Generell Topologi — Solutions to Exercise Sheet 2

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Guide

The topic of the questions on this exercise sheet is that of a basis, or more generally a sub-basis, of a topological space.

(1) The two facts which are asked to be proven in Question 3 are important, and are used in later questions. They will give you practise in working with a basis for theoretical purposes.

(2) Question 5 introduces the notion of a sub-basis of a topological space, and will check your understanding of Proposition 2.2 in Lecture 2.

(3) Questions 1, 2, 4, 6, 7, and 9 will all help you to gain familiarity with bases and sub-bases in different settings. Question 7 (b) and Question 9 (b) are probably the most difficult of these, and I encourage you to give them a go.

(4) Question 8 has a somewhat different feel, introducing second-countable topological spaces. Second-countability is an important technical notion — it crops up, for example, in the theory of manifolds. Part (b) especially may be quite challenging.

(5) Question 10 continues our investigation of Alexandroff spaces and pre-orders from Exercise Sheet 1. The question essentially asks to show, making use of our new tool of a basis, that Alexandroff topologies on a set X correspond exactly to pre-orders on X, by means of the constructions we became acquainted with on Exercise Sheet 1.

Questions and Solutions

1

Question. Let $(X, \mathcal{O})$ be a topological space. Prove that $\mathcal{O}' := \big\{\{x\} \mid x \in X\big\}$ is a basis for $(X, \mathcal{O})$ if and only if $\mathcal{O}$ is the discrete topology on X.

Solution. If $\mathcal{O}$ is the discrete topology on X, then $\{x\} \in \mathcal{O}$ for every $x \in X$. Moreover every subset of X may be obtained as a union of subsets of X belonging to $\mathcal{O}'$. Thus if $\mathcal{O}$ is the discrete topology on X, then $\mathcal{O}'$ is a basis for $(X, \mathcal{O})$.

If $\mathcal{O}'$ is a basis for $(X, \mathcal{O})$, then $\{x\} \in \mathcal{O}$ for every $x \in X$. Moreover, $\mathcal{O}$ consists exactly of unions of subsets of X belonging to $\mathcal{O}'$. Since every subset of X may be obtained as a union of subsets of X belonging to $\mathcal{O}'$, we deduce that every subset of X belongs to $\mathcal{O}$. Thus if $\mathcal{O}'$ is a basis for $(X, \mathcal{O})$, then $\mathcal{O}$ is the discrete topology on X.

2

Question. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\mathcal{O}'_X$ be a basis for $(X, \mathcal{O}_X)$. Let A be a subset of X, and let $\mathcal{O}_A$ denote the subspace topology upon A. Prove that

$$\mathcal{O}'_A := \left\{ A \cap U' \mid U' \in \mathcal{O}'_X \right\}$$

defines a basis for $(A, \mathcal{O}_A)$.

Solution. Since $U' \in \mathcal{O}_X$ for every $U' \in \mathcal{O}'$, we have that $A \cap U' \in \mathcal{O}_A$. Let U be a subset of A which belongs to $\mathcal{O}_A$. By definition of $\mathcal{O}_A$, we have that $U = A \cap U'$, where $U' \in \mathcal{O}_X$.

Since $\mathcal{O}'_X$ defines a basis for $(X, \mathcal{O}_X)$, there is a set J such that $U' = \bigcup_{j \in J} U'_j$, where $U'_j \in \mathcal{O}'_X$ for all $j \in J$. We then have that

$$\begin{aligned} U &= A \cap U' \\ &= A \cap \Big(\bigcup_{j \in J} U'_j \Big) \\ &= \bigcup_{j \in J} A \cap U'_j. \end{aligned}$$

Thus U can be obtained as a union of subsets of A belonging to $\mathcal{O}'_A$, as required.

3

Question.

(a) Let $(X, \mathcal{O})$ be a topological space, and let $\mathcal{O}'$ be a subset of $\mathcal{O}$. Then $\mathcal{O}'$ defines a basis for $\mathcal{O}$ if and only if for every $U \subset X$ which belongs to $\mathcal{O}$ and every $x \in U$ there is a $U' \in \mathcal{O}'$ such that $x \in U'$ and $U' \subset U$.

(b) Let $(X, \mathcal{O})$ be a topological space, and let $\mathcal{O}'$ be a basis for $(X, \mathcal{O})$. Then $U \subset X$ belongs to $\mathcal{O}$ if and only if for every $x \in U$ there is a $U' \in \mathcal{O}'$ such that $x \in U'$ and $U' \subset U$.

Solution.

(a) Suppose that $U \subset X$ belongs to $\mathcal{O}$. If $\mathcal{O}'$ defines a basis for $(X, \mathcal{O})$, then $U = \bigcup_{j \in J} U'_j$ for a set $\{U'_j\}_{j \in J}$ of subsets of X belonging to $\mathcal{O}'$. By definition of $\bigcup_{j \in J} U'_j$, for every $x \in U$ there is a $j \in J$ such that $x \in U'_j$.

Conversely, suppose that for every $x \in U$ there is $U'_x \subset X$ such that $x \in U'_x$ and $U'_x \subset U$. Then $\bigcup_{x \in X} U'_x \subset U$. But clearly also $U \subset \bigcup_{x \in X} U'_x$. Thus $\bigcup_{x \in X} U'_x = U$, and we have demonstrated that U can be obtained as a union of subsets of X which belong to $\mathcal{O}'$.

(b) Suppose that $U \in \mathcal{O}$. Since $\mathcal{O}'$ is a basis for $(X, \mathcal{O})$, it follows from (a) that if $U \subset X$ belongs to $\mathcal{O}$, then for every $x \in \mathcal{O}'$ there is a $U' \subset X$ such that $x \in U'$ and $U' \subset U$.

Conversely, suppose that for every $x \in U$ there is $U'_x \in \mathcal{O}'$ such that $x \in U'_x$ and $U'_x \subset U$. Then, as in (a), we have that $\bigcup_{x \in X} U'_x = U$. Since $\mathcal{O}'$ is a basis for $(X, \mathcal{O})$, we have that $U'_x \in \mathcal{O}$ for every $x \in X$. Thus $\bigcup_{x \in X} U'_x \in \mathcal{O}$, and we deduce that $U \in \mathcal{O}$.

4

Question.

(a) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that

$$\mathcal{O}' := \{U \times U' \mid U \in \mathcal{O}_X \text{ and } U' \in \mathcal{O}_Y\}$$

defines a basis for the product topology $\mathcal{O}_{X \times Y}$ upon $X \times Y$.

(b) Find a pair of topological spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ for which it is not true that $\mathcal{O}'$ as defined in (a) itself defines a topology on $X \times Y$. Find a pair of topological spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ for which it is true.

Solution.

(a) It is clear that $U \times U' \in \mathcal{O}_{X \times Y}$ for every $U \in \mathcal{O}_X$ and $U' \in \mathcal{O}_Y$. Let $W \in \mathcal{O}_{X \times Y}$. By definition of $\mathcal{O}_{X \times Y}$, for every $(x, y) \in X \times Y$ there exists $U_x \in \mathcal{O}_X$ and $U'_y \in \mathcal{O}_Y$ such that $(x, y) \in U_x \times U'_y$, and $U_x \times U'_y \subset W$. It follows immediately from Question 3 (a) that $\mathcal{O}'$ defines a basis for $(X \times Y, \mathcal{O}_{X \times Y})$.

(b) Almost any example that you try will give a pair of spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ for which $\mathcal{O}'$ does not define a topology on $X \times Y$. The simplest is the following. Let $X = Y = \{0, 1\}$, and let $\mathcal{O}_X = \mathcal{O}_Y = \big\{\emptyset, 1, \{0, 1\}\big\}$. Then

$$\mathcal{O}' = \big\{\emptyset, \{(1,1)\}, \{(0,1), (1,1)\}, \{(1,0), (1,1)\}, X \times Y\big\}$$

does not define a topology on $X \times Y$, since $\{(0,1), (1,1)\} \cup \{(1,0), (1,1)\} = \{(0,1), (1,0), (1,1)\}$, which does not belong to $\mathcal{O}'$.

For an example where $\mathcal{O}'$ does define a topology on $X \times Y$, we may take X and Y to be any pair of sets, each equipped with the discrete or indiscrete topologies.

5

Definition. Let $(X, \mathcal{O})$ be a topological space. A *sub-basis* for $(X, \mathcal{O})$ is a set $\mathcal{O}'$ of subsets of X belonging to $\mathcal{O}$ such that every subset of X belonging to $\mathcal{O}$ can be obtained as a (possibly infinite) union of finite intersections of subsets of X belonging to $\mathcal{O}'$.

Question. Let X be a set, and let $\mathcal{O}'$ be a set of subsets of X. Let $\mathcal{O}$ be the set of subsets of X which can be obtained as a (possibly infinite) union of finite intersections of subsets of X belonging to $\mathcal{O}'$. Suppose that $X \in \mathcal{O}$. Prove that $\mathcal{O}$ defines a topology on X with sub-basis $\mathcal{O}'$.

We refer to $\mathcal{O}$ as the topology *generated* by $\mathcal{O}'$.

Solution. Let us prove that $\mathcal{O}$ defines a topology on X.

(1) The empty set can be thought of as an 'empty union' of subsets of X belonging to $\mathcal{O}'$. If you're not comfortable with this, just include $\emptyset$ in the definition of $\mathcal{O}$.

(2) By assumption, $X \in \mathcal{O}$.

(3) Let $\{U_j\}_{j \in J}$ be a set of subsets of X belonging to $\mathcal{O}$. By definition of $\mathcal{O}$, we have that

$$U_j = \bigcup_{k \in K_j} U'_k$$

for a set $\{U'_k\}_{k \in K_j}$ of subsets of X with the property that for every $k \in K_j$ we have that

$$U'_k = \bigcap_{r \in R_k} U''_r$$

for a finite set $\{U''_r\}_{r \in R_k}$ of subsets of X belonging to $\mathcal{O}'$.

We then have that

$$\begin{aligned} \bigcup_{j \in J} U_j &= \bigcup_{j \in J} \Big(\bigcup_{k \in K_j} U'_k \Big) \\ &= \bigcup_{j \in J} \Big(\bigcup_{k \in K_j} \Big(\bigcap_{r \in R_k} U''_r \Big) \Big) \\ &= \bigcup_{k \in \bigcup_{j \in J} K_j} \Big(\bigcap_{r \in R_k} U''_r \Big). \end{aligned}$$

Thus $\bigcup_{j \in J} U_j$ is a union of finite intersections of subsets of X belonging to $\mathcal{O}'$, and hence $\bigcup_{j \in J} U_j$ belongs to $\mathcal{O}$.

(4) Let U and U' be subsets of X which belong to $\mathcal{O}$. By definition of $\mathcal{O}$ we have that

$$U = \bigcup_{j \in J} U'_j$$

for a set $\{U'_j\}_{j \in J}$ of subsets of X with the property that for every $j \in J$ we have that

$$U'_j = \bigcap_{k \in K_j} U''_k$$

for a finite set $\{U''_k\}_{k \in K_j}$ of subsets of X belonging to $\mathcal{O}'$.

We also have that

$$U' = \bigcup_{j' \in J'} U'_{j'}$$

for a set $\{U'_{j'}\}_{j' \in J'}$ of subsets of X with the property that for every $j' \in J'$ we have that

$$U'_{j'} = \bigcap_{k' \in K'_{j'}} U''_{k'}$$

for a finite set $\{U''_{k'}\}_{k' \in K'_j}$ of subsets of X belonging to $\mathcal{O}'$.

We then have that

$$\begin{aligned} U \cap U' &= \Big(\bigcup_{j \in J} U'_j\Big) \cap \Big(\bigcup_{j' \in J'} U'_{j'}\Big) \\ &= \Big(\bigcup_{j \in J}\Big(\bigcap_{k \in K_j} U''_k\Big)\Big) \cap \Big(\bigcup_{j' \in J'}\Big(\bigcap_{k' \in K_{j'}} U''_{k'}\Big)\Big) \\ &= \bigcup_{(j,j') \in J \times J'} \bigcap_{(k,k') \in K_j \times K_{j'}} U''_k \cap U''_{k'} \end{aligned}$$

Thus $U \cap U'$ is a union of finite intersections of subsets of X belonging to $\mathcal{O}'$, and hence $U \cap U' \in \mathcal{O}$.

It is clear that $\mathcal{O}'$ is a sub-basis for $(X, \mathcal{O}')$.

6

Question.

(a) Let $X = \{a, b, c\}$, and let $\mathcal{O}$ denote the topology

$$\big\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\big\}$$

upon X. Which sets of subsets of X define bases of $(X, \mathcal{O})$? Find a sub-basis of $(X, \mathcal{O}_X)$ which consists of three subsets of X, and to which $\{a, b\}$ and $\{b, c\}$ both belong. Does this sub-basis define a basis?

(b) Let $X = \{a, b, c, d\}$. List the subsets of X belonging to the topology $\mathcal{O}_1$ on X generated by $\mathcal{O}'_1 := \big\{\{a\}, \{d\}, \{b, d\}, \{c, d\}\big\}$. Do the same for the topology $\mathcal{O}_2$ on X generated by $\mathcal{O}'_2 := \big\{\{a\}, \{b, c\}, \{c, d\}\big\}$.

We have two ways to generate a topology on a set X from a set $\mathcal{O}'$ of subsets of X, namely as in Proposition 2.2 of the lecture notes and as in Question 5. We can apply Question 5 to an arbitrary $\mathcal{O}'$, but to apply Proposition 2.2 of the lecture notes to $\mathcal{O}'$, we must have that conditions (1) and (2) of Proposition 2.2 are satisfied.

Note that when both Question 5 and Proposition 2.2 can be applied, we obtain the same topology in both cases. The key thing is not to try to apply Proposition 2.2 when the required conditions are not satisfied!

Solution.

(a) The bases of $(X, \mathcal{O}_X)$ are those subsets of X which have $\{\{a\},\{b\},\{b,c\}\}$ as a subset.

The set $\mathcal{O}' := \{\{a\},\{a,b\},\{b,c\}\}$ defines a sub-basis of $(X,\mathcal{O})$ containing $\{a,b\}$ and $\{b,c\}$. It does not define a basis, since $\{b\}$ cannot be obtained as a union of subsets of X belonging to $\mathcal{O}'$.

(b) All non-empty intersections of subsets of X belonging to $\mathcal{O}'_1$ belong to $\mathcal{O}'_1$. Thus $\mathcal{O}_1$ is obtained by taking unions of subsets of X belonging to $\mathcal{O}'_1$, and we have that

$$\mathcal{O}_1 = \{\emptyset, \{a\}, \{d\}, \{b,d\}, \{c,d\}, \{a,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X\}.$$

Not all non-empty intersections of subsets of X belonging to $\mathcal{O}'_2$ belong to $\mathcal{O}'_2$. To obtain $\mathcal{O}_2$, we first enlarge $\mathcal{O}'_2$ to the set $\mathcal{O}''_2$ which consists of non-empty finite intersections of subsets of X belonging to $\mathcal{O}'_2$. Thus

$$\mathcal{O}''_2 = \{\{a\},\{c\},\{b,c\},\{c,d\}\}.$$

Then $\mathcal{O}_2$ consists of unions of subsets of X belonging to $\mathcal{O}''_2$, and we have that

$$\mathcal{O}_2 = \{\emptyset, \{a\}, \{c\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}, X\}.$$

7

Question.

(a) Let $\mathcal{O}' := \{(a,\infty) \mid a \in \mathbb{R}\} \cup \{(-\infty,b) \mid b \in \mathbb{R}\}$. Prove that $\mathcal{O}'$ defines a sub-basis for the standard topology $\mathcal{O}_{\mathbb{R}}$ on $\mathbb{R}$.

(b) Let $\mathcal{O}' := \{[a,\infty) \mid a \in \mathbb{R}\} \cup \{(-\infty,b) \mid b \in \mathbb{R}\}$. Let $\mathcal{O}$ denote the topology on $\mathbb{R}$ generated by $\mathcal{O}'$. Prove that $\mathcal{O}'' := \{[a,b) \mid a,b \in \mathbb{R}\}$ defines a basis for $(\mathbb{R},\mathcal{O})$. Prove that $\mathcal{O}_{\mathbb{R}} \subset \mathcal{O}$. Is it true that $\mathcal{O} = \mathcal{O}_{\mathbb{R}}$? Prove or disprove it!

Solution.

(a) Let $a, b \in \mathbb{R}$. It suffices to check that the open interval (a,b) can be obtained as a union of intersections of subsets of $\mathbb{R}$ belonging to $\mathcal{O}'$. Indeed, we have that $(a,b) = (-\infty,b) \cap (a,\infty)$.

(b) Let $a, b \in \mathbb{R}$. To prove that $\mathcal{O}''$ is a basis for $\mathcal{O}$ it is sufficient to check that:

(1) $[a, b) \in \mathbb{R}$ for every $a, b \in \mathbb{R}$,

(2) both $[a, \infty)$ and $(-\infty, b)$ can be obtained as a union of subsets of $\mathbb{R}$ belonging to $\mathcal{O}''$.

For (1), we have that $[a, b) = [a, \infty) \cap (-\infty, b)$. For (2), we have for example that $[a, \infty) = \bigcup_{n \in \mathbb{N}} [a, a+n)$ and that $(-\infty, b) = \bigcup_{n \in \mathbb{N}} (b-n, b)$.

To prove that $\mathcal{O}_{\mathbb{R}} \subset \mathcal{O}$, it suffices to check that $(a, b) \in \mathcal{O}$ for every $a, b \in \mathbb{R}$. Indeed, we have for example that $(a, b) = \bigcup_{n \in \mathbb{N}} [a + \frac{1}{n}, b)$, and by the above we have that $[a + \frac{1}{n}, b) \in \mathcal{O}$.

It is not true that $\mathcal{O} = \mathcal{O}_{\mathbb{R}}$. Indeed, $[a, b)$ does not belong to $\mathcal{O}_{\mathbb{R}}$ for any $a, b \in \mathbb{R}$.

For by definition of $\mathcal{O}_{\mathbb{R}}$, if $[a, b)$ belonged to $\mathcal{O}_{\mathbb{R}}$ we would have for some set J that $[a, b) = \bigcup_{j \in J} (a_j, b_j)$, where $a_j, b_j \in \mathbb{R}$ for all $j \in J$. By definition of $[a, b)$, we would then have that $a_j \geq a$ for all $j \in J$. But this contradicts that by definition of $\bigcup_{j \in J} (a_j, b_j)$ we would also have that $a \in (a_j, b_j)$ for some $j \in J$, and hence that $a_j < a$.

Remark. The topology $\mathcal{O}$ on $\mathbb{R}$ is known as the *lower limit topology*. The topological space $(\mathbb{R}, \mathcal{O})$ is sometimes known as the *Sorgenfrey line*.

8

Terminology. We will say that a set X is *countable* if there exists an injection

$$X \longrightarrow \mathbb{N}.$$

Otherwise we say that X is *uncountable*.

Definition. A topological space $(X, \mathcal{O})$ is *second-countable* if it admits a basis $\mathcal{O}'$ which is a countable set.

Remark. In particular, any topological space $(X, \mathcal{O})$ such that X is finite is second-countable. Indeed, $\mathcal{O}$ is then finite, and we may take all of $\mathcal{O}$ as a basis for $(X, \mathcal{O})$.

Remark. There is also a notion of a *first-countable* topological space. We will meet it later in the course.

Remark. Recall that $\mathbb{Z}$ is countable, since there is a bijection

$$\mathbb{Z} \longrightarrow \mathbb{N},$$

given for example by

$$z \mapsto \begin{cases} 2z+1 & \text{if } z \geq 0, \\ -z & \text{if } z < 0. \end{cases}$$

Explicitly, the image of this bijection may be described as: $0, 1, -1, 2, -2, \ldots$.

Recall also that $\mathbb{Q}$ is countable. One way to prove this is as follows.

(i) There is a bijection

$$\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

given by $(n, n') \mapsto 2^n 3^{n'}$.

(ii) Since $\mathbb{Z}$ is countable, we deduce from (i) that $\mathbb{Z} \times \mathbb{Z}$ is countable.

(iii) Let

$$\mathcal{P} := \{(z, z') \in \mathbb{Z} \times \mathbb{Z} \mid z \neq 0 \text{ and } \mathrm{hcf}(z, z') = 1\},$$

where $\mathrm{hcf}(z, z')$ denotes the highest common factor of z and z'.

Since the inclusion map

$$\mathcal{P} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$$

is injective, we deduce from (ii) that $\mathcal{P}$ is countable.

(iv) The map

$$\mathbb{Q} \longrightarrow \mathcal{P}$$

which sends $q \in \mathbb{Q}$ to the unique $(z, z') \in \mathcal{P}$ such that $q = \frac{z}{z'}$ is bijective. Thus $\mathbb{Q}$ is second-countable.

Remark. Let us prove that $\mathbb{R}$ is uncountable. Let I denote the unit interval. We will rely crucially on the fact that if we have a set $\{A_n\}_{n \in \mathbb{N}}$ of closed subsets of I such that

$$A_0 \supset A_1 \supset A_2 \supset \ldots$$

then $\bigcap_{n \in \mathbb{N}} A_n$ is non-empty. We will prove this in a later Exercise Sheet, after we discussed the notion of a compact topological space and proven that $(I, \mathcal{O}_I)$ is compact.

Let us here assume it. Suppose that

$$\mathbb{R} \xrightarrow{f} \mathbb{N}$$

is an injective map. Let

$$I \xrightarrow{i} \mathbb{R}$$

denote the inclusion map. Since both i and f are injective, we have that

$$I \xrightarrow{f \circ i} \mathbb{N}$$

is injective. Let us denote this map by g.

Inductively, we construct for any $n \in \mathbb{N}$ a subset A_n of I with the following properties.

(1) A_n is a closed subset of $(I, \mathcal{O}_I)$.

(2) $A_n \cap g^{-1}\big(\{0, \ldots, n\}\big) = \emptyset$.

(3) $A_{n+1} \subset A_n$.

We may construct a subset A_0 of I which satisfies conditions (1) – (3) by taking the complement in I of any neighbourhood of $g^{-1}(0)$. Suppose that we have constructed A_{n-1}.

Since g is injective, the $g^{-1}(i) \in I$ for $0 \leq i \leq n$ are distinct. Thus we may construct a subset U_n of I which is open in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ and which has the property that

$$U_n = U_n^0 \sqcup U_n^1 \sqcup \ldots \sqcup U_n^n,$$

where U_n^i is a neighbourhood of $g^{-1}(i)$ in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Check that you agree that the construction of such a U_n can be carried out!

Let $A = I \setminus U_n$. Then A is closed in $(I, \mathcal{O}_I)$, and $A \cap g^{-1}\big(\{0, \ldots, n\}\big) = \emptyset$. Define A_n to be $A \cap A_{n-1}$. Then A_n satisfies conditions (1) – (3) above.

Since A_n is closed in $(I, \mathcal{O}_I)$ for every $n \in \mathbb{N}$, we have by Question 4 (i) of Exercise Sheet 1 that $\bigcap_{n \in \mathbb{N}} A_n$ is closed in $(I, \mathcal{O}_I)$. Since A_n satisfies property (2) for every $n \in \mathbb{N}$, we also have that

$$\begin{aligned} \bigcap_{n \in \mathbb{N}} A_n &= \Big(\bigcap_{n \in \mathbb{N}} A_n \Big) \cap I \\ &= \Big(\bigcap_{n \in \mathbb{N}} A_n \Big) \cap g^{-1}(\mathbb{N}) \\ &= \emptyset. \end{aligned}$$

But since A_n satisfies property (3) for every $n \in \mathbb{N}$, we have that $\emptyset \neq \bigcap_{n \in \mathbb{N}} A_n$, as discussed at the beginning of this proof. Thus we have a contradiction.

Question.

(a) Prove that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is second-countable.

(b) Prove that the topology $\mathcal{O}$ on $\mathbb{R}$ defined in Question 7 (b) is not second-countable.

Solution.

(a) The set $\mathcal{O}' := \{(q, r) \mid q, r \in \mathbb{Q}\}$ defines a basis of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. To prove this, it suffices to demonstrate that for any $a, b \in \mathbb{R}$ the open interval (a, b) can be obtained as a union of subsets of $\mathbb{R}$ belonging to $\mathcal{O}'$.

Let $J = \{q \in \mathbb{Q} \mid q > a\}$, and let $J' = \{r \in \mathbb{Q} \mid r < b\}$. Then $\inf J = a$ and $\sup J' = b$. Thus $\bigcup_{(j,j') \in J \times J'} (q_j, r_j) = (a, b)$.

Let us prove that $\mathcal{O}'$ is countable. There is a bijection between $\mathcal{O}'$ and

$$\{(q, q') \in \mathbb{Q} \times \mathbb{Q} \mid q < q'\}.$$

The inclusion map from $\{(q, q') \in \mathbb{Q} \times \mathbb{Q} \mid q < q'\}$ to $\mathbb{Q} \times \mathbb{Q}$ is injective. In the prelude to the question we observed that $\mathbb{Q}$ and $\mathbb{N} \times \mathbb{N}$ are countable. Thus $\mathbb{Q} \times \mathbb{Q}$ is countable. Putting everything together, we have that there is an injection from $\mathcal{O}'$ to $\mathbb{N}$.

(b) Let $\mathcal{O}''$ be a basis of $\mathcal{O}$. By Question 3 (b) we have that for every $x \in \mathbb{R}$ there is a $U_x \in \mathcal{O}''$ such that $x \in U_x$ and $U_x \subset [x, \infty)$. For every $x, x' \in \mathbb{R}$, we have that either $x < x'$ or $x' < x$. Without loss of generality, since we may relabel x as x' and vice versa if necessary, suppose that $x < x'$.

Then $x \notin [x', \infty)$, and thus $x \notin U_{x'}$. Since $x \in U_x$, we deduce that $U_x \neq U_{x'}$.

Thus $x \mapsto U_x$ defines an injective map

$$\mathbb{R} \longrightarrow \mathcal{O}''.$$

We deduce that $\mathcal{O}''$ is uncountable.

9

Question.

(a) Which topology on $\mathbb{R}^2$ is generated by straight lines of infinite length? Does restricting to or allowing straight lines of finite length make any difference?

(b) Prove that there are topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ on $\mathbb{R}$ such that the product topology on $\mathbb{R}^2$ with respect to $(\mathbb{R}, \mathcal{O}_1)$ and $(\mathbb{R}, \mathcal{O}_2)$ is the topology generated by straight lines of infinite length in $\mathbb{R}^2$ parallel to the y-axis.

Solution.

(a) The discrete topology. Indeed, for any $(x', y') \in \mathbb{R}^2$ the intersection of the line $y = x'$ with the line $x = y'$ is $\{(x, y)\}$. Thus $\{(x, y)\}$ is open in the topology $\mathcal{O}$ on $\mathbb{R}^2$ generated by straight lines. By Question 1, we deduce that $\mathcal{O}$ is the discrete topology.

Restricting to or allowing straight lines of finite length does not make any difference.

(b) We may take $\mathcal{O}_1$ to be the indiscrete topology, and $\mathcal{O}_2$ to be the discrete topology. Indeed, let $\mathcal{O}$ denote the product topology with respect to $(\mathbb{R}, \mathcal{O}_1)$ and $(\mathbb{R}, \mathcal{O}_2)$.

Let $\mathcal{O}'$ denote the topology generated by the set $\mathcal{O}''$ of straight lines in $\mathbb{R}^2$ parallel to the y-axis. Then $\mathcal{O}''$ defines a basis for $(\mathbb{R}^2, \mathcal{O}')$. We also have that

$$\mathcal{O}''' := \big\{\{x\} \times \mathbb{R} \mid x \in \mathbb{R}\big\}$$

defines a basis for $(\mathbb{R}^2, \mathcal{O})$.

Since $\mathcal{O}'' = \mathcal{O}'''$, we deduce that $\mathcal{O} = \mathcal{O}'$.

10

This question builds upon Question 8 on Exercise Sheet 1.

Question.

(a) Let $(X,<)$ be a pre-order. Let $\mathcal{O}$ denote the corresponding topology upon X. For every $x \in X$, let $U^x := \{x' \in X \mid x < x'\}$. Prove that $\{U^x \mid x \in X\}$ defines a basis for $(X,\mathcal{O})$.

(b) Let $(X,\mathcal{O})$ be an Alexandroff space. As in Question 8 on Exercise Sheet 1, let U_x denote the intersection of all open subsets of X containing x. Prove that $\{U_x \mid x \in X\}$ defines a basis for $(X,\mathcal{O})$.

(c) Let $(X,\mathcal{O})$ be an Alexandroff space. Define $x << x'$ if $U_x \supset U_{x'}$. Prove that $<<$ defines a pre-ordering on X. This pre-ordering is the 'other way around' from the pre-ordering that was defined in Question 9 (f) of Exercise Sheet 1.

(d) Let $(X,<)$ be a pre-ordering, and let $\mathcal{O}$ denote the corresponding topology on X. Let $<<$ denote the pre-order on X of (c) corresponding to $(X,\mathcal{O})$. Prove that $<<$ coincides with $<$.

(e) Let $(X,\mathcal{O})$ be an Alexandroff space, and let $<$ denote the pre-order on X of (c) corresponding to $(X,\mathcal{O})$. Let $\mathcal{O}^<$ denote the topology on X corresponding to $<$. Prove that $\mathcal{O} = \mathcal{O}^<$.

Solution.

(a) Suppose that $x' \in U^x$, and that $x'' \in X$ has the property that $x' < x''$. By definition of U^x, we have that $x < x'$. Thus $x < x''$, and hence $x'' \in U^x$. We deduce that U^x belongs to $\mathcal{O}$.

Suppose now that $U \subset X$ belongs to $\mathcal{O}$, and $x \in U$. If $x' \in U^x$, then $x < x'$. Thus, by the defining property of U, we have that $x' \in U$. Hence $U_x \subset U$. then $x < x'$. By Question 3, we deduce that $\{U^x \mid x \in X\}$ defines a basis for $(X,\mathcal{O})$.

(b) Since $(X,\mathcal{O})$ is an Alexandroff space, we have that $U_x \in \mathcal{O}$ for every $x \in$X. Suppose that $U \subset X$ belongs to $\mathcal{O}$, and let $x \in U$. By definition of U_x, we have that $U_x \subset U$. By Question 3 (a), we deduce that $\{U_x \mid x \in X\}$ defines a basis for $(X,\mathcal{O})$.

(c) Suppose that $x, x', x'' \in X$ and that $x < x'$ and $x' < x''$. By definition, we then have that $U_x \supset U_{x'}$ and that $U_{x'} \supset U_{x''}$. Then $U_x \supset U_{x''}$, and thus $x < x''$.

(d) Suppose that $x, x' \in X$ and that $x < x'$. For every neighbourhood U of x in $(X,\mathcal{O})$ we have by definition of $\mathcal{O}$ that $x' \in U$, and thus that U is a neighbourhood of x'. It follows immediately that $U_x \supset U_{x'}$, and hence that $x << x'$.

Suppose instead that $x << x'$, so that $U_x \supset U_{x'}$. By definition, U_x is the intersection of all neighbourhoods of x in $(X,\mathcal{O})$. By (a) and the fact that $x < x$, we have that U^x is a neighbourhood of x in $(X,\mathcal{O})$. Thus $U_x \subset U^x$. Hence $U_{x'} \subset U^x$. Since $x' \in U_{x'}$, we deduce that $x' \in U^x$, and thus that $x < x'$.

(e) Suppose that $x' \in X$ has the property that $U_x \supset U_{x'}$. Then since $x' \in U_{x'}$ we have that $x' \in U_x$. Thus

$$\{x' \in X \mid U_x \supset U_{x'}\} \subset U_x.$$

Suppose instead that $x' \in U_x$. Then by definition of U_x, every neighbourhood of x in $(X, \mathcal{O})$ is a neighbourhood of x'. Thus $U_{x'} \subset U_x$, and we have that

$$U_x \subset \{x' \in X \mid U_x \supset U_{x'}\}.$$

We deduce that $U_x = \{x' \in X \mid U_x \supset U_{x'}\}$.

By (a), we have that $\left\{U^x \mid x \in X\right\}$ defines a basis for $(X, \mathcal{O}^<)$. In addition, we have that

$$\begin{aligned} U^x &= \{x' \in X \mid x < x'\} \\ &= \{x' \in X \mid U_x \supset U_{x'}\} \\ &= U_x. \end{aligned}$$

Moreover by (b) we have that $\{U_x \mid x \in X\}$ defines a basis for $(X, \mathcal{O})$. We deduce that $\mathcal{O}' = \mathcal{O}$.