NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **MA3002 General Topology**

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Examination date: Wednesday 4th June 2014

Examination time (from–to): 15:00 – 19:00

Permitted examination support material: D: No printed or hand-written support material is allowed. Permitted calculators: Hewlett Packard HP30S, Citizen SR-270X, Citizen SR-270X College, Casio fx-82ES PLUS.

Other information:
Answer at least four of the five problems. You are welcome to answer all five problems, in which case your four best answers will count. Each problem is worth 25 marks. The apporoximate marks for each part are indicated in brackets. Your grade will be determined by your mark and by a qualitative overall assessment of your answers. Follow carefully the following instructions.

(1) All answers must be justified, unless otherwise stated.

(2) You may assume that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected, and more generally that any interval (open, closed, or half open), equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, is connected. You may also assume that $(I, \mathcal{O}_I)$ is compact, and that $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is Hausdorff for all $n \geq 1$. Except for these facts, you must justify all assertions that you make regarding any examples of topological spaces that you consider, unless otherwise stated.

(3) Except that you must follow (2), you may quote without proof any results from the syllabus, unless otherwise stated.

(4) If a topological space that you consider is homeomorphic to another, you may assert this without proof, unless otherwise stated, or if you are asked to prove this.

(5) You may not make use of the Euler characteristic of a topological space, except in Problem 4.

Good luck!

Language: English

Number of pages: 8

Number pages enclosed: 0

Checked by:

Date Signature

Problem 1

a) Let $(X_0, \mathcal{O}_{X_0})$ and $(X_1, \mathcal{O}_{X_1})$ be topological spaces. Let

$$X_0 \times X_1 \xrightarrow{p_0} X_0$$

be the projection map onto X_0, given by $(x_0, x_1) \mapsto x_0$. Let

$$X_0 \times X_1 \xrightarrow{p_1} X_1$$

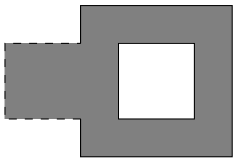
be the projection map onto X_1, given by $(x_0, x_1) \mapsto x_1$. Let $\mathcal{O}$ be the set of subsets W of $X_0 \times X_1$ such that either $W = p_0^{-1}(U_0)$, where U_0 belongs to $\mathcal{O}_{X_0}$, or $W = p_1^{-1}(U_1)$, where U_1 belongs to $\mathcal{O}_{X_1}$, or both. Does $\mathcal{O}$ define a topology on $X_0 \times X_1$? If you answer 'Yes', give a proof. If you answer 'No', give a counterexample. [5 marks]

b) Let Y be the set $\{a, b, c, d\}$. Let $\mathcal{O}_Y$ be the topology on Y given by

$$\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, Y\}.$$

Explain why $(Y, \mathcal{O}_Y)$ is not Hausdorff. You may not appeal to the fact that a finite topological space is Hausdorff if and only if it has the discrete topology! [5 marks]

c) Let Z be the union of $I^2 \setminus \left(\left]\frac{1}{4}, \frac{3}{4}\right[\times \left]\frac{1}{4}, \frac{3}{4}\right[\right)$ and $\left]-\frac{1}{2}, 0\right] \times \left]\frac{1}{4}, \frac{3}{4}\right[$. Let $\mathcal{O}_Z$ be the subspace topology on Z with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Give an example of an open covering of Z with respect to $\mathcal{O}_Z$ which does not admit a finite subcovering. You do not need to justify your answer. [5 marks]

d) There is a bijection

$$I^2 \xrightarrow{f} Z$$

given by

$$(x, y) \mapsto \begin{cases} (x, y) & \text{if } (x, y) \text{ belongs to } I^2 \setminus \left(\left]\frac{1}{4}, \frac{3}{4}\right[\times \left]\frac{1}{4}, \frac{3}{4}\right[\right), \\ (x - \frac{3}{4}, y) & \text{if } (x, y) \text{ belongs to } \left]\frac{1}{4}, \frac{3}{4}\right[\times \left]\frac{1}{4}, \frac{3}{4}\right[. \end{cases}$$

Let $\mathcal{O}$ be the topology on I^2 given by

$$\left\{ f^{-1}(U) \mid U \text{ belongs to } \mathcal{O}_Z \right\}.$$

Prove that $(I^2, \mathcal{O})$ is homeomorphic to $(Z, \mathcal{O}_Z)$. [5 marks]

e) Let $\mathcal{O}_{I^2}$ be the product topology on I^2 with respect to two copies of $(I, \mathcal{O}_I)$. Let $\mathcal{O}$ be the topology on I^2 of part d). Is $(I^2, \mathcal{O})$ homeomorphic to $(I^2, \mathcal{O}_{I^2})$? [5 marks]

Problem 2

a) Let $(X, \mathcal{O}_X)$ be a topological space. Let Y be the set $\{a, b\}$. Let $\mathcal{O}_Y$ be the discrete topology on Y, namely

$$\{\emptyset, \{a\}, \{b\}, Y\}.$$

Prove that if there exists a map

$$X \longrightarrow Y$$

which is both continuous and surjective, then $(X, \mathcal{O}_X)$ is not connected. You may not quote any result from the syllabus. [5 marks]

b) For each i which belongs to $\{-1, 0, 1\}$, let P_i be the subset of $\mathbb{R}^2$ given by

$$\left\{(x, y) \in \mathbb{R}^2 \mid \|(x - i, y)\| = \tfrac{1}{3} \text{ and } y \geq 0\right\}.$$

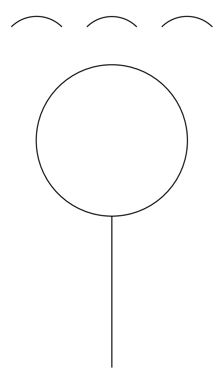
Let C be the subset of $\mathbb{R}^2$ given by

$$\left\{(x, y) \in \mathbb{R}^2 \mid \left\|(x, y + \tfrac{3}{2})\right\| = 1\right\}.$$

Let S be the subset of $\mathbb{R}^2$ given by

$$\left\{(0, y) \in \mathbb{R}^2 \mid -\tfrac{9}{2} \leq y \leq -\tfrac{5}{2}\right\}.$$

Let X be the union of these five sets. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

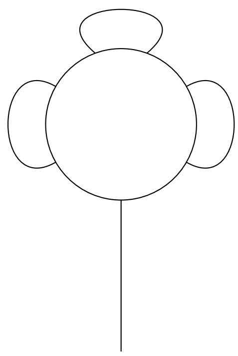


Prove that $(X, \mathcal{O}_X)$ is not connected. [5 marks]

c) Let $\sim$ be the equivalence relation on X generated by:

$$\begin{aligned}
\left(-\tfrac{4}{3}, 0\right) &\sim \left(-\tfrac{\sqrt{3}}{2}, -2\right) \\
\left(-\tfrac{2}{3}, 0\right) &\sim \left(-\tfrac{\sqrt{3}}{2}, -1\right) \\
\left(-\tfrac{1}{3}, 0\right) &\sim \left(-\tfrac{1}{2}, \tfrac{\sqrt{3}-3}{2}\right) \\
\left(\tfrac{1}{3}, 0\right) &\sim \left(\tfrac{1}{2}, \tfrac{\sqrt{3}-3}{2}\right) \\
\left(\tfrac{2}{3}, 0\right) &\sim \left(\tfrac{\sqrt{3}}{2}, -1\right) \\
\left(\tfrac{4}{3}, 0\right) &\sim \left(\tfrac{\sqrt{3}}{2}, -2\right).
\end{aligned}$$

Let $\mathcal{O}_{X/\sim}$ be the quotient topology on $X/\sim$ with respect to $(X, \mathcal{O}_X)$.



Let

$$X \xrightarrow{\pi} X/\sim$$

be the quotient map. Let U be the subset of X given by

$$\left\{(x,y) \in \mathbb{R}^2 \mid \left\|(x, y + \tfrac{3}{2})\right\| = 1 \text{ and } -\tfrac{3}{2} < y \leq -\tfrac{1}{2}\right\}.$$

Does $\pi(U)$ belong to $\mathcal{O}_{X/\sim}$? [5 marks]

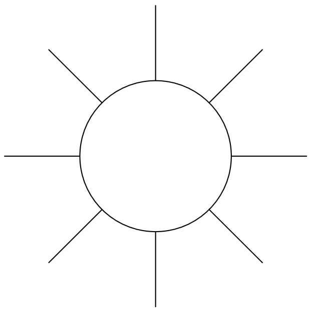
d) For $1 \leq i \leq 8$, where i is an integer, let L_i be the line defined to be the set of (x,y) which belong to $\mathbb{R}^2$, and which satisfy the following conditions.

Line	Condition on x	Condition on y
L_1	$-\frac{2}{\sqrt{2}} \leq x \leq -\frac{1}{\sqrt{2}}$	$y = x$
L_2	$-2 \leq x \leq -1$	$y = 0$
L_3	$-\frac{2}{\sqrt{2}} \leq x \leq -\frac{1}{\sqrt{2}}$	$y = -x$
L_4	$x = 0$	$1 \leq y \leq 2$
L_5	$\frac{1}{\sqrt{2}} \leq x \leq \frac{2}{\sqrt{2}}$	$y = x$
L_6	$1 \leq x \leq 2$	$y = 0$
L_7	$\frac{1}{\sqrt{2}} \leq x \leq \frac{2}{\sqrt{2}}$	$y = -x$
L_8	$x = 0$	$-2 \leq y \leq -1$

Let S^1 be the circle, namely the subset of $\mathbb{R}^2$ given by

$$\left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = 1\right\}.$$

Let Y be the set given by $S^1 \cup \left(\bigcup_{1 \leq i \leq 8} L_i\right)$. Let $\mathcal{O}_Y$ be the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Outline an argument to prove that $(Y, \mathcal{O}_Y)$ is connected. [5 marks]

e) Is $(X, \mathcal{O}_X)$ homeomorphic to $(Y, \mathcal{O}_Y)$? [5 marks]

Problem 3

a) Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X which is not closed in X with respect to $\mathcal{O}_X$. Let C be the closure of A in X with respect to $\mathcal{O}_X$. Let $\mathcal{O}_C$ be the subspace topology on C with respect to $(X, \mathcal{O}_X)$. Prove that $\mathcal{O}_C$ is not the discrete topology on C. In other words, prove that it is not the case that the singleton set $\{c\}$ belongs to $\mathcal{O}_C$ for every c which belongs to C. [5 marks]

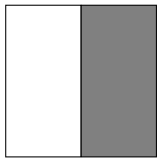
b) Let A be the set

$$\left\{\left(\frac{1}{m}, \frac{1}{n}\right) \mid m, n \in \mathbb{N}\right\}.$$

What is the closure of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$? You do not need to justify your answer. [5 marks]

c) Let A be as in part b). Let X be the closure of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ that you calculated in part b). Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Is $(X, \mathcal{O}_X)$ locally compact? [5 marks]

d) Let Y be the set $I^2 \setminus \left(\left]0, \frac{1}{2}\right[\times \left]0, 1\right[\right)$. Let $\mathcal{O}_Y$ be the subspace topology on Y with respect to $(I^2, \mathcal{O}_{I^2})$.



Let B be the set given by the union of

$$\left\{ (x, 0) \mid \tfrac{1}{8} < x < \tfrac{3}{8} \right\}$$

and $\left]\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right]$. What is the boundary of B in Y with respect to $\mathcal{O}_Y$? You do not need to justify your answer. [5 marks]

e) Let $\partial_{(Y, \mathcal{O}_Y)}(B)$ be the boundary of B in Y with respect to $\mathcal{O}_Y$ that you calculated in d). Let Z be the set

$$\left(B \cup \partial_{(Y, \mathcal{O}_Y)}(B) \right) \cap (\mathbb{Q} \times \mathbb{Q}) .$$

Let $\mathcal{O}_Z$ be the subspace topology on Z with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let $(X, \mathcal{O}_X)$ be as in part c). Does there exist a continuous bijection

$$X \longrightarrow Z?$$

[5 marks]

Problem 4

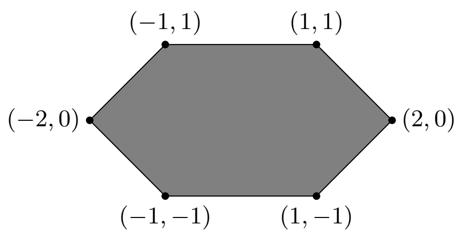
a) Let X be the subset of $\mathbb{R}^2$ given by the union of $[-1, 1] \times [-1, 1]$, the set

$$\left\{ (x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq -1 \text{ and } |y| \leq x + 2 \right\},$$

and the set

$$\left\{ (x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2 \text{ and } |y| \leq -x + 2 \right\}.$$

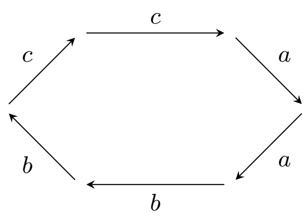
Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Let $\sim$ be the equivalence relation on X generated by:

$$\begin{aligned}
(x, -x+2) &\sim (3-x, 1-x) && \text{for } 1 \leq x \leq 2, \\
(x, -1) &\sim \left(\tfrac{x-3}{2}, \tfrac{-1-x}{2}\right) && \text{for } -1 \leq x \leq 1, \\
(x, x+2) &\sim (3+2x, 1) && \text{for } -2 \leq x \leq -1.
\end{aligned}$$

In other words, we glue together, without a twist, the edges with the same letter in the figure below.



Prove that $(X/\sim, \mathcal{O}_{X/\sim})$ is a surface. You must demonstrate that all the conditions in the definition of a surface are satisfied. Let

$$X \xrightarrow{\pi} X/\sim$$

be the quotient map. If, as part of your answer, you wish to describe a subset $\pi(A)$ of $X/\sim$, where A is a subset of X, it is sufficient to draw a picture of A, indicating any important aspects. You may not appeal to the fact that there is a subset Y of $\mathbb{R}^n$, for some n, such that $(X/\sim, \mathcal{O}_{X/\sim})$ is homeomorphic to $(Y, \mathcal{O}_Y)$, where $\mathcal{O}_Y$ is the subspace topology on Y with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. [12 marks]

b) Equip $(X/\sim, \mathcal{O}_{X/\sim})$ with the structure of a Δ-complex, and use this Δ-complex structure to calculate the Euler characteristic of $(X/\sim, \mathcal{O}_{X/\sim})$. [5 marks]

c) Which surface in the statement of the classification of surfaces is $(X/\sim, \mathcal{O}_{X/\sim})$ homeomorphic to? [3 marks]

d) Give an example of a subset C of $X/\sim$ which has the following properties.

(1) Let $\mathcal{O}_C$ be the subspace topology on C with respect to $(X/\sim, \mathcal{O}_{X/\sim})$. Then $(C, \mathcal{O}_C)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$.

(2) We can perform a surgery on $(X/\sim, \mathcal{O}_{X/\sim})$ with respect to C, giving a surface whose Euler characteristic is

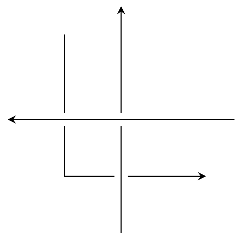
$$\chi\,(X/\sim) + 2,$$

where $\chi\,(X/\sim)$ is the Euler characteristic of $(X/\sim, \mathcal{O}_{X/\sim})$.

You are not required to prove that C has either of these properties. [5 marks]

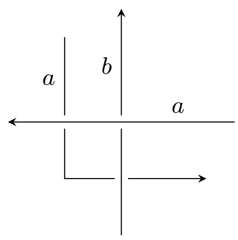
Problem 5

a) The figure below depicts a few arcs of the diagram of an oriented link L.



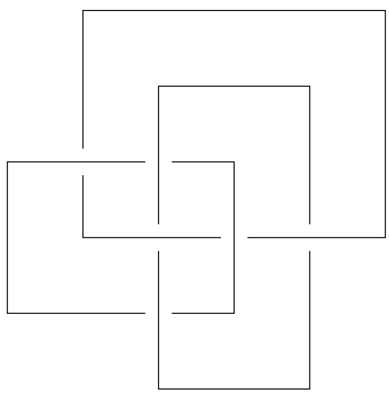
Describe an R3 move on the diagram of L which involves these arcs. [3 marks]

b) Suppose that the two arcs labelled a in the figure below belong to the same component of L. Suppose that the arc labelled b belongs to a different component.

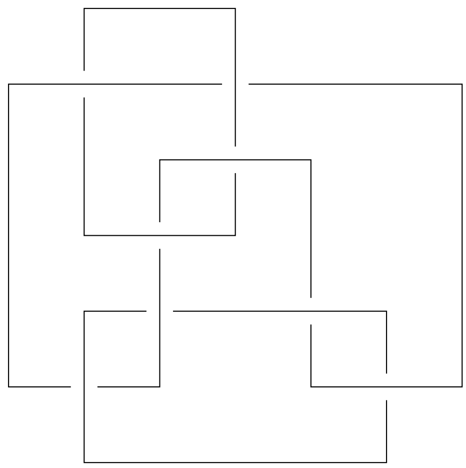


Prove that the linking number of L is unchanged by carrying out your R3 move of part a) upon these arcs. [5 marks]

c) Let 6^3_2 be the link depicted below, the Borromean rings.

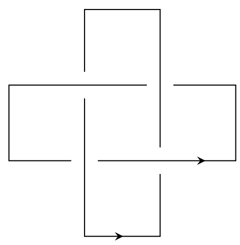


Let 8^3_4 be the link depicted below.



Can knot colouring be used to prove that 6^3_2 is not isotopic to 8^3_4? [8 marks]

d) Let 4^2_1 be the oriented link depicted below, Solomon's knot (sic!).



Using the Jones polynomial, prove that 4^2_1 is genuinely linked, namely that it is not isotopic to the unlink with two components.

You may wish to use the skein relations for an oriented link, which are as follows.

(1) $V_{\bigcirc}(t) = 1$

(2) $t^{-1}V_{L_+}(t) - tV_{L_-}(t) = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) V_{L_0}(t)$

You may appeal without proof to the fact that the Jones polynomial in t of the unlink with two components is $-t^{-\frac{1}{2}} - t^{\frac{1}{2}}$. [9 marks]