# Generell Topologi — Exercise Sheet 2

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February 11, 2013

## Guide

To help you to decide which questions to focus on, I have made a few remarks below. A question which is important for one of you may however be less important for another of you — if you need to work on your geometric intuition, for example, prioritise Question 7.

I encourage you to attempt all the questions if you have time — they have all been included for different reasons, to help your understanding. When you come to revise, you should check that you understand all of the solutions that I give.

(1) Questions 5 and 6 are essential, concerning constructions that we will make use of throughout the course.

(2) Questions 1 and 2 will help familiarise you with the axioms of a topological space.

(3) Question 3 tests your understanding of the part of Lecture 1 which approached the construction of a topology on $\mathbb{R}$, and the role of the completeness of $\mathbb{R}$ in this. It also motivates Question 1.4.

(4) Question 4 allows you to practise writing a proof which directly appeals to the axioms of a topological space. The argument is very typical of proofs in this early part of the course.

(5) Question 7 will help develop your geometric intuition, which is a vital aspect of the course. It will also help improve your understanding of subspace and product topologies.

(6) Question 8 and Question 9 give constructions of topological spaces different from those we have met in the lectures so far. Both will help with deepening your understanding of the axioms of a topological space. Both questions are also of wider significance. Questions on future Exercise Sheets will build upon Qeustion 8.

## Questions and Solutions

### 1

**Question.** Let $X := \{a, b, c, d\}$ be a set with four elements. Which of the following sets $\mathcal{O}$ of subsets of $X$ define a topology on $X$?

(1) $\mathcal{O}_1 := \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, d\}, X\}$.

(2) $\mathcal{O}_2 := \{\emptyset, \{a, c\}, \{d\}, \{b, d\}, \{a, c, d\}, X\}$.

(3) $\mathcal{O}_3 := \{\emptyset, \{a\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, X\}$.

**Solution.** The set $\mathcal{O}_2$ defines a topology on $X$.

The set $\mathcal{O}_1$ does not define a topology on $X$, since for example $\{a\} \cup \{b, d\} = \{a, b, d\}$, which does not belong to $\mathcal{O}_1$.

The set $\mathcal{O}_3$ also does not define a topology on $X$, since $\{b, d\} \cap \{a, c, d\} = \{d\}$, which does not belong to $\mathcal{O}_3$.

### 2

**Question.**

(a) Find five topologies on $X := \{a, b, c\}$.

(b) Check whether any of these topologies are the same up to a bijection

$$X \longrightarrow X,$$

namely a relabelling of the elements of $X$. If so, replace it by a different topology. Repeat until you end up with five topologies which are all distinct from each other up to a bijection

$$X \longrightarrow X.$$

(c) Let $\mathcal{O}$ be one of the five topologies that you ended up with in (b). Find all of the subsets of $X$ which are closed with respect to $\mathcal{O}$. Do this for each of the five topologies that you ended up with in (b).

**Solution.** There are exactly nine topologies on $X$ which are distinct from each other up to a bijection

$$X \longrightarrow X.$$

These are listed below. After relabelling if necessary, your topologies should agree with five of those on this list.

(1) $\mathcal{O}_1 := \{\emptyset, X\}$, the indiscrete topology.

(2) $\mathcal{O}_2 := \{\emptyset, \{a\}, X\}$.

(3) $\mathcal{O}_3 := \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$.

(4) $\mathcal{O}_4 := \{\emptyset, \{a\}, \{b, c\}, X\}$.

(5) $\mathcal{O}_5 := \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$.

(6) $\mathcal{O}_6 := \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$.

(7) $\mathcal{O}_7 := \{\emptyset, \{a, b\}, X\}$.

(8) $\mathcal{O}_8 := \{\emptyset, \{a\}, \{a, b\}, X\}$.

(9) $\mathcal{O}_9 := \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, the discrete topology.

The closed sets of $X$ with respect to each of these topologies are as follows.

(1) $\emptyset$, $X$.

(2) $\emptyset$, $\{b, c\}$, $X$.

(3) $\emptyset$, $\{b, c\}$, $\{a, c\}$, $\{c\}$, $X$.

(4) $\emptyset$, $\{b, c\}$, $\{a\}$, $X$.

(5) $\emptyset$, $\{b, c\}$, $\{c\}$, $\{b\}$, $X$.

(6) $\emptyset$, $\{b, c\}$, $\{a, c\}$, $\{c\}$, $\{b\}$, $X$.

(7) $\emptyset$, $\{c\}$, $X$.

(8) $\emptyset$, $\{b, c\}$, $\{c\}$, $X$.

(9) $\emptyset$, $\{b, c\}$, $\{a, c\}$, $\{a, b\}$, $\{c\}$, $\{b\}$,$\{a\}$, $X$.

## 3

**Question.**

(a) Let $\{\,[a_j, b_j]\,\}_{j\in J}$ be a set of (possibly infinitely many) closed intervals in $\mathbb{R}$. Prove that $\bigcap_{j\in J}[a_j, b_j]$ is either a closed interval in $\mathbb{R}$ or $\emptyset$.

(b) Let $a, a', b, b' \in \mathbb{R}$. Find a condition to express exactly when $[a, b] \cup [a', b']$ is disjoint, namely when $[a, b] \cap [a', b'] = \emptyset$. Suppose that $[a, b] \cup [a', b']$ is not disjoint. Prove that in this case $[a, b] \cap [a', b']$ is a closed interval in $\mathbb{R}$.

(c) Let $\{\,[a_j, b_j]\,\}_{j\in J}$ be a set of (possibly infinitely many) closed intervals in $\mathbb{R}$. Give an example to show that $\bigcup_{j\in J}[a_j, b_j]$ need not be a closed interval, even if this union cannot be expressed as a disjoint union of a pair of subsets of $\mathbb{R}$.

**Solution.**

(a) We claim that

$$\bigcap_{j\in J}[a_j,b_j]=\begin{cases}[\sup a_j,\inf b_j] & \text{if } \sup a_j\leq \inf b_j,\\ \emptyset & \text{otherwise.}\end{cases}$$

Let us prove this. If $x\in\bigcap_{j\in J}[a_j,b_j]$, then by definition we have the following.

(1) $x\geq a_j$ for all $j\in J$,

(2) $x\leq b_j$ for all $j\in J$.

By definition of $\sup a_j$, it follows from (1) that $x\geq \sup a_j$. By definition of $\inf b_j$, it follows from (2) that $x\leq \inf b_j$ for all $j\in J$. Thus if $\sup a_j\leq \inf b_j$, we have that $x\in[\sup a_j,\inf b_j]$. We deduce that $\bigcap_{j\in J}[a_j,b_j]\subset[\sup a_j,\inf b_j]$.

Conversely, suppose that $\sup a_j\leq\inf b_j$, and that $x\in[\sup a_j,\inf b_j]$, Then in particular $x\geq\sup a_j$ and $x\leq\inf a_j$. By definition of $\sup a_j$, we deduce that $x\geq a_j$ for all $j\in J$, and by definition of $\inf b_j$, we deduce that $x\leq b_j$ for all $j\in J$. Thus $x\in\bigcap_{j\in J}[a_j,b_j]$, and hence $[\sup a_j,\inf b_j]\subset\bigcap_{j\in J}[a_j,b_j]$.

This proves that if $\sup a_j\leq\inf b_j$, then $\bigcap_{j\in J}[a_j,b_j]=[\sup a_j,\inf b_j]$. Suppose now that $\sup a_j>\inf b_j$. As above, if $x\in\bigcap_{j\in J}[a_j,b_j]$, then:

(1) $x\geq\sup a_j$,

(2) $x\leq\inf b_j$.

But if $\sup a_j>\inf b_j$, then by (1) we have that $x>\inf b_j$. This contradicts (2). We deduce that if $\sup a_j>\inf b_j$, then $\bigcap_{j\in J}[a_j,b_j]=\emptyset$.

(b) We claim that $[a,b]\cup[a',b']$ is disjoint if and only if $\sup\{a,a'\}>\inf\{b,b'\}$, and that if $[a,b]\cup[a',b']$ is not disjoint, then $[a,b]\cup[a',b']=[\inf a_j,\sup b_j]$.

Let us prove this. Without loss of generality (we may relabel $a$ as $a'$ and $b$ as $b'$ if necessary), suppose that $\sup\{a,a'\}=a$ and $\sup\{b,b'\}=b$. Then $\inf\{a,a'\}=a'$, and $\inf\{b,b'\}=b'$.

Suppose that $x\in[a,b]\cup[a',b']$. By definition of $[a,b]\cup[a',b']$, either $x\in[a,b]$ or $x\in[a',b']$. If $x\in[a,b]$, then $x\geq a$. Since $a'=\inf\{a,a'\}$, we have that $a\geq a'$, and we deduce that $x\geq a'$. If $x\in[a',b']$, then again we have that $x\geq a'$.

In addition, if $x\in[a',b']$, then $x\leq b'$. Since $b=\sup\{b,b'\}$, we have that $b\geq b'$, and we deduce that $x\leq b$. If $x\in[a,b]$, then again we have that $x\leq b$.

Putting this together, we have proven that if $x\in[a,b]\cup[a',b']$, then $x\in[a',b]$.

Suppose that $a\leq b'$, and that $x\in[a',b]$. If $x\leq b'$, then $x\in[a',b']$. If $b'<x\leq b$, then $a\leq x<b$, so that $x\in[a,b]$. We have proven that if $x\in[a',b]$, then $x\in[a,b]\cup[a',b]$. We conclude that if $a\leq b'$, then $[a,b]\cup[a',b']=[a',b]$, as claimed.

Suppose now that $a > b'$, and that $x \in [a,b] \cap [a',b']$. Then in particular $x \leq b'$. But also $x \geq a$, so that $x > b'$. Thus we have a contradiction. We conclude that if $a > b'$, we have that $[a,b] \cap [a',b'] = \emptyset$, as claimed.

(c) We have that $\bigcup_{n \in \mathbb{N}} [-1 + 1/n, 1 - 1/n] = (-1,1)$. Check that you understand where the proof given in (b) breaks down!

## 4

**Question.** Motivated by Question 3, consider a pair $(X, \mathcal{C})$ of a set $X$ and a set $\mathcal{C}$ of subsets of $X$ such that the following conditions are satisfied.

(1) $\emptyset$ belongs to $\mathcal{C}$.

(2) $X$ belongs to $\mathcal{C}$.

(3) An intersection of (possibly infinitely many) subsets of $X$ belonging to $\mathcal{C}$ belongs to $\mathcal{C}$.

(4) Let $V$ and $V'$ be subsets of $X$ belonging to $\mathcal{C}$. Then $V \cup V'$ belongs to $\mathcal{C}$.

Then:

(i) Let $(X, \mathcal{O})$ be a topological space. Let $\mathcal{C}$ denote the set of closed subsets of $X$ with respect to $\mathcal{O}$. Prove that $(X, \mathcal{C})$ satisfies the four conditions above.

(ii) Suppose that $(X, \mathcal{C})$ satisfies the four conditions above. Let $\mathcal{O}$ denote the set of subsets $U$ of $X$ such that $X \setminus U$ belongs to $\mathcal{C}$. Prove that $(X, \mathcal{O})$ defines a topological space.

**Solution.**

(i) (1) Since $X \in \mathcal{O}$, we have that $\emptyset = X \setminus X$ belongs to $\mathcal{C}$.

(2) Since $\emptyset \in \mathcal{O}$, we have that $X = X \setminus \emptyset$ belongs to $\mathcal{C}$.

(3) Let $\{V_j\}_{j \in J}$ be a set of closed subsets of $X$. Since $V_j$ is a closed subset of $X$, $X \setminus V_j$ belongs to $\mathcal{O}$ for every $j \in J$. Since $\mathcal{O}$ defines a topology on $X$ in the sense of the lectures, we deduce that $\bigcup_{j \in J} (X \setminus V_j)$ belongs to $\mathcal{O}$.

Now $X \setminus \left( \bigcap_{j \in J} V_j \right) = \bigcup_{j \in J} (X \setminus V_j)$. Thus $X \setminus \left( \bigcap_{j \in J} V_j \right)$ belongs to $\mathcal{O}$, so $\bigcap_{j \in J} V_j$ is a closed subset of $X$.

(4) Since $V$ and $V'$ are closed subsets of $X$, we have that $X \setminus V$ and $X \setminus V'$ belong to $\mathcal{O}$. Since $\mathcal{O}$ defines a topology on $X$, we deduce that $(X \setminus V) \cap (X \setminus V')$ belongs to $\mathcal{O}$.

Now $X \setminus (V \cup V') = (X \setminus V) \cap (X \setminus V')$. Thus $X \setminus (V \cup V')$ belongs to $\mathcal{O}$, so $V \cup V'$ is a closed subset of $X$.

(ii) (1) Since $X$ belongs to $\mathcal{C}$, we have that $\emptyset = X \setminus X$ belongs to $\mathcal{O}$.

(2) Since $\emptyset$ belongs to $\mathcal{C}$, we have that $X = X \setminus \emptyset$ belongs to $\mathcal{O}$.

(3) Let $\{U_j\}_{j \in J}$ be a set of subsets of $X$ such that $X \setminus U_j$ belongs to $\mathcal{C}$ for every $j \in J$. By the third condition which $(X, \mathcal{C})$ satisfies, $\bigcap_{j \in J}(X \setminus U_j)$ belongs to $\mathcal{C}$.

   Now $X \setminus (\bigcup_{j \in J} U_j) = \bigcap_{j \in J}(X \setminus U_j)$. Thus $X \setminus (\bigcup_{j \in J} U_j)$ belongs to $\mathcal{C}$, so $\bigcup_{j \in J} U_j$ belongs to $\mathcal{O}$.

(4) Let $U$ and $U'$ be subsets of $X$ such that $X \setminus U$ and $X \setminus U'$ belong to $\mathcal{C}$. By the fourth condition which $(X, \mathcal{C})$ satisfies, $(X \setminus U) \cup (X \setminus U')$ belongs to $\mathcal{C}$.

   Now $X \setminus (U \cap U') = (X \setminus U) \cup (X \setminus U')$. Thus $X \setminus (U \cap U')$ belongs to $\mathcal{C}$, so $U \cap U'$ belongs to $\mathcal{O}$.

## 5

Let $(Y, \mathcal{O}_Y)$ be a topological space, and let $X$ be a subset of $Y$. Prove that $(X, \mathcal{O}_X)$ defines a topological space, where

$$\mathcal{O}_X := \{X \cap U \mid U \in \mathcal{O}_Y\}.$$

**Solution.** (1) We have that $X \cap \emptyset = \emptyset$, so that $\emptyset \in \mathcal{O}_X$.

(2) We have that $X \cap Y = X$, so that $X \in \mathcal{O}_X$.

(3) Let $\{U_j\}_{j \in J}$ be a set of subsets of Y belonging to $\mathcal{O}_Y$. Since $\mathcal{O}_Y$ defines a topology on $Y$, $\bigcup_{j \in J} U_j$ belongs to $\mathcal{O}_Y$. Thus $X \cap \left(\bigcup_{j \in J} U_j\right)$ belongs to $\mathcal{O}_X$.

   Now $\bigcup_{j \in J}(X \cap U_j) = X \cap \left(\bigcup_{j \in J} U_j\right)$. Hence $\bigcup_{j \in J}(X \cap U_j)$ belongs to $\mathcal{O}_X$.

(4) Let $U$ and $U'$ be subsets of $Y$ belonging to $\mathcal{O}_Y$. Since $\mathcal{O}_Y$ defines a topology on $Y$, $U \cap U'$ belongs to $\mathcal{O}_Y$. Thus $X \cap (U \cap U')$ belongs to $\mathcal{O}_X$.

   Now $(X \cap U) \cap (X \cap U') = X \cap (U \cap U')$. Hence $(X \cap U) \cap (X \cap U')$ belongs to $\mathcal{O}_X$.

## 6

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that $(X \times Y, \mathcal{O}_{X \times Y})$ defines a topological space, where $\mathcal{O}_{X \times Y}$ denote the set of subsets $W$ of $X \times Y$ such that for every $(x, y) \in W$ there exist $U \in \mathcal{O}_X$ and $U' \in \mathcal{O}_Y$ with $x \in U$, $y \in U'$, and $U \times U' \subset W$.

**Solution.** The following argument is quite general. We will see another manifestation of it in Question 8 (a).

(1) We have that $\emptyset$ belongs to $\mathcal{O}_{X \times Y}$. Indeed in this case the defining condition holds vacuously.

(2) We have that $X \times Y$ belongs to $\mathcal{O}_{X \times Y}$. Indeed for every $(x, y) \in X \times Y$, the sets $X \in \mathcal{O}_X$ and $Y \in \mathcal{O}_Y$ have the property that $x \in X$, $y \in Y$, and certainly $X \times Y \subset X \times Y$. (Recall that our definition of $\subset$ included the possibility of equality!).

(3) Let $\{W_j\}_{j \in J}$ be a set of subsets of $X \times Y$ belonging to $\mathcal{O}_{X \times Y}$. Let $(x, y) \in \bigcup_{j \in J} W_j$. By definition of $\bigcup_{j \in J} W_j$, $(x, y)$ belongs to $W_j$ for some $j \in J$.

Since $W_j \in \mathcal{O}_{X \times Y}$, there exist $U \in \mathcal{O}_X$ and $U' \in \mathcal{O}_Y$ such that $x \in U$, $y \in U'$, and $U \times U' \subset W_j$. It remains only to note that $W_j \subset \bigcup_{j \in J} W_j$, so that $U \times U' \subset \bigcup_{j \in J} W_j$.

(4) Let $W$ and $W'$ be subsets of $X \times Y$ belonging to $\mathcal{O}_{X \times Y}$. Let $(x, y) \in W \cap W'$. Since $(x, y) \in W$, there exist $U_0 \in \mathcal{O}_X$ and $U_1 \in \mathcal{O}_Y$ such that $x \in U_0$, $y \in U_1$, and $U_0 \times U_1 \subset W$.

Since $(x, y) \in W'$, there exist $U'_0 \in \mathcal{O}_X$ and $U'_1 \in \mathcal{O}_Y$ such that $x \in U'_0$, $y \in U'_1$, and $U'_0 \times U'_1 \subset W'$. We make the following observations.

(i) $U_0 \cap U'_0 \in \mathcal{O}_X$, since $\mathcal{O}_X$ defines a topology on $X$, and since $U_0, U'_0 \in \mathcal{O}_X$.

(ii) $U_1 \cap U'_1 \in \mathcal{O}_Y$, since $\mathcal{O}_Y$ defines a topology on $Y$, and since $U_1, U'_1 \in \mathcal{O}_Y$.

(iii) $x \in U_0 \cap U'_0$.

(iv) $y \in U_1 \cap U'_1$.

(v) $(U_0 \cap U'_0) \times (U_1 \cap U'_1) = (U_0 \times U_1) \cap (U'_0 \times U'_1) \subset W \cap W'$.

We deduce that $W \cap W'$ belongs to $\mathcal{O}_{X \times Y}$.

## 7

**Question.**

(a) Equip the subset $X := [1, 2] \cup [4, 5)$ of $\mathbb{R}$ with the subspace topology $\mathcal{O}_X$ with respect to $\mathcal{O}_{\mathbb{R}}$. Give and draw an example of a subset $U$ of $X$ which belongs to $\mathcal{O}_X$ in each of the following cases.

(i) $U \cap [4, 5) = \emptyset$, and neither 1 nor 2 belongs to $U$.

(ii) $U \cap [1, 2] = \emptyset$, and 4 does not belong to $U$.

(iii) $U \cap [4, 5) = \emptyset$, and 1 belongs to $U$.

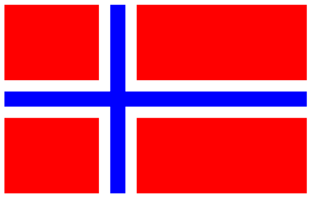
(iv) $U \cap [1, 2] = \emptyset$, and 4 belongs to $U$.

(v) 2 and 4 both belong to $U$.

(vi) $U \cap [1, 2] \neq \emptyset$, $U \cap [4, 5] \neq \emptyset$, and 1, 2, and 4 all do not belong to $U$.

(b) Let $0 < k < 1$ be a real number. Recall the topological spaces $(A_k, \mathcal{O}_{A_k})$ and $(I, \mathcal{O}_I)$ from Lecture 1. Equip $A_k \times I$ with the product topology $(A_k \times I, \mathcal{O}_{A_k \times I})$. Draw $A_k \times I$, and visualise (draw if you can!) some subsets of $A_k \times I$ belonging to $\mathcal{O}_{A_k \times I}$.

(c) Let $X$ be a subset of $\mathbb{R}^2$ consisting of the red and blue parts of the Norwegian flag shown below.



Equip $X$ with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. Draw an example of a subset $U$ of $X$ belonging to $\mathcal{O}_X$ in each of the following cases.

(i) $U$ is contained in the upper right red rectangle.

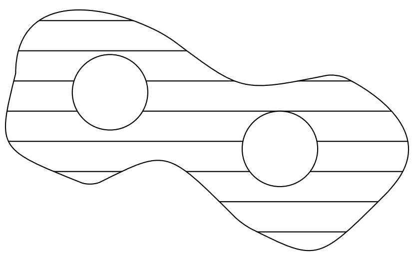
(ii) $U$ intersects all four red rectangles, and both of the blue rectangles.

(iii) $U$ intersects both of the blue rectangles, but none of the red rectangles.

(iv) $U$ intersects only the horizontal blue rectangle, the upper left red rectangle, and the lower left red rectangle.

(v) $U$ intersects only the vertical blue rectangle and the two upper red rectangles.

(d) Let $X$ be a subset of $\mathbb{R}^2$ as shown below, a 'blob' with two open discs cut out.



Equip $X$ with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}\times\mathbb{R}})$. Draw an example of a subset $U$ of $X$ belonging to $\mathcal{O}_X$ in each of the following cases.

(i) $U$ intersects part but not all of one of the circles, and does not intersect the other circle.

(ii) $U$ intersects all of one circle, and part but not all of the other.

(iii) $U$ intersects part but not all of both circles.

(iv) $U$ intersects neither of the two circles.

(v) $U$ intersects all of both circles, but not all of $X$.

**Solution.** Omitted for now as the pictures have not yet been typeset — to come!

A *pre-order* on a set $X$ consists for every ordered pair $(x, x')$ of distinct elements of $X$ of either one or zero arrows from $x$ to $x'$. We require that for any ordered triple $(x, x', x'')$ of distinct elements of $X$, the following condition is satisfied: if there is an arrow from $x$ to $x'$, and an arrow from $x'$ to $x''$, then there is an arrow from $x$ to $x''$.

**Examples.**

(1) Let $X = \{0, 1\}$. There are four pre-orders on $X$, pictured below.

$$\begin{tikzcd} 0 \arrow[r] & 1 \end{tikzcd} \qquad \begin{tikzcd} 0 & 1 \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, shift left] & 1 \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} 0 & 1 \end{tikzcd}$$

The rightmost pre-order should be interpreted as the case that there zero arrows from 0 to 1 and from 1 to 0.

(2) Let $X := \{0, 1, 2\}$. There are 29 possible pre-orders on $X$. A few of them are pictured below.

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \\ & 2 \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, shift left] & 1 \arrow[l, shift left] \\ & 2 \arrow[u] \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 1 \\ & 2 \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 & 1 \arrow[d] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 & 1 \\ & 2 \end{tikzcd}$$

The following are not examples of pre-orders on $X$. Check that you understand why!

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[d] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, shift left] \arrow[dr] & 1 \arrow[l, shift left] \arrow[d, shift left] \\ & 2 \arrow[u, shift left] \end{tikzcd}$$

(3) Let $X := \mathbb{N}$, the set of natural numbers. The following defines a pre-order on $X$.

$$0 \longrightarrow 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5 \longleftarrow 6 \longrightarrow \quad \cdots\cdots\cdots$$

**Question.** Let $X$ be a set equipped with a pre-order. For a pair $(x, x')$ of elements of $X$, we write $x < x'$ if there is an arrow from $x$ to $x'$ or if $x = x'$. Let $\mathcal{O}_X$ denote the set consisting of the subsets $U$ of $X$ with the property that if $x \in U$ and $x'$ has the property that $x < x'$, then $x' \in U$.

(a) Prove that $(X, \mathcal{O}_X)$ defines a topological space.

(b) Which of the four pre-orders on $X := \{0, 1\}$ corresponds to the topology defining the Sierpiński interval? Which corresponds to the discrete topology? Which to the indiscrete topology?

(c) Find a pre-order on $X := \{a, b, c\}$ which corresponds to the topology

$$\mathcal{O} := \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$$

on $X$.

(d) List all of the subsets of $X := \{a, b, c, d\}$ which belong to the topology $\mathcal{O}$ on $X$ corresponding to the following pre-order.

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b & & c \\ & d \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The topological space $(X, \mathcal{O})$ is sometimes known as the *pseudo-circle*.

(e) Let $(X, <)$ be a set equipped with a pre-order, and let $\mathcal{O}_X$ denote the corresponding topology on $X$. Prove that for any set $\{U_j\}_{j \in J}$ of subsets of $X$ belonging to $\mathcal{O}_X$ we have that $\bigcap_{j \in J} U_j \in \mathcal{O}_X$. In particular, this holds even if $J$ is infinite.

**Definition.** A topological space $(X, \mathcal{O})$ is an *Alexandroff space* if for any set $\{U_j\}_{j \in J}$ if subsets of $X$ belonging to $\mathcal{O}$ we have that $\bigcap_{j \in J} U_j \in \mathcal{O}$. In particular this holds even if $J$ is infinite.

**Observation.** Every finite space is an Alexandroff space.

By (a) and (e) we may cook up an Alexandroff space from a pre-order $(X, <)$. We now proceed to establish a converse.

Let $(X, \mathcal{O})$ be an Alexandroff space. For any $x \in X$, let $U_x$ denote the intersection of all subsets of $X$ which contain $x$ and which belong to $\mathcal{O}$. For any $x' \in X$, define $x < x'$ if $U_x \subset U_{x'}$.

**Question (continued).**

(f) Prove that $<$ defines a pre-order on $X$.

(g) Draw the pre-order corresponding to the topology on $X := \{a, b, c, d, e\}$ given by

$$\mathcal{O} := \big\{\emptyset, \{a, b\}, \{c\}, \{d, e\}, \{a, b, c\}, \{c, d, e\}, \{a, b, d, e\}, X\big\}.$$

**Solution.**

(a) (1) Vacuously, we have that $\emptyset \in \mathcal{O}_X$.

(2) It is clear that $X \in \mathcal{O}_X$.

(3) Let $\{U_j\}_{j \in J}$ be a set of subsets of $X$ belonging to $\mathcal{O}_X$. Let $x \in \bigcup_{j \in J} U_j$, and suppose that $x' \in X$ has the property that $x < x'$.

By definition of $\bigcup_{j \in J} U_j$, there is a $j \in J$ such that $x \in U_j$. By definition of $U_j$, we deduce that $x'$ belongs to $U_j$. Thus $x'$ belongs to $\bigcup_{j \in J} U_j$. We conclude that $\bigcup_{j \in J} U_j \in \mathcal{O}_X$.

(4) Let $U$ and $U'$ be subsets of $X$ belonging to $\mathcal{O}_X$. Let $x \in U \cap U'$, and suppose that $x' \in X$ has the property that $x < x'$.

Since $x \in U$, we have that $x' \in U$. Since $x \in U'$, we have that $x' \in U'$. Thus $x' \in U \cap U'$. We conclude that $U \cap U' \in \mathcal{O}_X$.

(b) (1) The pre-order

$$0 \longrightarrow 1$$

corresponds to the Sierpiński interval.

(2) The pre-order

$$0 \qquad\quad 1$$

corresponds to the discrete topology.

(3) The pre-order

$$\begin{tikzcd} 0 \arrow[r, shift left] & 1 \arrow[l, shift left] \end{tikzcd}$$

corresponds to the indiscrete topology.

(c) The pre-order shown below corresponds to $\mathcal{O}$.

$$\begin{tikzcd} a \arrow[r] & b \\ & c \arrow[u] \end{tikzcd}$$

(d) The topology $\mathcal{O}$ on $X$ is the set

$$\{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

(e) Let $x \in \bigcap_{j \in J} U_j$, and suppose that $x' \in X$ has the property that $x < x'$. For every $j \in J$, we have that $x \in U_j$, and hence that $x' \in U_j$. Thus $x' \in \bigcap_{j \in J}$. We deduce that $\bigcap_{j \in J} U_j$ belongs to $\mathcal{O}_X$.

(f) We must prove that if $x < x'$ and $x' < x''$, then $x < x''$. If $x < x'$, we have that $U_x \subset U_{x'}$. If $x' < x''$, we have that $U_{x'} \subset U_{x''}$. Thus if both $x < x'$ and $x' < x''$, we have that $U_x \subset U_{x''}$ as required.

(g) The pre-order corresponding to $(X, \mathcal{O})$ is depicted below.

$$\begin{tikzcd} a \arrow[r, shift left] & b \arrow[l, shift left] & c & d \arrow[r, shift left] & e \arrow[l, shift left] \end{tikzcd}$$

## 9

**Question.** Let $\mathbb{Z}$ denote the set of integers. Let us denote the set of prime numbers by $\mathsf{Spec}(\mathbb{Z})$. For any integer $n$, let

$$V(n) := \{p \in \mathbb{Z} \mid p \text{ is prime, and } p \mid n\}.$$

Prove that

$$\mathcal{O} := \{\mathsf{Spec}(\mathbb{Z}) \setminus V(n) \mid n \in \mathbb{Z}\}$$

defines a topology on $\mathsf{Spec}(\mathbb{Z})$. If you wish you may appeal to Question 4.

**Remark.** This topology is known as the *Zariski topology* on $\mathbb{Z}$. A generalisation defines a topology on the set of prime ideas in any commutative ring, which is the starting point of algebraic geometry.

**Solution.**
Let us prove that $\mathcal{C} := \{V(n) \mid n \in \mathbb{Z}\}$ satisfies conditions (1)-(4) of Question 4.

(1) $V(1) = \emptyset$.

(2) $V(0) = \mathsf{Spec}(\mathbb{Z})$.

(3) Let $J$ be a set of integers, which we allow to be infinite. Recall that the *greatest common divisor* of $J$ is the largest positive integer $m$ such that $m \mid z$ for every $z \in J$. Then $\bigcap_{n \in J} V(n) = V(m)$.

Indeed, if $p$ is a prime which divides $m$, then $p \mid z$ for every $z \in J$. If $p \in \bigcap_{n \in J} V(n)$, then by definition $p \mid n$ for all $n \in J$. By the uniqueness of prime factorisation, this implies that $p \mid m$.

(4) Let $n$ and $n'$ be integers. Then $V(n) \cup V(n') = V(nm)$.

Indeed, if $p \in V(n\mathbb{Z}) \cup V(n'\mathbb{Z})$, then $p \mid n$ or $p \mid n'$. Thus $p \mid nn'$. Conversely, if $p \mid nn'$ then the fact that $p$ is prime implies that $p \mid n$ or $p \mid n'$.