MA3002 Generell Topologi — Vår 2014

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Part I.

Point-set foundations

1. Monday 6th January

1.1. Definition of a topological space

Definition 1.1.1. Let X be a set, and let $\mathcal{O}$ be a set of subsets of X. Then $(X, \mathcal{O})$ is a *topological space* if the following hold.

(1) The empty set $\emptyset$ belongs to $\mathcal{O}$.

(2) The set X belongs to $\mathcal{O}$.

(3) Let U be a union of (possibly infinitely many) subsets of X which belong to $\mathcal{O}$. Then U belongs to $\mathcal{O}$.

(4) Let U and U' be subsets of X which belong to $\mathcal{O}$. Then $U \cap U'$ belongs to $\mathcal{O}$.

Remark 1.1.2. By induction, the following holds if and only if (4) holds.

(4') Let J be a finite set, and let $\{U_j\}_{j \in J}$ be a set of subsets of X such that U_j belongs to $\mathcal{O}$ for all $j \in J$. Then $\bigcap_j U_j$ belongs to $\mathcal{O}$.

Terminology 1.1.3. Let $(X, \mathcal{O})$ be a topological space. We refer to $\mathcal{O}$ as a *topology* on X.

A set may be able to be equipped with many different topologies! See §1.4.

1.2. Open and closed subsets

Notation 1.2.1. Let X be a set. By $A \subset X$ we shall mean that A is a subset of X, allowing that A may be equal to X. In the past, you may instead have written $A \subseteq X$.

Terminology 1.2.2. Let $(X, \mathcal{O})$ be a topological space.

(1) Let U be a subset of X. Then U is *open* with respect to $\mathcal{O}$ if U belongs to $\mathcal{O}$.

(2) Let V be a subset of X. Then V is *closed* with respect to $\mathcal{O}$ if $X \setminus V$ is an open subset of X with respect to $\mathcal{O}$.

1.3. Discrete and indiscrete topologies

Example 1.3.1. We can equip any set X with the following two topologies.

(1) The *discrete topology*, consisting of all subsets of X. In other words, the power set of X.

(2) The *indiscrete topology*, given by $\{\emptyset, X\}$.

Remark 1.3.2. By (1) and (2) of Definition 1.1.1, every topology on a set X must contain both $\emptyset$ and X. Thus the indiscrete topology is the smallest topology with which X may be equipped.

1.4. Finite examples of topological spaces

Example 1.4.1. Let $X = \{a\}$ be a set with one element. Then X can be equipped with exactly one topology, given by $\{\emptyset, X\}$. In particular, the discrete topology on X is the same as the indiscrete topology on X.

Remark 1.4.2. The topological space of Example 1.4.1 is important! It is known as the *point*.

Example 1.4.3. Let $X = \{a, b\}$ be a set with two elements. We can define exactly four topologies upon X.

(1) The discrete topology, given by $\big\{\emptyset, \{a\}, \{b\}, X\big\}$.

(2) The topology given by $\big\{\emptyset, \{a\}, X\big\}$.

(3) The topology given by $\big\{\emptyset, \{b\}, X\big\}$.

(4) The indiscrete topology, given by $\big\{\emptyset, X\big\}$.

Remark 1.4.4. Up to the bijection

$$X \xrightarrow{f} X$$

given by $a \mapsto b$ and $b \mapsto a$, or in other words up to relabelling the elements of X, the topologies of (2) and (3) are the same.

Terminology 1.4.5. The topological space $(X, \mathcal{O})$, where $\mathcal{O}$ is the topology of (2) or (3), is known as the *Sierpiński interval*, or *Sierpiński space*.

Remark 1.4.6. In fact (1) – (4) is a list of every possible set of subsets of X which contains $\emptyset$ and X. In other words, every set of subsets of X which contains $\emptyset$ and X defines a topology on X.

Example 1.4.7. Let $X = \{a, b, c\}$ be a set with three elements. We can equip X with exactly twenty nine topologies! Up to relabelling, there are exactly nine.

(1) The set
$$\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$$
defines a topology on X.

(2) The set $\mathcal{O}_X$ given by
$$\{\emptyset, \{a\}, \{c\}, X\}$$
does not define a topology on X. This is because
$$\{a\} \cup \{c\} = \{a, c\}$$
does not belong to $\mathcal{O}_X$, so (3) of Definition 1.1.1 is not satisfied.

(3) The set $\mathcal{O}_X$ given by
$$\{\emptyset, \{a, b\}, \{a, c\}, X\}$$
does not define a topology on X. This is because
$$\{a, b\} \cap \{a, c\} = \{a\}$$
does not belong to $\mathcal{O}_X$, so (4) of Definition 1.1.1 is not satisfied.

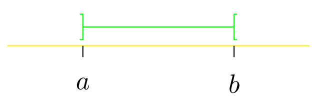
Remark 1.4.8. There are quite a few more 'non-topologies' on X.

1.5. Open, closed, and half open intervals

Notation 1.5.1. Let $\mathbb{R}$ denote the set of real numbers.

Notation 1.5.2. Let $a, b \in \mathbb{R}$.

(1) We denote by $]a, b[$ the set
$$\{x \in \mathbb{R} \mid a < x < b\}.$$



(2) We denote by $]a, \infty[$ the set

$$\{x \in \mathbb{R} \mid x > a\}.$$

(3) We denote by $]-\infty, b[$ the set

$$\{x \in \mathbb{R} \mid x < b\}.$$

(4) We sometimes denote $\mathbb{R}$ by $]-\infty, \infty[$.

Terminology 1.5.3. We shall refer to any of (1) – (4) in Notation 1.5.2 as an *open interval*.

Remark 1.5.4. We shall never use the notation (a, b), (a, ∞), $(-\infty, b)$, or $(-\infty, \infty)$ for an open interval. In particular, for us (a, b) will always mean an ordered pair of real numbers a and b.

Notation 1.5.5. Let $a, b \in \mathbb{R}$. We denote by $[a, b]$ the set

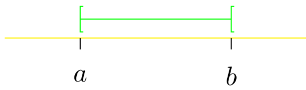
$$\{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

Terminology 1.5.6. We shall refer to $[a, b]$ as a *closed interval*.

Notation 1.5.7. Let $a, b \in \mathbb{R}$.

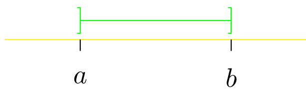
(1) We denote by $[a, b[$ the set

$$\{x \in \mathbb{R} \mid a \leq x < b\}.$$



(2) We denote by $]a, b]$ the set

$$\{x \in \mathbb{R} \mid a < x \leq b\}.$$



(3) We denote by $[a, \infty[$ the set

$$\{x \in \mathbb{R} \mid x \geq a\}.$$

(4) We denote by $]-\infty, b]$ the set

$$\{x \in \mathbb{R} \mid x \leq b\}.$$

Terminology 1.5.8. We shall refer to any of (1) – (4) of Notation 1.5.7 as a *half open interval*.

Terminology 1.5.9. By an *interval* we shall mean a subset of $\mathbb{R}$ which is either an open interval, a closed interval, or a half open interval.

1.6. Standard topology on $\mathbb{R}$

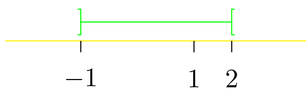
Definition 1.6.1. Let $\mathcal{O}_{\mathbb{R}}$ denote the set of subsets U of $\mathbb{R}$ with the property that, for every $x \in U$, there is an open interval I such that $x \in I$ and $I \subset U$.

Observation 1.6.2. We have that $\mathbb{R}$ belongs to $\mathcal{O}_{\mathbb{R}}$. Moreover $\emptyset$ belongs to $\mathcal{O}_{\mathbb{R}}$, since the required property vacuously holds.

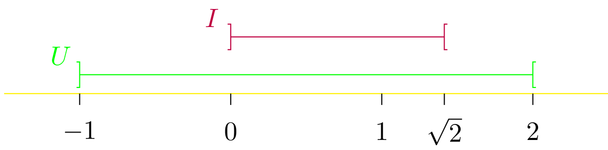
Example 1.6.3. Let U be an open interval $]a, b[$.

Then U belongs to $\mathcal{O}_{\mathbb{R}}$. For every $x \in U$, we can take the corresponding open interval I such that $x \in I$ and $I \subset U$ to be U itself.

There are infinitely many other possibilities for I. For instance, suppose that U is the open interval $]-1, 2[$. Let $x = 1$.



We can take I to be $]-1, 2[$, but also for example $\left]0, \sqrt{2}\right[$.



E1. Exercises for Lecture 1

E1.1. Exam questions

Task E1.1.1. Let $X = \{a, b, c, d\}$. Which of the following defines a topology on X?

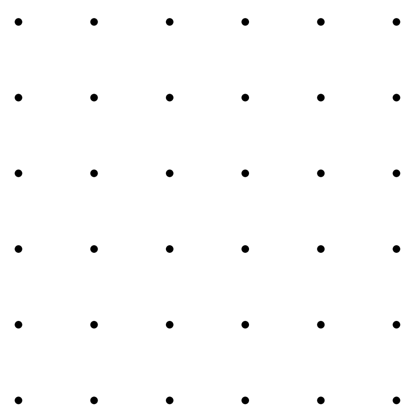
(1) $\{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, d\}, X\}$

(2) $\{\emptyset, \{a, c\}, \{d\}, \{b, d\}, \{a, c, d\}, X\}$

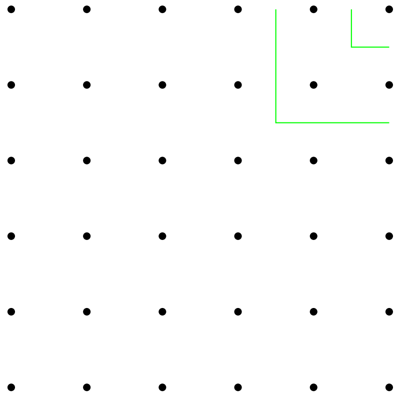
(3) $\{\emptyset, \{a\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, X\}$

Task E1.1.2.

(1) Let X be an $n \times n$ grid of integer points in $\mathbb{R}^2$, where $n \in \mathbb{N}$.

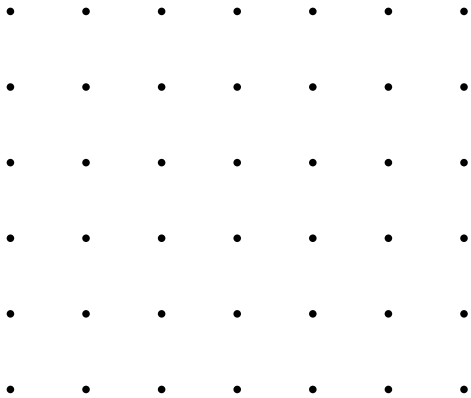


Let $\mathcal{O}$ be the set of subsets of X which are $m \times m$ grids, for $0 \leq m \leq n$, at the top right corner. Think of the case $m = 0$ as the empty set.

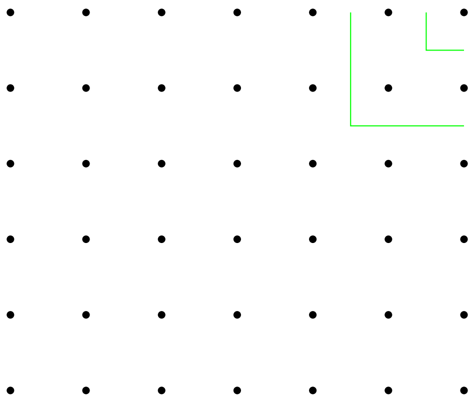


Does $(X, \mathcal{O})$ define a topological space?

(2) Let Y be an $(n+1) \times n$ grid of integer points in $\mathbb{R}^2$.



Let $\mathcal{O}$ be the set of subsets of Y which are $m \times m$ grids, for $0 \leq m \leq n$, at the top right corner. Again, think of the case $m = 0$ as the empty set.

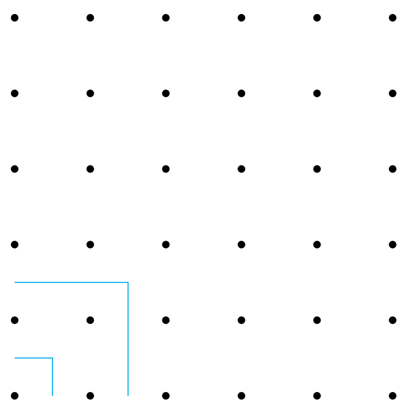


Does $(Y, \mathcal{O})$ define a topological space?

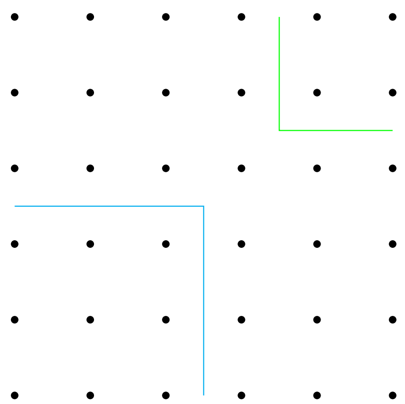
(3) Let X be as in (1). Suppose that $n \geq 3$. Let $\mathcal{O}'$ be the union of the following sets of subsets of X.

(a) $\mathcal{O}$.

(b) The set of subsets of X which are $m \times m$ grids, for $0 \leq m \leq n$, at the bottom left corner.



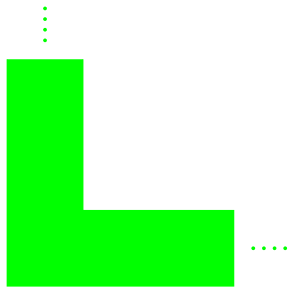
(c) Unions of subsets of X of the kind considered in (a) and (b). For instance, the union of a 3×3 grid at the bottom left corner, and a 2×2 grid at the top right corner.



Does $(X, \mathcal{O}')$ define a topological space?

Task E1.1.3 (Continuation Exam, August 2013)**.** Let X denote the set

$$([0,1] \times]0,\infty]) \cup (]0,\infty] \times [0,1]) .$$



Let $\mathcal{O}$ be the union of $\{\emptyset, X\}$, the set

$$\{[0,1] \times [0,n] \mid n \in \mathbb{N}\}$$

and the set

$$\{[0,n] \times [0,1] \mid n \in \mathbb{N}\}.$$

Is $(X, \mathcal{O})$ a topological space?

E1.2. In the lecture notes

Task E1.2.1. Let X be a set.

(1) Verify that conditions (1) – (4) of Definition 1.1.1 are satisfied by the discrete topology on X.

(2) Verify that conditions (1) – (4) of Definition 1.1.1 are satisfied by the indiscrete topology on X.

Task E1.2.2.

(1) Check that you agree that (1) of Example 1.4.3 is the discrete topology.

(2) Verify that (2) and (3) of Example 1.4.3 define topologies.

Task E1.2.3.

(1) Verify that (1) of Example 1.4.7 defines a topology.

(2) Can you find the nine different topologies, up to relabelling, on a set with three elements?

(3) Find four examples of non-topologies on a set with three elements, in addition to (2) and (3) of Example 1.4.7.

E1.3. For a deeper understanding

Task E1.3.1. Let X be a set. Let $\mathcal{C}$ be a set of subsets of X such that the following hold.

(1) The empty set $\emptyset$ belongs to $\mathcal{C}$.

(2) The set X belongs to $\mathcal{C}$.

(3) Let V be an intersection of (possibly infinitely many) subsets of X which belong to $\mathcal{C}$. Then V belongs to $\mathcal{C}$.

(4) Let V and V' be subsets of X which belong to $\mathcal{C}$. Then $V \cup V'$ belongs to $\mathcal{C}$.

Let $\mathcal{O}$ be given by

$$\{X \setminus V \mid V \text{ belongs to } \mathcal{C}\}.$$

Prove that $(X, \mathcal{O})$ is a topological space.

Remark E1.3.2. Conversely, let $(X, \mathcal{O})$ be a topological space. Let $\mathcal{C}$ denote the set of closed subsets of X. Then $\mathcal{C}$ satisfies (1) – (4) of Task E1.3.1.

Task E1.3.3 (Longer)**.** Let I be a subset of $\mathbb{R}$. Prove that I is an interval if and only if it has the following property: if $x < y < x'$ for $x, x' \in I$ and $y \in \mathbb{R}$, then $y \in I$. For proving that I is an interval if this condition is satisfied, you may wish to proceed as follows.

(1) Suppose that I is bounded. Denote the greatest lower bound of I by a, and denote the least upper bound of I by b. Prove that if $a < y < b$, then $y \in I$.

(2) Using this, deduce that I is $]a, b[$, $[a, b]$, $[a, b[$, or $]a, b]$.

(3) Give a proof when I is not bounded.

Remark E1.3.4. Task E1.3.3 relies crucially on the existence of a least upper bound for a subset of $\mathbb{R}$ which is bounded above, and on the existence of a greatest lower bound for a subset of $\mathbb{R}$ which is bounded below. This is known as the *completeness* of $\mathbb{R}$.

We shall demonstrate in later lectures that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ has important properties. To use a couple of terms which we shall define later, it is *connected* and *locally compact*. The proofs ultimately rest upon the completeness of $\mathbb{R}$, via Task E1.3.3.

Task E1.3.5. Let I_0 and I_1 be intervals. Prove that $I_0 \cap I_1$ is an interval. You may wish to appeal to Task E1.3.3.

E1.4. Exploration — Alexandroff topological spaces

Definition E1.4.1. Let X be a set, and let $X \Diamond X$ denote the set of ordered pairs (x_0, x_1) of X such that x_0 is not equal to x_1. A *pre-order* on X is the data of a map

$$X \Diamond X \xrightarrow{\chi} \{0, 1\},$$

or, in other words, for every ordered pair (x_0, x_1) of distinct elements of X, an element of the set $\{0, 1\}$. We require that for any ordered triple (x_0, x_1, x_2) of mutually distinct elements of X, such that $\chi(x_0, x_1) = 1$ and $\chi(x_1, x_2) = 1$, we have that $\chi(x_0, x_2) = 1$.

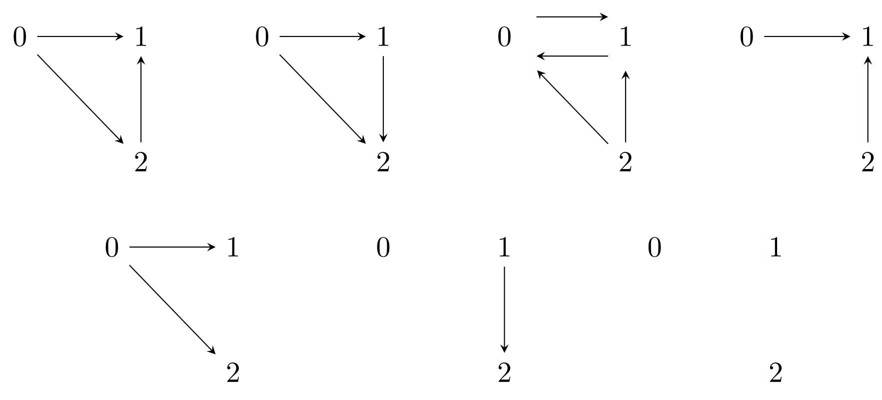
Terminology E1.4.2. There is an *arrow from x_0 to x_1* if $\chi(x_0, x_1) = 1$. We depict this as follows.

$$x_0 \longrightarrow x_1$$

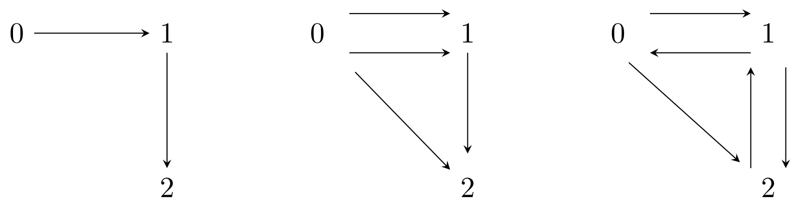
Example E1.4.3. Let $X = \{0, 1\}$. There are four pre-orders on X, pictured below.

The rightmost diagram should be interpreted as: $\chi(0, 1) = 0$ and $\chi(1, 0) = 0$.

Example E1.4.4. Let $X = \{0, 1, 2\}$. There are 29 pre-orders on X. A few are pictured below.



Example E1.4.5. The following are not examples of pre-orders on X.



Task E1.4.6. Why do the diagrams of Example E1.4.5 not define pre-orders.?

Example E1.4.7. The following defines a pre-order on $\mathbb{N}$.

$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5 \longrightarrow 6 \longleftarrow 7 \longrightarrow \quad \cdots\cdots\cdots$$

Notation E1.4.8. Let X be a set, and let χ be a pre-order on X. For any pair (x_0, x_1) of elements of X, we write $x_0 < x_1$ if eiher there is an arrow from x_0 to x_1 or $x_0 = x_1$.

Definition E1.4.9. Let $\mathcal{O}_<$ denote the set of subsets U of X with the property that if $x \in U$ and x' has the property that $x < x'$, then $x' \in U$.

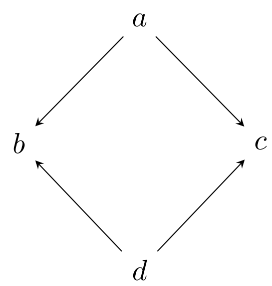
Task E1.4.10. Prove that $(X, \mathcal{O}_<)$ is a topological space.

Task E1.4.11. Which of the four pre-orders of Example E1.4.3 corresponds to the topology defining the Sierpiński interval? Which corresponds to the discrete topology? Which to the indiscrete topology?

Task E1.4.12. Find a pre-order on $X = \{a, b, c\}$ which corresponds to the topology $\mathcal{O}$ on X given by

$$\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}\,.$$

Task E1.4.13. List all the subsets of $X = \{a, b, c, d\}$ which belong to the topology $\mathcal{O}$ on X corresponding to the following pre-order.



The topological space $(X, \mathcal{O})$ is sometimes known as the *pseudo-circle*.

Task E1.4.14. Let $(X, <)$ be a set equipped with a pre-order, and let $\mathcal{O}_<$ denote the corresponding topology on X. Prove that, for any set $\{U_j\}_{j \in J}$ of subsets of X belonging to $\mathcal{O}_X$, we have that $\bigcap_{j \in J} U_j$ belongs to $\mathcal{O}_<$. In particular, this holds even if J is infinite.

Remark E1.4.15. In other words, $(X, \mathcal{O}_<)$ is an Alexandroff topological space.

Notation E1.4.16. Let $(X, \mathcal{O})$ be an Alexandroff topological space. For any $x \in X$, let U_x denote the intersection of all subsets of X which contain x and which belong to $\mathcal{O}$.

Definition E1.4.17. Let $(X, \mathcal{O})$ be an Alexandroff topological space. For any $x_0, x_1 \in X$, define $x_0 < x_1$ if $U_{x_1} \subset U_{x_0}$.

Task E1.4.18. Prove that $<$ defines a pre-order on X.

Task E1.4.19. Let $X = \{a, b, c, d, e\}$, and let $\mathcal{O}$ denote the topology on X given by

$$\{\emptyset, \{a, b\}, \{c\}, \{d, e\}, \{a, b, c\}, \{c, d, e\}, \{a, b, d, e\}, X\}.$$

Draw the pre-order corresponding to $(X, \mathcal{O})$.

E1.5. Exploration — Zariski topologies

Notation E1.5.1. Let $\mathbb{Z}$ denote the set of integers.

Notation E1.5.2. Let $\mathsf{Spec}(\mathbb{Z})$ denote the set of prime numbers.

Notation E1.5.3. For any integer n, let $V(n)$ denote the set

$$\{p \in \mathbb{Z} \mid p \text{ is prime, and } p \mid n\}.$$

Definition E1.5.4. Let $\mathcal{O}$ denote the set

$$\{\mathsf{Spec}(\mathbb{Z}) \setminus V(n) \mid n \in \mathbb{Z}\}.$$

Task E1.5.5. Prove that $(\mathsf{Spec}(\mathbb{Z}), \mathcal{O})$ is a topological space. You may wish to make use of Task E1.3.1.

Terminology E1.5.6. The topology $\mathcal{O}$ on $\mathsf{Spec}(\mathbb{Z})$ is known as the *Zariski topology*.

Remark E1.5.7 (Ignore if you have not met the notion of a ring before)**.** Generalising this, one can define a topology on the set of prime ideals of any commutative ring. This is a point of departure for *algebraic geometry*.