MA3002 Generell Topologi — Vår 2014

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10.1 Connectedness in finite examples

Example 10.1.1. Let $X = \{a, b\}$ be a set with two elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{b\}, X\}.$$

The only way to express X as a disjoint union of subsets which are not empty is:

$$X = \{a\} \sqcup \{b\}.$$

However, $\{a\}$ does not belong to $\mathcal{O}_X$. We conclude that $(X, \mathcal{O}_X)$ is connected.

Example 10.1.2. Let $X = \{a, b, c, d, e\}$ be a set with five elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{c, d, e\}, \{a, b, c, d\}, \{a, c, d, e\}, X\}.$$

The following hold.

(1) We have that $X = \{a, b\} \sqcup \{c, d, e\}$.

(2) Both $\{a, b\}$ and $\{c, d, e\}$ belong to $\mathcal{O}_X$.

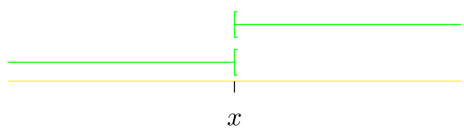
We conclude that $(X, \mathcal{O}_X)$ is not connected.

10.2 $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not connected

Example 10.2.1. Let $\mathbb{Q}$ denote the rational numbers. Let $\mathcal{O}_{\mathbb{Q}}$ denote the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $x \in \mathbb{R}$ be irrational. For instance, we can take x to be $\sqrt{2}$. The following hold.

(1) Since x is irrational, we have that

$$\mathbb{Q} = (\mathbb{Q} \cap]-\infty, x[) \sqcup (\mathbb{Q} \cap]x, \infty[).$$



(2) By Example 1.6.3, we have that $]-\infty, x[$ belongs to $\mathcal{O}_{\mathbb{R}}$. By definition of $\mathcal{O}_{\mathbb{Q}}$, we deduce that $\mathbb{Q} \cap]-\infty, x[$ belongs to $\mathcal{O}_{\mathbb{Q}}$.

(3) By Example 1.6.3, we have that $]x, \infty[$ belongs to $\mathcal{O}_{\mathbb{R}}$. By definition of $\mathcal{O}_{\mathbb{Q}}$, we thus have that $\mathbb{Q} \cap]x, \infty[$ belongs to $\mathcal{O}_{\mathbb{Q}}$.

We conclude that $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not connected.

10.3 A characterisation of connectedness

Proposition 10.3.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{0, 1\}$ be equipped with the discrete topology. A topological space $(X, \mathcal{O}_X)$ is connected if and only if there does not exist a surjective, continuous map

$$X \longrightarrow \{0, 1\}.$$

Proof. Suppose that there exists a surjective continuous map

$$X \xrightarrow{f} \{0, 1\}.$$

The following hold.

(1) Both $\{0\}$ and $\{1\}$ belong to the discrete topology on $\{0, 1\}$. Since f is continuous, we thus have that both $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ belong to $\mathcal{O}_X$.

(2) Since f is surjective, neither $f^{-1}(\{0\})$ nor $f^{-1}(\{1\})$ is empty.

(3) We have that

$$\begin{aligned} f^{-1}(\{0\}) \cup f^{-1}(\{1\}) &= f^{-1}(\{0, 1\}) \\ &= X. \end{aligned}$$

(4) We have that

$$f^{-1}(\{0\}) \cap f^{-1}(\{1\}) = \{x \in X \mid f(x) = 0 \text{ and } f(x) = 1\}.$$

Since f is a well-defined map, the set

$$\{x \in X \mid f(x) = 0 \text{ and } f(x) = 1\}$$

is empty. We deduce that

$$f^{-1}(\{0\}) \cap f^{-1}(\{1\})$$

is empty.

By (3) and (4), we have that

$$X = f^{-1}(\{0\}) \sqcup f^{-1}(\{1\}).$$

We conclude, by (1) and (2), that $(X, \mathcal{O}_X)$ is not connected.

Conversely, suppose that $(X, \mathcal{O}_X)$ is not connected. Then there are subsets X_0 and X_1 of X with the following properties.

(1) Neither X_0 nor X_1 is empty, and both belong to $\mathcal{O}_X$.

(2) We have that $X = X_0 \sqcup X_1$.

Let

$$X \xrightarrow{f} \{0,1\}$$

be the map given by

$$\begin{cases} x \mapsto 0 & \text{if } x \in X_0, \\ x \mapsto 1 & \text{if } x \in X_1. \end{cases}$$

By (2), we have that f is well-defined. Since neither X_0 nor X_1 is empty, we have that f is surjective. Moreover we have that $f^{-1}(\{0\}) = X_0$, and that $f^{-1}(\{1\}) = X_1$. Since both X_0 and X_1 belong to $\mathcal{O}_X$, we deduce that f is continuous. □

Remark 10.3.2. For theoretical purposes, it is often very powerful to have a characterisation of a mathematical concept in terms of maps. We shall see that Proposition 10.3.1 is very useful for carrying out proofs involving connected topological spaces.

10.4 $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected

Proposition 10.4.1. The topological space $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.

Remark 10.4.2. This is one of the most important facts in the course! It is a 'low-level' result, which relies fundamentally on the completeness of $\mathbb{R}$. Task E10.2.1 guides you through a proof.

To put it another way, Proposition 10.4.1 is the bridge between set theory and topology upon which connectedness rests. After we have proven it, we shall not need again to work in a 'low-level' way with $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ in matters concerning connectedness. We shall be able to argue entirely topologically.

Remark 10.4.3. Nevertheless Proposition 10.4.1 is intuitively clear. Something would be wrong with our notion of a connected topological space if it did not hold! It is for this very reason that Proposition 10.4.1 requires a 'low-level' proof. We have to think very carefully about how our intuitive understanding that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected can be captured rigorously within the framework in which we are working.

10.5 Continuous surjections with a connected source

Proposition 10.5.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that $(X, \mathcal{O}_X)$ is connected. Suppose that there exists a continuous, surjective map

$$X \xrightarrow{f} Y.$$

Then $(Y, \mathcal{O}_Y)$ is connected.

Proof. Let $\{0, 1\}$ be equipped with the discrete topology. Suppose that

$$Y \xrightarrow{g} \{0, 1\}$$

is a continuous, surjective map. Since f is continuous, we have by Proposition 5.3.1 that

$$X \xrightarrow{g \circ f} \{0, 1\}$$

is continuous. Since f is surjective, we moreover have that $g \circ f$ is surjective. By Proposition 10.3.1, this contradicts our hypothesis that $(X, \mathcal{O}_X)$ is connected.

We deduce there does not exist a continuous, surjective map

$$Y \xrightarrow{g} \{0, 1\}.$$

By Proposition 10.3.1, we conclude that $(Y, \mathcal{O}_Y)$ is connected. □

Corollary 10.5.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Suppose that $(X, \mathcal{O}_X)$ is connected. Then $(Y, \mathcal{O}_Y)$ is connected.

Proof. Since f is a homeomorphism, f is in particular a continuous bijection. By Task E7.2.1, a bijection in the sense of Definition 7.1.1 is in particular surjective. By Proposition 10.5.1, we deduce that $(Y, \mathcal{O}_Y)$ is connected. □

Corollary 10.5.3. Let $(X, \mathcal{O}_X)$ be a connected topological space. Let $\sim$ be an equivalence relation on X. Then $(X/\sim, \mathcal{O}_{X/\sim})$ is connected.

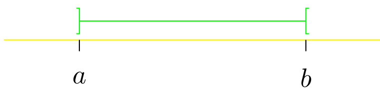
Proof. Let

$$X \xrightarrow{\pi} X/\sim$$

denote the quotient map with respect to $\sim$. By Remark 6.1.9, we have that π is continuous. Moreover π is surjective. By Proposition 10.5.1, we deduce that $(X/\sim, \mathcal{O}_{X/\sim})$ is connected. □

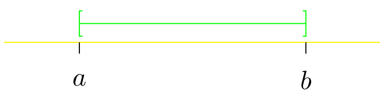
10.6 Geometric examples of connected topological spaces

Example 10.6.1. Let $]a, b[$ be an open interval. Let $\mathcal{O}_{]a,b[}$ denote the subspace topology on $]a, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



By Example 7.3.10, we have that $(]a, b[\,, \mathcal{O}_{]a,b[})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Corollary 10.5.2, we deduce that $(]a, b[\,, \mathcal{O}_{]a,b[})$ is connected.

Example 10.6.2. Let $[a, b]$ be a closed interval, where $a < b$. Let $\mathcal{O}_{[a,b]}$ denote the subspace topology on $[a, b]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

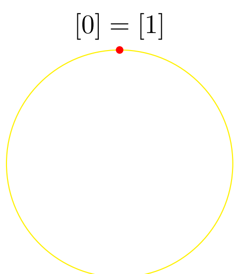


We have that $\mathsf{cl}_{(\mathbb{R}, \mathcal{O}_{\mathbb{R}})}(]a, b[)$ is $[a, b]$. By Example 10.6.1 and Corollary E10.3.4, we deduce that $([a, b]\,, \mathcal{O}_{[a,b]})$ is connected.

Remark 10.6.3. We can go beyond Example 10.6.1 and Example 10.6.2. Let X be a subset of $\mathbb{R}$, and let $\mathcal{O}_X$ be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then $(X, \mathcal{O}_X)$ is connected if and only if X is an interval. To prove this is the topic of Task E10.3.5.

Example 10.6.4. As in Example 6.3.1, let $\sim$ be the equivalence relation on I generated by $0 \sim 1$.

By Example 10.6.2, we have that $(I, \mathcal{O}_I)$ is connected. By Corollary 10.5.3, we deduce that $(I/\sim, \mathcal{O}_{I/\sim})$ is connected.



By Task E7.3.10, there is a homeomorphism

$$I/\sim \longrightarrow S^1.$$

By Corollary 10.5.2, we deduce that $(S^1, \mathcal{O}_{S^1})$ is connected.

10.7 Products of connected topological spaces

Proposition 10.7.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be connected topological spaces. Then $(X \times Y, \mathcal{O}_{X \times Y})$ is connected.

Proof. Let $\{0,1\}$ be equipped with the discrete topology. Let

$$X \times Y \xrightarrow{f} \{0,1\}$$

be a continuous map. Our argument has two principal steps.

(1) Suppose that x belongs to X. By Task E5.1.5, we have that the map

$$Y \xrightarrow{c_x} X$$

given by $y \mapsto x_0$ for all y which belong to Y is continuous. By Task E5.1.3, we also have that the map

$$Y \xrightarrow{id} Y$$

is continuous. By Task E5.3.17, we deduce that the map

$$Y \xrightarrow{c_x \times id} X \times Y$$

given by $y \mapsto (x, y)$ for all y which belong to Y is continuous. By Proposition 5.3.1, we deduce that the map

$$Y \xrightarrow{f \circ (c_x \times id)} \{0,1\}$$

given by $y \mapsto f(x, y)$ for all y which belong to Y is continuous. Since $(Y, \mathcal{O}_Y)$ is connected, we deduce, by Proposition 10.3.1, that $f \circ (c_x \times id)$ is not surjective. Since $\{0,1\}$ has only two elements, we deduce that $f \circ (c_x \times id)$ is constant. In other words, we have that

$$f(x, y_0) = f(x, y_1)$$

for all y_0 and y_1 which belong to Y.

(2) Suppose that y belongs to Y. Let

$$X \xrightarrow{c_y} Y$$

denote the map given by $x \mapsto y$ for all x which belong to X. Arguing as in (1), we have that the map

$$X \xrightarrow{f \circ (id \times c_y)} \{0,1\}$$

given by $x \mapsto f(x,y)$ for all x which belong to X is continuous. To carry out this argument is the topic of Task E10.2.2. Since $(X, \mathcal{O}_X)$ is connected, we deduce, by Proposition 10.3.1, that $f \circ (id \times c_y)$ is not surjective. Since $\{0,1\}$ has only two elements, we deduce that $f \circ (id \times c_y)$ is constant. In other words, we have that

$$f(x_0, y) = f(x_1, y)$$

for all x_0 and x_1 which belong to X.

Suppose now that x_0 and x_1 belong to X, and that y_0 and y_1 belong to Y. By (1), taking x to be x_0, we have that

$$f(x_0, y_0) = f(x_0, y_1).$$

By (2), taking y to be y_1, we have that

$$f(x_0, y_1) = f(x_1, y_1).$$

We deduce that

$$f(x_0, y_0) = f(x_1, y_1).$$

Thus f is constant. In particular, f is not surjective. We have thus demonstrated that there does not exist a continuous surjection

$$X \times Y \longrightarrow \{0,1\}.$$

By Proposition 10.3.1, we conclude that $(X \times Y, \mathcal{O}_{X \times Y})$ is connected.

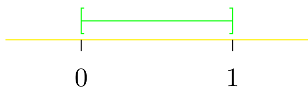
□

Remark 10.7.2. Suppose that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are topological spaces. The converse to Proposition 10.7.1 holds: if $(X \times Y, \mathcal{O}_{X \times Y})$ is connected, then both $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are connected. To prove this is the topic of Task E10.3.8.

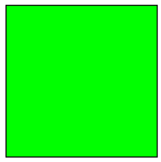
10.8 Further geometric examples of connected topological spaces

Example 10.8.1. By Proposition 10.4.1, we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected. Applying Proposition 10.7.1 repeatedly, we deduce that $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is connected, for any $n \in \mathbb{N}$.

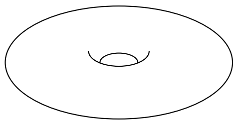
Example 10.8.2. By Example 10.6.2, we have that $(I, \mathcal{O}_I)$ is connected.



By Proposition 10.7.1, we deduce that $(I^2, \mathcal{O}_{I^2})$ is connected.



Example 10.8.3. By Example 10.8.2, we have that $(I^2, \mathcal{O}_{I^2})$ is connected. By Corollary 10.5.3, we deduce that $(T^2, \mathcal{O}_{T^2})$ is connected.

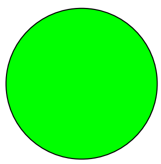


Remark 10.8.4. By a similar argument, $(M^2, \mathcal{O}_{M^2})$ and $(K^2, \mathcal{O}_{K^2})$ are connected. To check that you understand how we have built up to being able to prove this is the topic of Task E10.1.3.

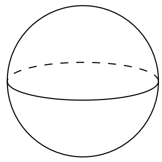
Example 10.8.5. By Example 10.8.2, we have that $(I^2, \mathcal{O}_{I^2})$ is connected. By Task E7.2.9, there is a homeomorphism

$$I^2 \longrightarrow D^2.$$

By Corollary 10.5.2, we deduce that $(D^2, \mathcal{O}_{D^2})$ is connected.



Example 10.8.6. By Example 10.8.5, we have that $(D^2, \mathcal{O}_{D^2})$ is connected. By Corollary 10.5.3, we deduce that $(S^2, \mathcal{O}_{S^2})$ is connected.



E10 Exercises for Lecture 10

E10.1 Exam questions

Task E10.1.1. Let $X = \{a, b, c, d\}$ be a set with four elements.

(1) Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

Is $(X, \mathcal{O}_X)$ connected?

(2) Let $\mathcal{O}_X$ be the topology on X given by

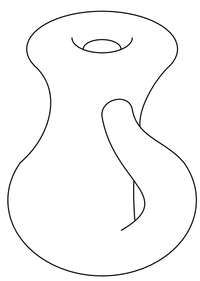
$$\{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}.$$

Is $(X, \mathcal{O}_X)$ connected?

(3) Find an equivalence relation $\sim$ on X with the property that $(X/\sim, \mathcal{O}_{X/\sim})$ is connected, where $\mathcal{O}_{X/\sim}$ is the quotient topology on $X/\sim$ with respect to the topology $\mathcal{O}_X$ on X of (2).

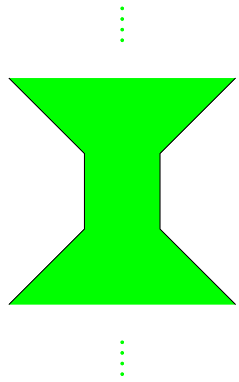
Task E10.1.2. Let $\mathbb{R} \setminus \mathbb{Q}$ be equipped with the subspace topology $\mathcal{O}_{\mathbb{R}\setminus\mathbb{Q}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Prove that $(\mathbb{R} \setminus \mathbb{Q}, \mathcal{O}_{\mathbb{R}\setminus\mathbb{Q}})$ is not connected.

Task E10.1.3. Prove that $(K^2, \mathcal{O}_{K^2})$ is connected. You may appeal without proof to any results from the lecture, but may not assert without justification that any topological space except $(I, \mathcal{O}_I)$ is connected.

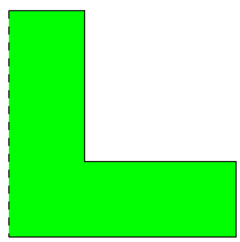


Task E10.1.4. Prove that the following topological spaces are connected. Where possible, give both a proof which makes use of Task E10.3.9, and a proof which does not. You may appeal to any results from the lectures or tasks. In addition, if you may assert the existence of homeomorphisms without proofs or explicit definitions.

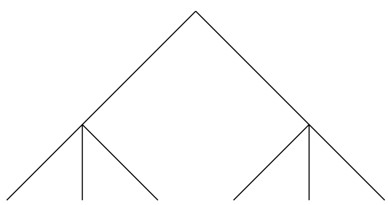
(1) The subset X of $\mathbb{R}^2$ depicted below, equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



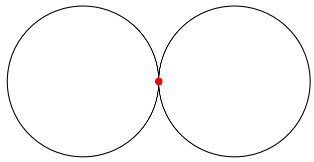
(2) The subset X of $\mathbb{R}^2$ depicted below, equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



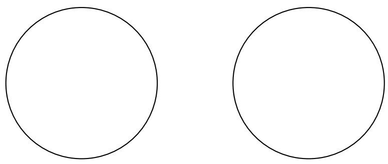
(3) The subset X of $\mathbb{R}^2$ depicted below, equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



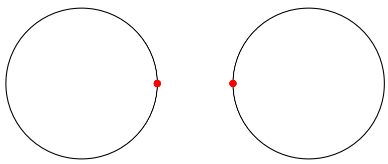
(4) The subset of $\mathbb{R}^2$ depicted below, consisting of two circles joined at a point, equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



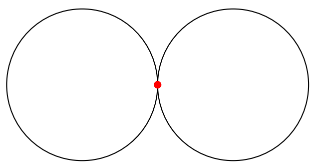
Task E10.1.5. Let X be a disjoint union of two circles of radius 1 in $\mathbb{R}^2$, centred at $(0,0)$ and $(3,0)$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



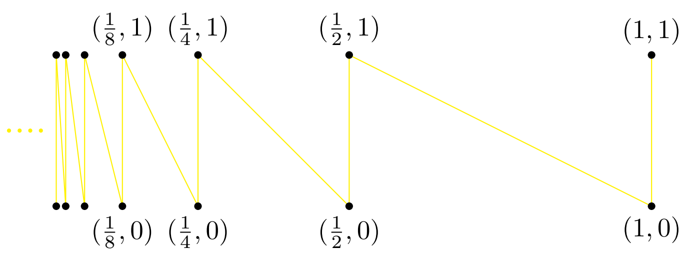
Let $\sim$ be the equivalence relation on X generated by $(1,0) \sim (2,0)$.



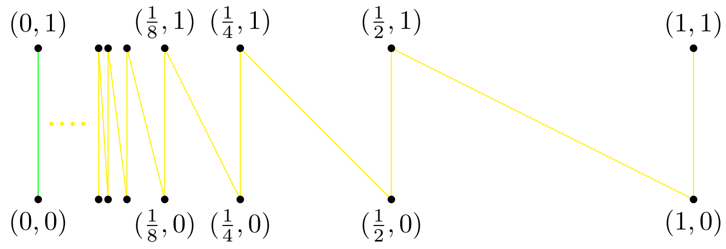
Without appealing to the fact that $(X/\sim, \mathcal{O}_{X/\sim})$ is homeomorphic to the topological space of Task E10.1.4 (5), prove that $(X/\sim, \mathcal{O}_{X/\sim})$ is connected.



Task E10.1.6. Let A be the subset of $\mathbb{R}^2$ of Task E8.1.7.



Let X be the closure of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$, which, as you were asked to prove in Task E8.1.7, is the union of X and the line $\{0\} \times [0,1]$.



Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Prove that $(X, \mathcal{O}_X)$ is connected. You may wish to proceed as follows.

(1) Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Prove that $(A, \mathcal{O}_A)$ is connected by appealing to Task E2.3.1, Task E7.1.8, Example 10.6.2, Corollary 10.5.2, and Task E10.3.9.

(2) Deduce that $(X, \mathcal{O}_X)$ is connected by Task E10.3.4.

Task E10.1.7. Let $\mathbb{R}$ be equipped with its standard topology $\mathcal{O}_{\mathbb{R}}$. Let $\mathcal{O}_{\mathbb{Q}}$ be the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Can there be a continuous map

$$\mathbb{R} \longrightarrow \mathbb{Q}$$

which is a surjection?

E10.2 In the lecture notes

Task E10.2.1. Prove that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected, by filling in the details of the following argument. Let U be a subset of $\mathbb{R}$ which belongs to $\mathcal{O}_{\mathbb{R}}$. By Task E2.3.7, there is a set I and an open interval U_i for each $i \in I$ such that $U = \bigsqcup_{i \in I} U_i$. Suppose that U is neither $\emptyset$ nor $\mathbb{R}$. Then there is an $i \in I$ such that one of the following holds.

(1) We have that U_i is $]a, \infty[$, where $a \in \mathbb{R}$.

(2) We have that U_i is $]-\infty, b[$, where $b \in \mathbb{R}$,.

(3) We have that U_i is $]a, b[$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Treat each of the cases separately, as follows.

(1) Then a is a limit point of U in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, and a does not belong to U.

(2) Then b is a limit point of U in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, and b does not belong to U.

(3) Then both a and b are limit points of U in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, and neither a nor b belongs to U.

By Proposition 9.1.1, deduce in each case that U is not closed with respect to $\mathcal{O}_{\mathbb{R}}$. By Task E10.3.1, conclude that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.

Task E10.2.2. Carry out the argument needed for (2) of the proof of Proposition 10.7.1.

E10.3 For a deeper understanding

Task E10.3.1. Let $(X, \mathcal{O}_X)$ be a topological space. Prove that $(X, \mathcal{O}_X)$ is connected if and only if the only subsets of X which both belong to $\mathcal{O}_X$ and are closed with respect to $\mathcal{O}_X$ are $\emptyset$ and X. You may wish to proceed as follows.

(1) Suppose that $(X, \mathcal{O}_X)$ is connected. Let X_0 be a subset of X which belongs to $\mathcal{O}_X$. If X_0 is closed with respect to $\mathcal{O}_X$, we have that $X \setminus X_0$ belongs to $\mathcal{O}_X$. Moreover $X_0 \cap (X \setminus X_0)$ is empty. Since $(X, \mathcal{O}_X)$ is connected, conclude that X_0 is either $\emptyset$ or X.

(2) Suppose that X_0 is a subset of X which is neither $\emptyset$ nor X. Observe that $X \setminus X_0$ is then neither $\emptyset$ nor X. We have that $X = X_0 \sqcup (X \setminus X_0)$. If both X_0 and $X \setminus X_0$ belong to $\mathcal{O}_X$, deduce that $(X, \mathcal{O}_X)$ is not connected. Conclude that if X_0 both belongs to $\mathcal{O}_X$ and is closed with respect to $\mathcal{O}_X$, then $(X, \mathcal{O}_X)$ is not connected.

Task E10.3.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that $(X, \mathcal{O}_X)$ is connected. Let

$$X \xrightarrow{f} Y$$

be a continuous map. Let $\mathcal{O}_{f(X)}$ denote the subspace topology on $f(X)$ with respect to $(Y, \mathcal{O}_Y)$. Prove that $\big(f(X), \mathcal{O}_{f(X)}\big)$ is connected. You may wish to proceed as follows.

(1) Let

$$X \xrightarrow{g} f(X)$$

be the map given by $x \mapsto f(x)$. By Task E5.1.9, observe that g is continuous.

(2) Moreover we have that g is surjective. By Proposition 10.5.1, conclude that $\big(f(X), \mathcal{O}_{f(X)}\big)$ is connected.

Task E10.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X, and let $\mathcal{O}_A$ denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Suppose that $(A, \mathcal{O}_A)$ is connected. Let B be a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ with the property that A is a subset of B. Let $\mathcal{O}_B$ denote the subspace topology on B with respect to $(X, \mathcal{O}_X)$. Prove that $(B, \mathcal{O}_B)$ is connected. You may wish to proceed as follows.

(1) Let $\{0,1\}$ be equipped with the discrete topology $\mathcal{O}_{\mathsf{discrete}}$. Suppose that

$$B \xrightarrow{f} \{0,1\}$$

is continuous. Let

$$A \xrightarrow{i} B$$

denote the inclusion map. By Proposition 5.2.2, we have that i is continuous. By Proposition 5.3.1, deduce that

$$A \xrightarrow{f \circ i} \{0,1\}$$

is continuous.

(2) Since $(A, \mathcal{O}_A)$ is connected, deduce by Proposition 10.3.1 that $f \circ i$ is not surjective. Since $\{0,1\}$ has only two elements, deduce that $f \circ i$ is constant.

(3) By Task E8.3.13, we have that $\mathsf{cl}_{(B,\mathcal{O}_B)}(A)$ is $B \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(A)$. Since B is a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ by assumption, deduce that $\mathsf{cl}_{(B,\mathcal{O}_B)}(A)$ is B.

(4) By Task E9.3.9, we have that $f\left(\mathsf{cl}_{(B,\mathcal{O}_B)}(A)\right)$ is a subset of

$$\mathsf{cl}_{(\{0,1\},\mathcal{O}_{\mathsf{discrete}})}(f(A)).$$

By (3), deduce that $f(B)$ is a subset of

$$\mathsf{cl}_{(\{0,1\},\mathcal{O}_{\mathsf{discrete}})}(f(A)).$$

(5) Demonstrate that $\mathsf{cl}_{(\{0,1\},\mathcal{O}_{\mathsf{discrete}})}(f(A))$ is $f(A)$. By (4), deduce that $f(B)$ is a subset of $f(A)$.

(6) By (2) and (5), we have that f is constant. In particular, we have that f is not surjective. By Proposition 10.3.1, conclude that $(B, \mathcal{O}_B)$ is connected.

Corollary E10.3.4. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X, and let $\mathcal{O}_A$ denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Suppose that $(A, \mathcal{O}_A)$ is connected. Let $\mathcal{O}_{\mathsf{cl}_{(X,\mathcal{O}_X)}(A)}$ denote the subspace topology on $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then $\left(\mathsf{cl}_{(X,\mathcal{O}_X)}(A), \mathcal{O}_{\mathsf{cl}_{(X,\mathcal{O}_X)}(A)}\right)$ is connected.

Proof. Follows immediately from Task E10.3.3, taking B to be $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$. □

Task E10.3.5. Let X be a subset of $\mathbb{R}$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Prove that $(X, \mathcal{O}_X)$ is connected if and only if X is an interval. You may wish to proceed as follows.

(1) Suppose that X is an interval. The difference possibilities for X are listed below, where a and b belong to $\mathbb{R}$, and $a < b$. In each case, fill in the details of the outlined proof that $(X, \mathcal{O}_X)$ is connected.

Interval X	Proof that $(X, \mathcal{O}_X)$ is connected
$\mathbb{R}$	Proposition 10.4.1
$]a, b[$	Example 10.6.1
$[a, b]$	Example 10.6.2
$[a, b[$	Example 10.6.1 and Task E10.3.3
$]a, b]$	Example 10.6.1 and Task E10.3.3
$\emptyset$	By inspection.
$[a, a]$	By inspection.
$]a, \infty[$	Task E7.1.5, Corollary 10.5.2, and Proposition 10.4.1.
$]-\infty, b[$	Task E7.1.6, Corollary 10.5.2, and Proposition 10.4.1.
$[a, \infty[$	Corollary E10.3.4 and the case that X is $]a, \infty[$.
$]-\infty, b]$	Corollary E10.3.4 and the case that X is $]-\infty, b[$.

(2) Suppose that X is not an interval. By Task E1.3.3, there is an $x_0 \in X$, an $x_1 \in X$, and a $y \in \mathbb{R} \setminus X$, such that $x_0 < y < x_1$. Let X_0 be $X \cap]-\infty, y[$, and let X_1 be $X \cap]y, \infty[$. Observe that both X_0 and X_1 belong to $\mathcal{O}_X$, and that $X = X_0 \sqcup X_1$. Conclude that $(X, \mathcal{O}_X)$ is not connected.

Task E10.3.6. Let $(X, \mathcal{O}_X)$ be a connected topological space. Let $\mathbb{R}$ be equipped with the standard topology $\mathcal{O}_{\mathbb{R}}$. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Suppose that x_0 and x_1 belong to X, and that $f(x_0) \leq f(x_1)$. Prove that, for every $x \in \mathbb{R}$ such that $f(x_0) \leq y \leq f(x_1)$, there is an $x_2 \in X$ such that $f(x_2) = y$. You may wish to proceed as follows.

(1) Let $\mathcal{O}_{f(X)}$ denote the subspace topology on $f(X)$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Since $(X, \mathcal{O}_X)$ is connected, deduce by Task E10.3.2 that $\left(f(X), \mathcal{O}_{f(X)}\right)$ is connected.

(2) By Task E10.3.5, deduce that $f(X)$ is an interval.

(3) Appeal to Task E1.3.3.

Remark E10.3.7. Taking $(X, \mathcal{O}_X)$ to be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, or to be an interval equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, the conclusion of Task E10.3.6 is exactly the *intermediate value theorem.* As you may recall from earlier courses, this is one of the handful of crucial facts upon which analysis rests.

Task E10.3.8. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that $(X \times Y, \mathcal{O}_{X \times Y})$ is connected. Prove that both $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are connected. You may wish to appeal to Proposition 5.4.3 and Proposition 10.5.1.

Task E10.3.9. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{A_j\}_{j \in J}$ be a set of subsets of X such that the following hold.

(1) For every $j \in J$, we have that $(A_j, \mathcal{O}_{A_j})$ is connected, where $\mathcal{O}_{A_j}$ denotes the subspace topology on A_j with respect to $(X, \mathcal{O}_X)$.

(2) We have that $\bigcup_{j \in J} A_j$ is X.

(3) We have that $\bigcap_{j \in J} A_j$ is not empty.

Prove that $(X, \mathcal{O}_X)$ is connected. You may wish to proceed as follows.

(1) Let $\{0, 1\}$ be equipped with the discrete topology. Let

$$X \xrightarrow{f} \{0, 1\}$$

be a continuous map. Suppose that j belongs to J. Let

$$A_j \xrightarrow{i_j} X$$

denote the inclusion map, given by $a \mapsto a$. By Proposition 5.2.2, we have that i_j is continuous. By Proposition 5.3.1, deduce that the map

$$A_j \xrightarrow{f \circ i_j} \{0, 1\}$$

given by $a \mapsto f(a)$ is continuous.

(2) Since $(A_j, \mathcal{O}_{A_j})$ is connected, deduce by Proposition 10.3.1 that $f \circ i_j$ is constant.

(3) Observe that the fact that $\bigcup_{j \in J} A_j$ is X, that $\bigcap_{j \in J} A_j$ is not empty, and that $f \circ i_j$ is constant for every $j \in J$, implies that f is constant.

(4) In particular, f is not surjective. Thus we have demonstrated that there does not exist a continuous surjection

$$X \longrightarrow \{0, 1\}.$$

By Proposition 10.3.1, conclude that $(X, \mathcal{O}_X)$ is connected.