude that $(T^2, \mathcal{O}_{T^2})$ is connected.

(6) We can apply methods B and C here. Let us discuss method C.

We have that $(S^1, \mathcal{O}_{S^1})$ is connected by method A, since it is homeomorphic to $(I/\sim, \mathcal{O}_{I/\sim})$ where $\sim$ is the equivalence relation of Examples 3.9 (1), and we know that a quotient of a connected topological space is connected.

Thus a union of two topological spaces homeomorphic to $(S^1, \mathcal{O}_{S^1})$ which intersect in a point



is connected by method C.

Question 2.4. Construct a continuous bijection

$$[0,1) \longrightarrow S^1.$$

Can it be a homeomorphism?

Solution 2.4. One way to construct a continuous bijection

$$[0,1) \longrightarrow S^1$$

is as follows.

Let

$$[0,1) \xrightarrow{i} I$$

be the inclusion map, which we know to be continuous by Proposition 2.15 in the Lecture Notes.

Let

$$I \xrightarrow{\pi} S^1$$

be the quotient map, thinking of S^1 as $I/\sim$ where $\sim$ is the equivalence relation of Examples 3.9 (1) in the Lecture Notes. By Observation 3.7 in the Lecture Notes we have that π is continuous.

By Proposition 2.16 in the Lecture Notes we deduce that

$$[0,1) \xrightarrow{\pi \circ i} S^1$$

is continuous. Moreover $\pi \circ i$ is a bijection.

It is not a homeomorphism. There are at least three ways to see this.

(1) Observe that $[0,1)$ is not homeomorphic to S^1 since S^1 is compact and $[0,1)$ is not compact.

(2) Observe that $[0,1)$ is not homeomorphic to S^1 since removing the point 0 from $[0,1)$ we obtain a topological space with only one connected component, whereas removing any point from S^1 we obtain a topological space with two connected components.

(3) Observe directly that $\pi \circ i$ is not a homeomorphism by exhibiting an open subset U of $[0,1)$ such that $\big(\pi \circ i\big)(U)$ is not open in S^1.

For this we can for example take U to be $[0, \frac{1}{2})$. By definition, $\big(\pi \circ i\big)(U)$ is open in S^1 if and only if $\pi^{-1}\Big(\big(\pi \circ i\big)(U)\Big)$ is open in $(I, \mathcal{O}_I)$. But

$$\pi^{-1}\Big(\big(\pi \circ i\big)(U)\Big) = \big[0, \frac{1}{2}\big) \cup \{1\},$$

which is not open in $(I, \mathcal{O}_I)$.

Question 2.5. Prove directly — without appealing to the fact that a product of Hausdorff topological spaces is Hausdorff — that $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$ is Hausdorff.

Solution 2.5. Let $(x_0, x_1, x_2) \in \mathbb{R}^3$ and $(y_0, y_1, y_2) \in \mathbb{R}^3$ be distinct. At least one of the following holds.

(1) $x_0 \neq y_0$. Then let us define $\epsilon_0 \in \mathbb{R}$ to be

$$\frac{|y_0 - x_0|}{2},$$

and define $\epsilon_1 > 0$ and $\epsilon_2 > 0$ arbitrarily.

(2) $x_1 \neq y_1$. Then let us define $\epsilon_1 \in \mathbb{R}$ to be

$$\frac{|y_1 - x_1|}{2},$$

and define $\epsilon_0 > 0$ and $\epsilon_2 > 0$ arbitrarily.

(3) $x_2 \neq y_2$. Then let us define $\epsilon_2 \in \mathbb{R}$ to be

$$\frac{|y_2 - x_2|}{2},$$

and define $\epsilon_0 > 0$ and $\epsilon_1 > 0$ arbitrarily.

If more than (1), (2), and (3) hold, we just choose one.

Let

$$U_0 = (x_0 - \epsilon_0, x_0 + \epsilon_0) \times (x_1 - \epsilon_1, x_1 + \epsilon_1) \times (x_2 - \epsilon_2, x_2 + \epsilon_2)$$

Let

$$U_1 = (y_0 - \epsilon_0, y_0 + \epsilon_0) \times (y_1 - \epsilon_1, y_1 + \epsilon_1) \times (y_2 - \epsilon_2, y_2 + \epsilon_2).$$

We make the following observations.

(1) $U_0,\ U_1,\ U_2 \in \mathcal{O}_{\mathbb{R}^2}$.

(2) $(x_0, x_1, x_2) \in U_0$ and $(y_0, y_1, y_2) \in U_1$.

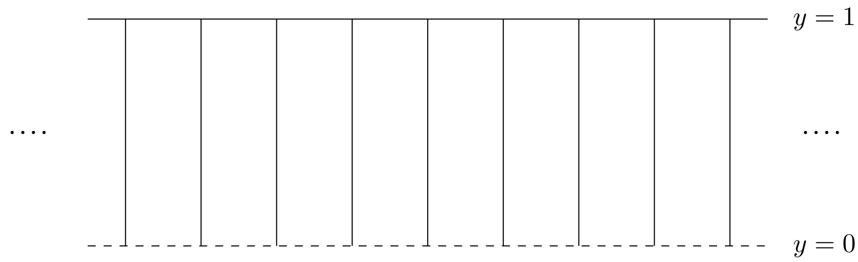
(3) $U_0 \cap U_1 = \emptyset$.

If when answering this kind of question on an exam you see the idea but cannot find a way to express it rigorously, you will pick up marks by drawing a clear picture with a caption.

Question 2.6. Is the set

$$X = \{(x, y) \in \mathbb{R}^2 \mid 0 < y \leq 1\}$$

closed, open, or neither in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$?

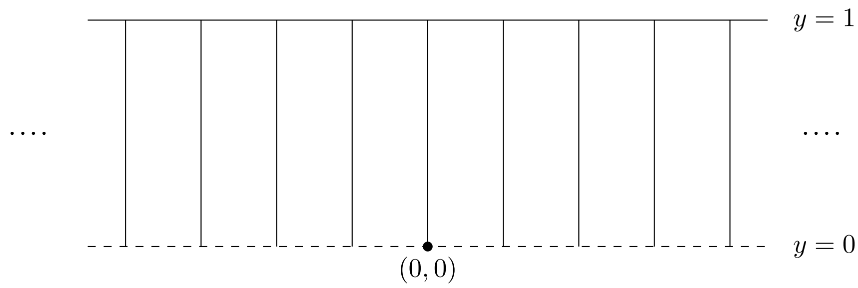


If it is not closed, what is its closure in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$?

Equip X with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Can you express it as a product of topological spaces?

Is $(X, \mathcal{O}_X)$ connected? Is $(X, \mathcal{O}_X)$ compact?

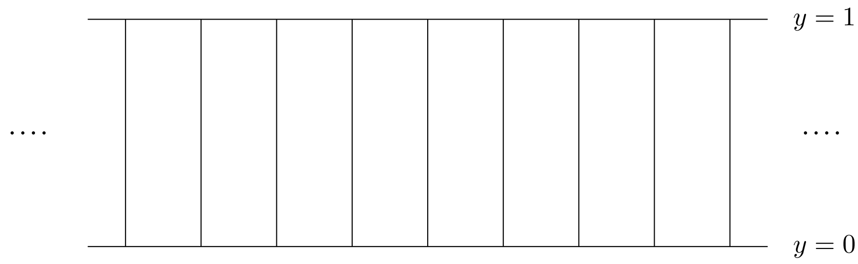
Let X' be the set obtained by adding a point at $(0, 0)$ to X, and let $\mathcal{O}_{X'}$ denote the subspace topology on X' with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Is $(X', \mathcal{O}_{X'})$ locally compact?

Solution 2.6. The set X is neither closed nor open in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Its closure is

$$\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}.$$



We have that $(X, \mathcal{O}_X)$ is $(\mathbb{R} \times I, \mathcal{O}_{\mathbb{R} \times I})$.

Yes, $(X, \mathcal{O}_X)$ is connected. For example by Proposition 7.13 we have that a product of connected topological spaces is connected, and we know by Proposition 7.9 that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ and $(I, \mathcal{O}_I)$ are connected.

Alternatively since we can join any two points in X by a straight line we have that $(X, \mathcal{O}_X)$ is path connected, and hence connected.

No, $(X, \mathcal{O}_X)$ is not compact. For example we know that a subset of $\mathbb{R}^2$ is compact if and only if it is closed in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ and bounded, and X satisfies neither of these conditions.

No, $(X', \mathcal{O}_{X'})$ is not locally compact. Every neighbourhood U of $(0, 0)$ contains a set

$$\Big((a, b) \times (0, c)\Big) \cup \{(0, 0)\}$$

for some $a < 0$, $b > 0$, and $0 < c \leq 1$. If the closure $\overline{U}$ of U in $(X', \mathcal{O}_{X'})$ were a compact subset of $(X', \mathcal{O}_{X'})$ we would have by Proposition 13.7 in the Lecture Notes that the closure of

$$\Big((a, b) \times (0, c)\Big) \cup \{(0, 0)\}$$

in $(X', \mathcal{O}_{X'})$, namely

$$\Big([a, b] \times (0, c]\Big) \cup \{(0, 0)\},$$

is a compact subset of $(X', \mathcal{O}_{X'})$. We can conclude the argument in two ways.

(1) Observe that

$$\Big([a, b] \times (0, c]\Big) \cup \{(0, 0)\}$$

would then be a compact subset of $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. But

$$\Big([a, b] \times (0, c]\Big) \cup \{(0, 0)\}$$

is not closed in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$, and every compact subset of $\mathbb{R}^2$ is closed in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

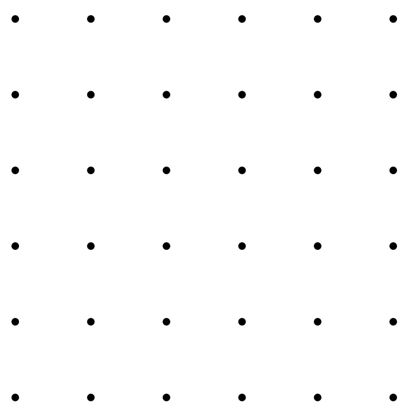
(2) Give an open covering of

$$\Big([a,b] \times (0,c]\Big) \cup \{(0,0)\}$$

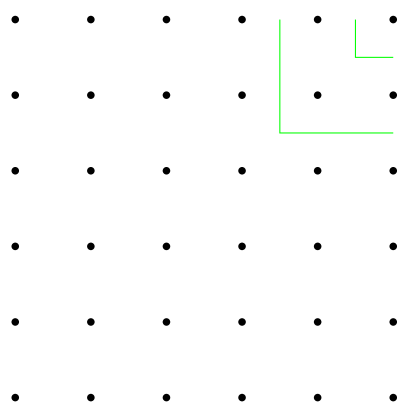
in $(X', \mathcal{O}_{X'})$ which does admit a finite subcovering, such as the following.

$$\left\{ \left([a,b] \times (\frac{1}{n}, c]\right) \cup \{(0,0)\} \right\}_{n \in \mathbb{N} \text{ and } \frac{1}{n} < c}$$

Question 2.7. Let X be an $n \times n$ grid of integer points in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



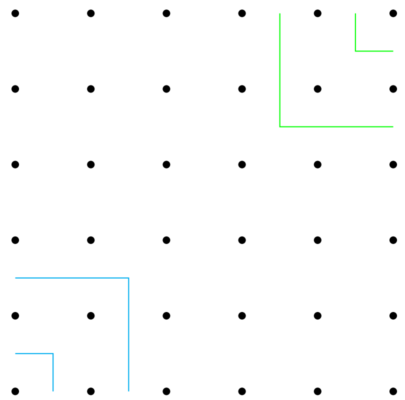
Let $\mathcal{O}$ be the set of $m \times m$ grids for $0 \leq m \leq n$ which start at the top right corner, thinking of the case $m = 0$ as the empty set.



Does $(X, \mathcal{O})$ define a topological space?

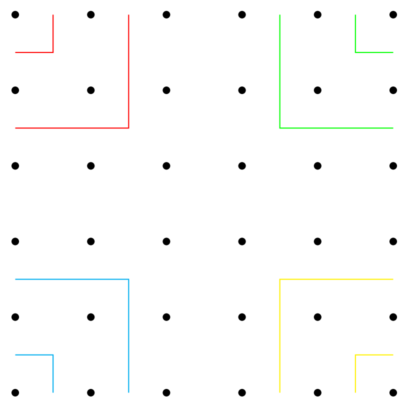
If we instead let X be an $(n+1) \times n$ grid, does $(X, \mathcal{O})$ define a topological space?

Let X again be an $n \times n$ grid. Let $\mathcal{O}'$ be obtained by adding $m \times m$ grids for $0 \leq m \leq n$ which begin at the bottom left corner to $\mathcal{O}$.



Does $(X, \mathcal{O}')$ define a topological space for $n \geq 3$?

Let $\mathcal{O}''$ be the topology on X with sub-basis given by the set of $m \times m$ grids which begin at any corner.



Is this a basis for $(X, \mathcal{O}'')$?

Is $(X, \mathcal{O}'')$ connected when n is even?

Is $(X, \mathcal{O})$ connected?

Are either $(X, \mathcal{O})$ or $(X, \mathcal{O}'')$ compact?

Is $(X, \mathcal{O})$ Hausdorff?

Is $(X, \mathcal{O})$ a T0 topological space?

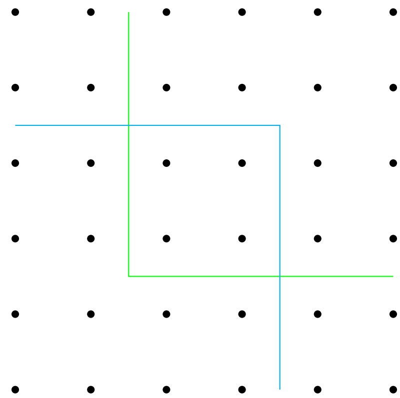
Solution 2.7. Yes, $(X, \mathcal{O})$ defines a topological space.

If X is an $(n \times 1) \times n$ grid then $(X, \mathcal{O})$ does not define a topological space, since X does not belong to $\mathcal{O}$.

No, $(X, \mathcal{O}')$ does not define a topological space. Let U for example be a $q \times q$ grid which starts at the top right corner for some $\lceil \frac{n}{2} \rceil < q < n$, and let U' be an $r \times r$ grid which starts at the bottom left corner for some $0 < r < \lfloor \frac{n}{2} \rfloor$.

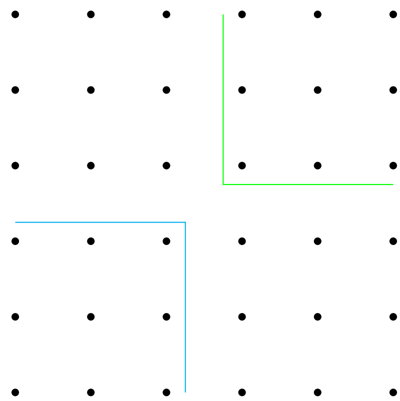
Here $\lceil \frac{n}{2} \rceil$ is the smallest integer which is greater than $\frac{n}{2}$, and $\lfloor \frac{n}{2} \rfloor$ is the largest integer which is smaller than $\frac{n}{2}$,

Then $U \cap U'$ is not empty and does not belong to $\mathcal{O}'$.

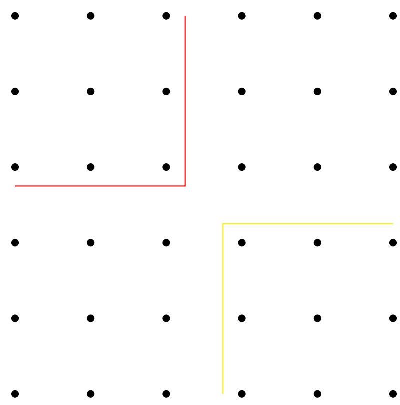


No, the set of $m \times m$ grids which begin at any corner is not a basis for $(X, \mathcal{O}'')$. The intersection of the sets U and U' above does not belong to it.

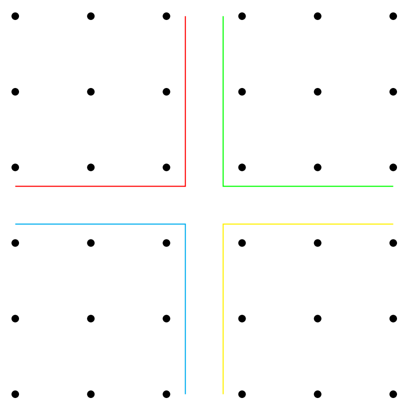
No, $(X, \mathcal{O}'')$ is not connected when n is even. Let U be the union of $\frac{n}{2} \times \frac{n}{2}$ grids starting at two of the corners



and let U' be the union of $\frac{n}{2} \times \frac{n}{2}$ grids starting at the other two of the corners,



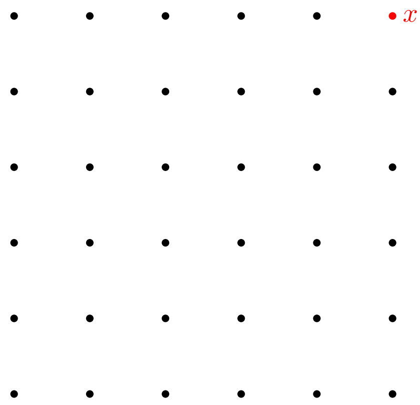
Then U and U' are open in $(X, \mathcal{O}'')$, and $X = U \sqcup U'$.



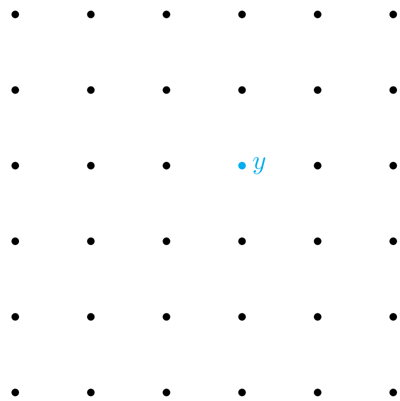
Yes, $(X, \mathcal{O})$ is connected. There are no non-empty $U, U' \in \mathcal{O}$ such that $U \cap U = \emptyset$.

Yes, both $(X, \mathcal{O})$ and $(X, \mathcal{O}'')$ are compact since X is finite.

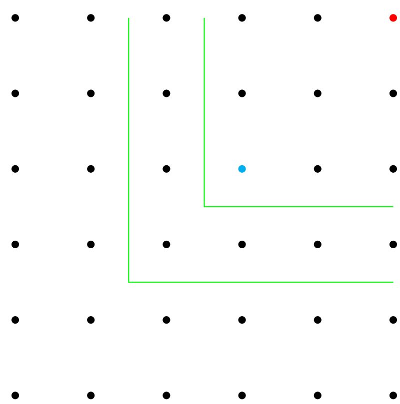
No, $(X, \mathcal{O})$ is not Hausdorff. It is not even T0. Let x be the integer at the top right hand corner.



Let $y \in X$ be such that $x \neq y$.



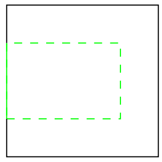
Then every neighbourhood of y contains x.



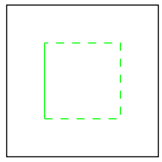
Question 2.8. Is the set

$$X = \{(x, y) \in I^2 \mid 0 \leq x < \frac{3}{4} \text{ and } \frac{1}{4} < y < \frac{3}{4}\}$$

open in $(I^2, \mathcal{O}_{I^2})$?



What if we restrict to $\frac{1}{4} \leq x < \frac{3}{4}$?



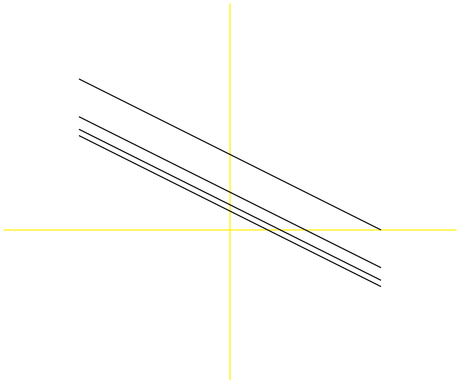
Can there be a non-empty subset of I^2 which is not I^2 itself and which is both open and closed?

Solution 2.8. Yes, X is open in $(I^2, \mathcal{O}_{I^2})$.

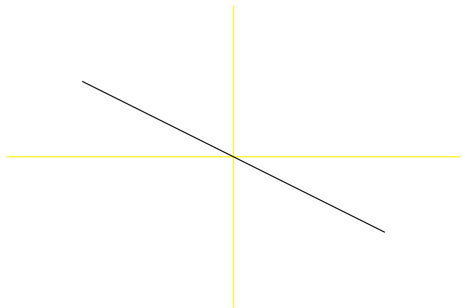
If we restrict to $\frac{1}{4} \leq x < \frac{3}{4}$ then X is not open.

Since $(I^2, \mathcal{O}_{I^2})$ is connected there cannot be a non-empty subset of I^2 which is not I^2 itself and which is both open and closed, by Proposition 6.3 in the Lecture Notes.

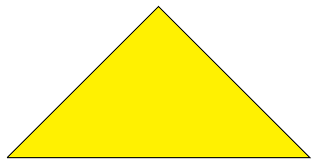
Question 2.9. What is the closure of the set $\{L_n\}_{n \in \mathbb{N}}$, where L_n is the line in $\mathbb{R}^2$ of gradient $-\frac{1}{2}$ which passes through $(0, \frac{1}{n})$?



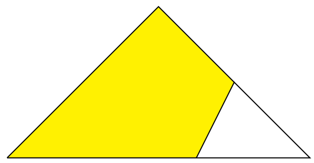
Solution 2.9. The line of gradient $-\frac{1}{2}$ through zero.



Question 2.10. Let X be the subset of $\mathbb{R}^2$ shown below, namely a 2-simplex X'



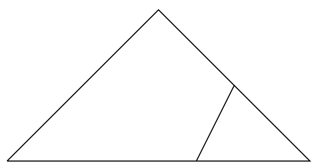
with the interior of a smaller 2-simplex cut out.



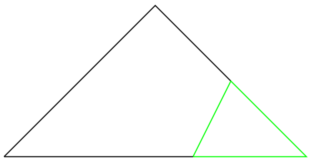
What is the boundary of X in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$?

What is the boundary of X in $(X', \mathcal{O}_{X'})$, where $\mathcal{O}_{X'}$ is the subspace topology on X' with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$?

Solution 2.10. We have that $\partial_{\mathbb{R}^2} X$ is the collection of lines shown below.



We have that $\partial_{X'} X$ is the collection of lines shown in green below.



Question 2.11. Let

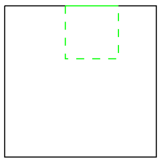
$$I^2 \xrightarrow{\pi} K^2$$

denote the quotient map.

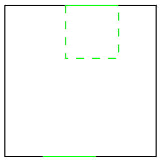
Find an open subset U of I^2 such that $\pi(U)$ is not open in $(K^2, \mathcal{O}_{K^2})$.

Find an open subset U' of I^2 for which $\pi(U)$ is open in $(K^2, \mathcal{O}_{K^2})$.

Solution 2.11. Let be the open subset of I^2 shown below.

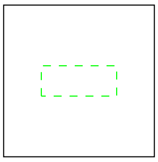


Then $\pi^{-1}\big(\pi(U)\big)$ is as shown below.



This set is not open in $(I^2, \mathcal{O}_{I^2})$. Thus $\pi(U)$ is not open in $(K^2, \mathcal{O}_{K^2})$.

An open subset U' of I^2 for which $\pi(U')$ is open is shown below.



Indeed we have that $\pi^{-1}\big(\pi(U')\big) = U'$.