MA3002 Generell Topologi — Vår 2014

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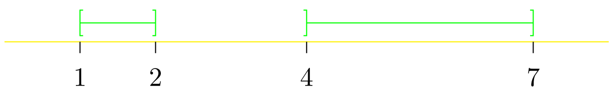
11 Monday 10th February

11.1 Using connectedness to prove that two topological spaces are not homeomorphic

Remark 11.1.1. To prove that a given topological space $(X, \mathcal{O}_X)$ is not homeomorphic to a particular topological space $(Y, \mathcal{O}_Y)$ is typically hard. In geometric examples, when X and Y are infinite, there are many infinitely many maps from X to Y. Thus we cannot simply list them all, and then check whether or not there is a homeomorphism amongst them.

We must proceed in a more sophisticated way. The theory of connectedness which we have developed furnishes us with our first powerful tool for proving that two topological spaces are not homeomorphic.

Example 11.1.2. Let $X = [1,2] \cup]4,7]$.



Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Arguing as in Example 9.6.2, we have that $(X, \mathcal{O}_X)$ is not connected.

Let $\mathcal{O}_{]1,5[}$ be the subspace topology on $[1,5]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

By Task E10.3.5, we have that $\left([1,5], \mathcal{O}_{]1,5]}\right)$ is connected. Suppose that

$$[1,5] \xrightarrow{f} X$$

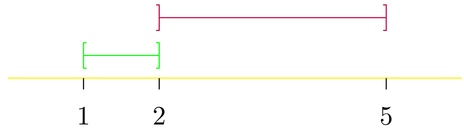
is a homeomorphism. By Corollary 10.5.2, we then have that $(X, \mathcal{O}_X)$ is connected.

Thus we have a contradiction. We conclude that there does not exist a homeomorphism

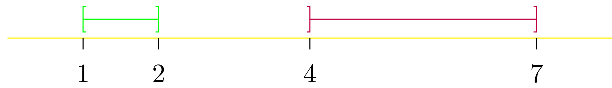
$$[1,5] \xrightarrow{f} X.$$

In other words, we have that $(X, \mathcal{O}_X)$ is not homeomorphic to $\left([1,5], \mathcal{O}_{]1,5]}\right)$.

Remark 11.1.3. We can 'snap off' the half open interval $]2, 5]$ from $[1, 5]$.



We can then 'move' this half open interval to $]4, 7]$.



This defines a bijection

$$[1, 5] \xrightarrow{f} X.$$

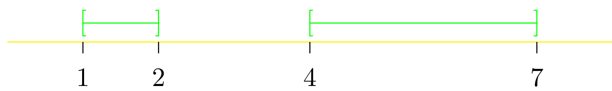
However, this bijection is not continuous. To 'snap off' is not allowed in topology! The details of this are the topic of Task E11.2.1.

It is very important to appreciate that to distinguish between $(X, \mathcal{O}_X)$ and

$$\left([1, 5], \mathcal{O}_{]1,5]}\right),$$

we must give a topological argument. From the point of view of set theory, $[1, 5]$ and X are 'the same'.

Remark 11.1.4. Let $X = [1, 2] \cup [4, 7]$.

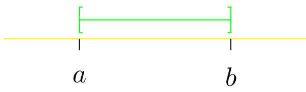


Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Exactly the same kind of argument as in Example 11.1.2 proves that $(X, \mathcal{O}_X)$ is not homeomorphic to $\left([1, 5], \mathcal{O}_{]1,5]}\right)$.

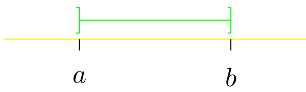
There is a bijection between $[1, 5]$ and X, though it is harder to find than the bijection of Remark 11.1.3. This is the topic of Task E11.4.2. Once more, we see that it is necessary to give a topological argument to distinguish between $(X, \mathcal{O}_X)$ and $\left([1, 5], \mathcal{O}_{]1,5]}\right)$.

11.2 Using connectedness to distinguish between topological spaces by removing points

Example 11.2.1. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Let $\mathcal{O}_{[a,b]}$ be the subspace topology on $[a, b]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



We have that $[a, b] \setminus \{a\}$ is $]a, b]$.



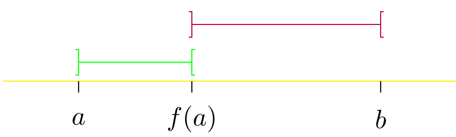
Let $\mathcal{O}_{]a,b]}$ be the subspace topology on $]a, b]$ with respect to $\left([a, b], \mathcal{O}_{[a,b]}\right)$. By Task E2.3.1 and Task E10.3.5, we have that $\left(]a, b], \mathcal{O}_{]a,b]}\right)$ is connected.

Let $\mathcal{O}_{]a,b[}$ be the subspace topology on $]a, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Suppose that

$$[a, b] \xrightarrow{f}]a, b[$$

is a homeomorphism. Let $\mathcal{O}_{]a,b[\setminus\{f(a)\}}$ be the subspace topology on $]a, b[\setminus \{f(a)\}$ with respect to $\left(]a, b[, \mathcal{O}_{]a,b[}\right)$. We have that $]a, b[\setminus \{f(a)\}$ is the union of $]a, f(a)[$ and $]f(a), b[$. This union is disjoint.



Moreover, both $]a, f(a)[$ and $]f(a), b[$ belong to $\mathcal{O}_{]a,b[\setminus\{f(a)\}}$. Thus

$$\left(]a, b[\setminus \{f(a)\}, \mathcal{O}_{]a,b[\setminus\{f(a)\}}\right)$$

is not connected. To generalise this argument is the topic of Task E11.2.5.

By Task E7.1.20, since f is a homeomorphism, the map

$$]a, b] \longrightarrow]a, b[\setminus \{f(a)\}$$

given by $x \mapsto f(x)$ is a homeomorphism. Since $(]a, b], \mathcal{O}_{]a,b]})$ is connected, we deduce, by Corollary 10.5.2, that

$$\left(]a, b[\setminus \{f(a)\}, \mathcal{O}_{]a,b[\setminus\{f(a)\}}\right)$$

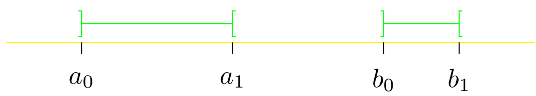
is connected.

Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$[a, b] \xrightarrow{f}]a, b[.$$

In other words, $\left([a, b], \mathcal{O}_{[a,b]}\right)$ is not homeomorphic to $\left(]a, b[, \mathcal{O}_{]a,b[}\right)$.

Remark 11.2.2. Suppose that $a_0 < a_1 < b_0 < b_1$ belong to $\mathbb{R}$. Let X be the union of $]a_0, a_1[$ and $]b_0, b_1[$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



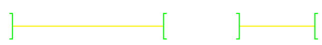
Let

$$\left(]a, b[\setminus \{f(a)\}, \mathcal{O}_{]a,b[\setminus\{f(a)\}}\right)$$

be as in Example 11.2.1. By Task E11.2.3, we have that

$$\left(]a, b[\setminus \{f(a)\}, \mathcal{O}_{]a,b[\setminus\{f(a)\}}\right)$$

is homeomorphic to $(X, \mathcal{O}_X)$. Thus we can picture $\left(]a, b[\setminus \{f(a)\}, \mathcal{O}_{]a,b[\setminus\{f(a)\}}\right)$ as follows.



Example 11.2.3. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Let $\mathcal{O}_{[a,b]}$ be the subspace topology on $[a, b]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

We have that $[a, b] \setminus \{a, b\}$ is $]a, b[$.

Let $\mathcal{O}_{]a,b[}$ be the subspace topology on $]a,b[$ with respect to $\left([a,b[\,,\mathcal{O}_{[a,b[}\right)$. By Task E2.3.1 and Task E10.3.5, we have that $\left(]a,b[\,,\mathcal{O}_{]a,b[}\right)$ is connected. Let $\mathcal{O}_{[a,b[}$ be the subspace topology on $[a,b[$ with respect to $(\mathbb{R},\mathcal{O}_{\mathbb{R}})$.

Let

$$[a,b] \xrightarrow{f} [a,b[$$

be a homeomorphism. Let $\mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}$ be the subspace topology on $[a,b[\setminus\{f(a),f(b)\}$ with respect to $\left([a,b[\,,\mathcal{O}_{[a,b[}\right)$. One of the following two possibilities must hold.

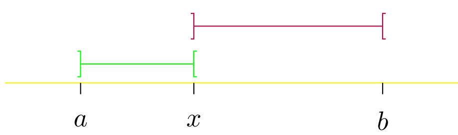
(I) One of $f(a)$ or $f(b)$ is a.

(II) Neither $f(a)$ nor $f(b)$ is a.

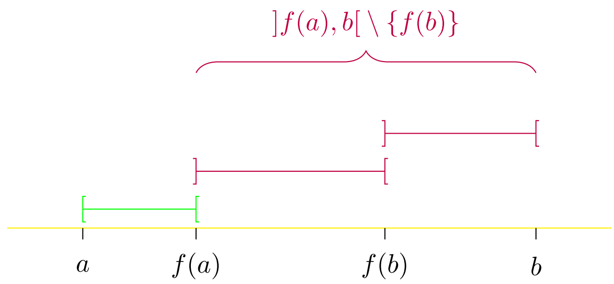
Suppose that (I) holds. Since f is bijective, one of $f(a)$ or $f(b)$ is not a. Let us denote whichever of $f(a)$ or $f(b)$ is not a by x. Then $[a,b[\setminus\{f(a),f(b)\}$ is $]a,b[\setminus\{x\}$. As in Example 11.2.1, we deduce that

$$\left([a,b[\setminus\{f(a),f(b)\}\,,\mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

is not connected.



Suppose now that (II) holds. Since f is bijective, either $f(a) < f(b)$ or $f(a) > f(b)$. Suppose that $f(a) < f(b)$. We have that $[a,b[\setminus\{f(a),f(b)\}$ is the union of $[a,f(a)[$ and $]f(a),b[\setminus\{f(b)\}$, and this union is disjoint.



Moreover, both $[a, f(a)[$ and $]f(a), b[\setminus \{f(b)\}$ belong to $\mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}$. Thus

$$\left([a, b[\setminus \{f(a), f(b)\}, \mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

is not connected. A similar argument establishes that

$$\left([a, b[\setminus \{f(a), f(b)\}, \mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

is not connected if $f(a) > f(b)$. This is the topic of Task E11.2.2.

By Task E7.1.20, since f is a homeomorphism, the map

$$]a, b[\longrightarrow [a, b[\setminus \{f(a), f(b)\}$$

given by $x \mapsto f(x)$ is a homeomorphism. Since $(]a, b[, \mathcal{O}_{]a,b[})$ is connected, we deduce, by Corollary 10.5.2, that

$$\left([a, b[\setminus \{f(a), f(b)\}, \mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

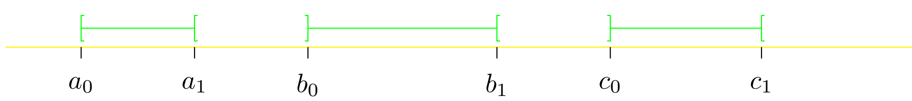
is connected.

Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$[a, b] \xrightarrow{f} [a, b[.$$

In other words, $\left([a, b], \mathcal{O}_{[a,b]}\right)$ is not homeomorphic to $\left([a, b[, \mathcal{O}_{[a,b[}\right)$.

Remark 11.2.4. Suppose that $a_0 < a_1 < b_0 < b_1 < c_0 < c_1$ belong to $\mathbb{R}$. Let X be the union of $]a_0, a_1[$, $]b_0, b_1[$, and $]c_0, c_1[$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Let

$$\left([a, b[\setminus \{f(a), f(b)\}, \mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

be as in case (II) of Example 11.2.3. By Task E11.2.4, we have that

$$\left([a,b[\setminus\{f(a),f(b)\}\,,\mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

is homeomorphic to $(X,\mathcal{O}_X)$. Thus we can picture

$$\left([a,b[\setminus\{f(a),f(b)\}\,,\mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

as follows.

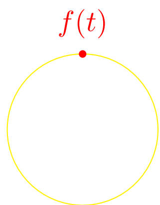
Example 11.2.5. Let $(I,\mathcal{O}_I)$ be the unit interval. Suppose that $0<t<1$.

Let $\mathcal{O}_{I\setminus\{t\}}$ be the subspace topology on $I\setminus\{t\}$ with respect to $(I,\mathcal{O}_I)$. Then $\left(I\setminus\{t\},\mathcal{O}_{I\setminus\{t\}}\right)$ is not connected.

Suppose that

$$I \xrightarrow{f} S^1$$

is a homeomorphism.



Let $\mathcal{O}_{S^1\setminus\{f(t)\}}$ be the subspace topology on $S^1\setminus\{f(t)\}$ with respect to $(S^1,\mathcal{O}_{S^1})$. We have that $\left(S^1\setminus\{f(t)\}\,,\mathcal{O}_{S^1\setminus\{f(t)\}}\right)$ is connected.

Since f is a homeomorphism, we have by Task E7.1.20 that there is a homeomorphism

$$I \setminus \{t\} \longrightarrow S^1 \setminus \{f(t)\}.$$

By Task E7.3.2, we deduce that there is a homeomorphism

$$S^1 \setminus \{f(t)\} \longrightarrow I \setminus \{t\}.$$

By Corollary 10.5.2, since

$$\left(S^1 \setminus \{f(t)\}, \mathcal{O}_{S^1 \setminus \{f(t)\}}\right)$$

is connected, we deduce that

$$\left(I \setminus \{t\}, \mathcal{O}_{I \setminus \{t\}}\right)$$

is connected. Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$I \longrightarrow S^1.$$

In other words, $(I, \mathcal{O}_I)$ is not homeomorphic to $\left(S^1, \mathcal{O}_{S^1}\right)$.

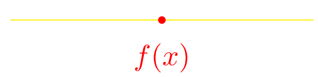
Remark 11.2.6. To prove the assertion that $\left(I \setminus \{t\}, \mathcal{O}_{I \setminus \{t\}}\right)$ is not connected, and the assertion that $\left(S^1 \setminus \{f(t)\}, \mathcal{O}_{S^1 \setminus \{f(t)\}}\right)$ is connected, is the topic of Task E11.2.11.

Remark 11.2.7. There exists a bijection between I and S^1. This is the topic of Task E11.4.3. Hence I and S^1 are 'the same' from the point of view of set theory. Thus, just as in Remark 11.1.3, a topological argument, such as that of Example 11.2.5, must be given to prove that $(I, \mathcal{O}_I)$ is not homeomorphic to $(S^1, \mathcal{O}_{S^1})$.

Example 11.2.8. Suppose that $n > 1$ belongs to $\mathbb{N}$. Let $\mathbb{R}$ be equipped with the standard topology $\mathcal{O}_{\mathbb{R}}$. Let $\mathbb{R}^n$ be equipped with the product topology $\mathcal{O}_{\mathbb{R}^n}$ of Notation E3.3.8. Suppose that x belongs to $\mathbb{R}^n$. Suppose that

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}$$

is a homeomorphism. Let $\mathcal{O}_{\mathbb{R} \setminus \{f(x)\}}$ be the subspace topology on $\mathbb{R} \setminus \{f(x)\}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



By Task E11.2.5, we have that $(\mathbb{R} \setminus \{f(x)\}, \mathcal{O}_{\mathbb{R} \setminus \{f(x)\}})$ is not connected.

Let $\mathcal{O}_{\mathbb{R}^n \setminus \{x\}}$ be the subspace topology on $\mathbb{R}^n \setminus \{x\}$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. By Task E11.3.1, we have that $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n \setminus \{x\}})$ is connected. Since f is a homeomorphism, we have by Task E7.1.20 that there is a homeomorphism

$$\mathbb{R}^n \setminus \{x\} \longrightarrow \mathbb{R} \setminus \{f(x)\}.$$

By Corollary 10.5.2, we deduce that

$$\left(\mathbb{R} \setminus \{f(x)\}, \mathcal{O}_{\mathbb{R} \setminus \{f(x)\}}\right)$$

is connected. Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$\mathbb{R}^n \longrightarrow \mathbb{R}.$$

In other words, $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not homeomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.

Remark 11.2.9. Example 11.2.8 is intuitively evident. We cannot bend or squash ourselves in such a way that we become a line! However, there is a bijection between $\mathbb{R}$ and $\mathbb{R}^n$, for any $n \geq 1$! This is the topic of Task E11.4.4.

Moreover, to prove that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ when $m \neq n$, $m \geq 2$, and $n \geq 2$, is much harder. One needs more powerful techniques.

11.3 Connected components

Terminology 11.3.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X, and let $\mathcal{O}_A$ be the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Then A is a *connected subset* of X with respect to $\mathcal{O}_X$ if $(A, \mathcal{O}_A)$ is a connected.

Terminology 11.3.2. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Let A be a connected subset of X with respect to $\mathcal{O}_X$ such that the following hold.

(1) We have that x belongs to A.

(2) For every connected subset B of X with respect to $\mathcal{O}_X$ to which x belongs, we have that B is a subset of A.

We refer to A as the *largest* connected subset of X with respect to $\mathcal{O}_X$ to which x belongs.

We do not yet know whether, for a given x which belongs to X, there is a subset A of X which has the property that it is the largest connected subset of X with respect to $\mathcal{O}_X$ to which x belongs. We shall now demonstrate this.

Remark 11.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Let A_0 and A_1 be connected subsets of X with respect to $\mathcal{O}_X$ which both satisfy (1) and (2) of Terminology 11.3.2. Then $A_0 = A_1$. To check that you understand this is the topic of Task E11.2.6.

Definition 11.3.4. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. The *connected component* of x in $(X, \mathcal{O}_X)$ is the union of all connected subsets of X with respect to $\mathcal{O}_X$ to which x belongs.

Notation 11.3.5. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. We denote the connected component of x in $(X, \mathcal{O}_X)$ by $\Gamma^x_{(X,\mathcal{O}_X)}$.

Remark 11.3.6. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Then $\{x\}$ is a connected subset of X with respect to $\mathcal{O}_X$. This is the topic of Task E11.2.7. Thus x belongs to $\Gamma^x_{(X,\mathcal{O}_X)}$.

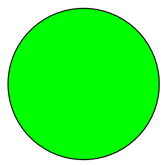
Proposition 11.3.7. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Then $\Gamma^x_{(X,\mathcal{O}_X)}$ is a connected subset of X.

Proof. Let $\{A_i\}_{i \in I}$ be the set of connected subsets of X with respect to $\mathcal{O}_X$ to which x belongs. We have that $\{x\}$ is a subset of $\bigcap_{i \in I} A_i$. By Task E10.3.9, we deduce that $\Gamma^x_{(X,\mathcal{O}_X)} = \bigcup_{i \in I} A_i$ is a connected subset of X with respect to $\mathcal{O}_X$. $\square$

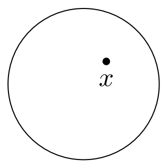
Remark 11.3.8. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Let A be a connected subset of X with respect to $\mathcal{O}_X$ to which x belongs. By definition of $\Gamma^x_{(X,\mathcal{O}_X)}$, we have that A is a subset of $\Gamma^x_{(X,\mathcal{O}_X)}$. By Proposition 11.3.7, we conclude that $\Gamma^x_{(X,\mathcal{O}_X)}$ is the largest connected subset of X with respect to $\mathcal{O}_X$ to which x belongs.

11.4 Examples of connected components

Example 11.4.1. Let $(X, \mathcal{O}_X)$ be a connected topological space. For example, we can take $(X, \mathcal{O}_X)$ to be $(D^2, \mathcal{O}_{D^2})$.



Suppose that x belongs to X.



We have that X is a connected subset of X with respect to $\mathcal{O}_X$, for which (1) and (2) of Terminology 11.3.2 hold. By Remark 11.3.3 and Remark 11.3.8, we conclude that $\Gamma^x_{(X,\mathcal{O}_X)} = X$.

Example 11.4.2. Let X be a set. Let $\mathcal{O}_X$ be the discrete topology on X. Suppose that x belongs to X. Let A be a subset of X to which x belongs. Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Then $\mathcal{O}_A$ is the discrete topology on A. To verify this is Task E11.2.8.

Suppose that A has more than one element, so that $A \setminus \{x\}$ is not empty. We have that

$$A = \{x\} \sqcup (A \setminus \{x\}) .$$

Since $\mathcal{O}_A$ is the discrete topology on A, every subset of A belongs to $\mathcal{O}_A$. In particular, both $\{x\}$ and $A \setminus \{x\}$ belong to $\mathcal{O}_A$, Thus $(A, \mathcal{O}_A)$ is not connected. In other words, A is not a connected subset of X with respect to $\mathcal{O}_X$.

We conclude that $\Gamma^x_{(X,\mathcal{O}_X)} = \{x\}$.

Example 11.4.3. Let $X = \{a, b, c, d\}$ be a set with four elements. Let $\mathcal{O}_X$ be the topology on X given by

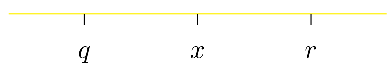
$$\{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\} .$$

Table 11.1 lists the connected subsets of X with respect to $\mathcal{O}_X$. By inspecting Table 11.1, and by Remark 11.3.3 and Remark 11.3.8, we conclude that the connected components in $(X, \mathcal{O}_X)$ of the elements of X are as follows.

Element	Connected component
a	$\{a\}$
b	$\{b, c\}$
c	$\{b, c\}$
d	$\{d\}$

Example 11.4.4. Let $\mathbb{Q}$ be the set of rational numbers. Let $\mathcal{O}_{\mathbb{Q}}$ be the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that q belongs to $\mathbb{Q}$. Let A be a subset of $\mathbb{Q}$ to which q belongs. Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$. By Task E2.3.1, we have that $\mathcal{O}_{\mathbb{Q}}$ is the subspace topology on A with respect to $\mathcal{O}_{\mathbb{R}}$.

Suppose that r belongs to A, and that r is not equal to q. There is an irrational number x with $q < x < r$.



The following hold.

(1) Since x is irrational, and thus does not belong to A, we have that

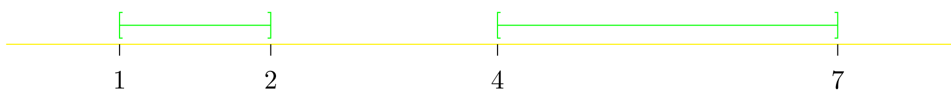
$$A = (A \cap \,]-\infty, x[) \sqcup (A \cap \,]x, \infty[) \,.$$

(2) We have that q belongs to $A \cap \,]-\infty, x[$, and that r belongs to $A \cap \,]x, \infty[$. In particular, neither $A \cap \,]-\infty, x[$ nor $A \cap \,]x, \infty[$ is empty.

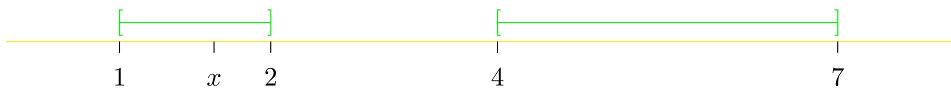
(3) Since both $]-\infty, x[$ and $]x, \infty[$ belong to $\mathcal{O}_{\mathbb{R}}$, and since $\mathcal{O}_A$ is the subspace topology on A with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, we have that both $A \cap \,]-\infty, x[$ and $A \cap \,]x, \infty[$ belong to $\mathcal{O}_A$.

Thus $(A, \mathcal{O}_A)$ is not connected. In other words, A is not a connected subset of $\mathbb{Q}$ with respect to $\mathcal{O}_{\mathbb{Q}}$. We conclude that $\Gamma^q_{(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})}$ is $\{q\}$.

Example 11.4.5. Let $X = [1, 2] \cup [4, 7]$.

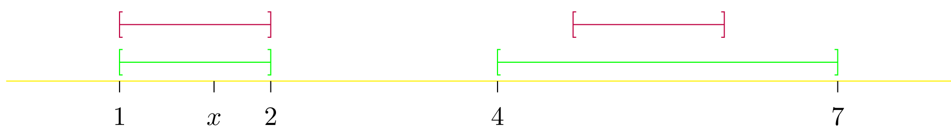


Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that x belongs to $[1, 2]$.



By Task E2.3.1 and Task E10.3.5, we have that $[1, 2]$ is a connected subset of $(X, \mathcal{O}_X)$.

Suppose that A is a subset of X to which x belongs, and which has the property that $A \cap [4, 7]$ is not empty.



Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(X, \mathcal{O}_X)$. The following hold.

(1) We have that $A = (A \cap [1, 2]) \sqcup (A \cap [4, 7])$.

(2) We have that x belongs to $A \cap [1, 2]$. In particular, $A \cap [1, 2]$ is not empty. By assumption, we also have that $A \cap [4, 7]$ is not empty.

(3) As demonstrated in Example 9.6.2, both $[1, 2]$ and $[4, 7]$ belong to $\mathcal{O}_X$. Thus both $A \cap [1, 2]$ and $A \cap [4, 7]$ belong to $\mathcal{O}_A$.

Thus $(A, \mathcal{O}_A)$ is not connected. In other words, A is not a connected subset of X with respect to $\mathcal{O}_X$. By Remark 11.3.3 and Remark 11.3.8, we conclude that $\Gamma^x_{(X,\mathcal{O}_X)}$ is $[1, 2]$.

A similar argument demonstrates that if x belongs to $[4, 7]$, then $\Gamma^x_{(X,\mathcal{O}_X)}$ is $[4, 7]$. To fill in the details is the topic of Task E11.2.9.

11.5 Number of distinct connected components as an invariant

Remark 11.5.1. If two topological spaces are homeomorphic, then they have the same number of distinct connected components. This is the topic of Task E11.3.18.

Therefore, to prove that two topological spaces are not homeomorphic, it suffices to count their respective numbers of distinct connected components, and to observe that they are different. This is a gigantic simplification! It is so much of a simplification that it is only useful to a certain extent, as we shall see.

In particular, it is most definitely not the case that two topological spaces are homeomorphic if and only if they have the same number of distinct connected components. There are many connected topological spaces which are not homeomorphic!

Nevertheless, the idea that we can associate to complicated gadgets, such as topological spaces, simpler *invariants*, which we can calculate with more easily, is of colossal importance in mathematics. These invariants might be: numbers; algebraic gadgets such as groups, vector spaces, or rings; or other structures.

Thus the number of distinct connected components of a topological space is the beginning of a fascinating story!

Example 11.5.2. Let T be the subset of $\mathbb{R}^2$ given by the union of

$$\{(0, y) \mid -1 \leq y \leq 1\}$$

and

$$\{(x, 1) \mid -1 \leq x \leq 1\}.$$

Let $\mathcal{O}_{\mathsf{T}}$ be the subspace topology on T with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let I be the subset of $\mathbb{R}^2$ given by

$$\{(0, y) \mid 0 \leq y \leq 1\}.$$

Let $\mathcal{O}_{\mathsf{I}}$ be the subspace topology on I with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Suppose that

$$\mathsf{T} \xrightarrow{f} \mathsf{I}$$

is a homeomorphism. Let x be the point $(0,1)$ of T.

Let $\mathcal{O}_{\mathsf{T}\setminus\{x\}}$ be the subspace topology on $\mathsf{T} \setminus \{x\}$ with respect to $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$. Then

$$\left(\mathsf{T} \setminus \{x\}, \mathcal{O}_{\mathsf{T}\setminus\{x\}}\right)$$

has three distinct connected components.

Let $\mathcal{O}_{\mathsf{I}\setminus\{f(x)\}}$ be the subspace topology on $\mathsf{I} \setminus \{f(x)\}$ with respect to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$. Suppose that $f(x)$ is $(0,0)$ or $(0,1)$.

Then $\left(\mathsf{I} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{I}\setminus\{f(x)\}}\right)$ is connected. Suppose that $f(x)$ is not $(0,0)$ or $(0,1)$.

Then $\left(\mathsf{I} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{I}\setminus\{f(x)\}}\right)$ has two distinct connected components.

Since f is a homeomorphism, we have by Task E7.1.20 that there is a homeomorphism

$$\mathsf{T} \setminus \{x\} \longrightarrow \mathsf{I} \setminus \{f(x)\}.$$

By Corollary E11.3.19, since

$$\left(\mathsf{T} \setminus \{x\}, \mathcal{O}_{\mathsf{T}\setminus\{x\}}\right)$$

has three distinct connected components, we deduce that

$$\left(\mathsf{I} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{I}\setminus\{f(x)\}}\right)$$

has three distinct connected components. Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$\mathsf{T} \longrightarrow \mathsf{I}.$$

In other words, $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ is not homeomorphic to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$.

Remark 11.5.3. To fill in the details of the three calculations of numbers of distinct connected components in Example 11.5.2 is the topic of Task E12.2.1.

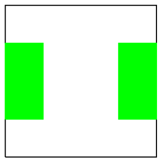
Subset A	Connected?	Reason
$\emptyset$	✓	
$\{a\}$	✓	
$\{b\}$	✓	
$\{c\}$	✓	
$\{d\}$	✓	
$\{a,b\}$	✗	$A = \{a\} \sqcup \{b\}$, and both $\{a\} = A \cap \{a\}$ and $\{b\} = A \cap \{b\}$ belong to $\mathcal{O}_A$.
$\{a,c\}$	✗	$A = \{a\} \sqcup \{c\}$, and both $\{a\} = A \cap \{a\}$ and $\{c\} = A \cap \{b,c,d\}$ belong to $\mathcal{O}_A$.
$\{a,d\}$	✗	$A = \{a\} \sqcup \{d\}$, and both $\{a\} = A \cap \{a\}$ and $\{d\} = A \cap \{d\}$ belong to $\mathcal{O}_A$.
$\{b,c\}$	✓	
$\{b,d\}$	✗	$A = \{b\} \sqcup \{d\}$, and both $\{b\} = A \cap \{b\}$ and $\{d\} = A \cap \{d\}$ belong to $\mathcal{O}_A$.
$\{c,d\}$	✗	$A = \{c\} \sqcup \{d\}$, and both $\{c\} = A \cap \{b,c\}$ and $\{d\} = A \cap \{b,d\}$ belong to $\mathcal{O}_A$.
$\{a,b,c\}$	✗	$A = \{a\} \sqcup \{b,c\}$, and both $\{a\} = A \cap \{a\}$ and $\{b,c\} = A \cap \{b,c\}$ belong to $\mathcal{O}_A$.
$\{a,b,d\}$	✗	$A = \{a\} \cup \{b,d\}$, and both $\{a\} = A \cap \{a\}$ and $\{b,d\} = A \cap \{b,d\}$ belong to $\mathcal{O}_A$.
$\{a,c,d\}$	✗	$A = \{a\} \cup \{c,d\}$, and both $\{a\} = A \cap \{a\}$ and $\{c,d\} = A \cap \{b,c,d\}$ belong to $\mathcal{O}_A$.
$\{b,c,d\}$	✗	$A = \{b,c\} \cup \{d\}$, and both $\{b,c\} = A \cap \{b,c\}$ and $\{d\} = A \cap \{d\}$ belong to $\mathcal{O}_A$.
X	✗	$X = \{a\} \cup \{b,c,d\}$, and both $\{a\}$ and $\{b,c,d\}$ belong to $\mathcal{O}_X$.

Table 11.1: Connected subsets of the topological space $(X, \mathcal{O}_X)$ of Example 11.4.3. For each subset A of X, we denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$ by $\mathcal{O}_A$.

E11 Exercises for Lecture 11

E11.1 Exam questions

Task E11.1.1. Let A be the subset of I^2 given by the union of $\left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right] \times \left[\frac{1}{4}, \frac{3}{4}\right]$.

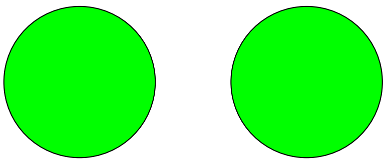


Let

$$I^2 \xrightarrow{\pi} T^2$$

be the quotient map. Let $\mathcal{O}_A$ be the subspace topology on $\pi(A)$ with respect to $(T^2, \mathcal{O}_{T^2})$. Let X be the subset of $\mathbb{R}^2$ given by the union of D^2 and

$$\left\{(x, y) \in \mathbb{R}^2 \mid \|(x-3, y)\| \leq 1\right\}.$$



Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Is $\left(\pi(A), \mathcal{O}_{\pi(A)}\right)$ homeomorphic to $(X, \mathcal{O}_X)$?

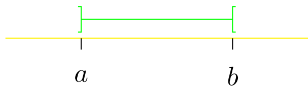
Task E11.1.2. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Let $\mathcal{O}_{[a,b[}$ be the subspace topology on $[a, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let $\mathcal{O}_{]a,b[}$ be the subspace topology on $]a, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Prove that $\left([a,b[\,,\mathcal{O}_{[a,b[}\right)$ is not homeomorphic to $\left(]a,b[\,,\mathcal{O}_{]a,b[}\right)$.

Task E11.1.3. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Let $\mathcal{O}_{]a,b]}$ be the subspace topology on $]a,b]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let $\mathcal{O}_{]a,b[}$ be the subspace topology on $]a,b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Prove that $\left(]a,b]\,,\mathcal{O}_{]a,b]}\right)$ is not homeomorphic to $\left(]a,b[\,,\mathcal{O}_{]a,b[}\right)$. You may wish to proceed by appealing to Task E11.1.2 and to Task E7.1.4.

Task E11.1.4. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Let $\mathcal{O}_{[a,b]}$ be the subspace topology on $[a,b]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let $\mathcal{O}_{]a,b]}$ be the subspace topology on $]a,b]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Prove that $\left([a,b]\,,\mathcal{O}_{[a,b]}\right)$ is not homeomorphic to $\left(]a,b]\,,\mathcal{O}_{]a,b]}\right)$. You may wish to proceed by appealing to Example 11.2.3 and to Task E7.1.4.

Remark E11.1.5. Suppose that $a < b$ belong to $\mathbb{R}$. Let $\mathcal{O}_{[a,b]}$, $\mathcal{O}_{]a,b[}$, $\mathcal{O}_{[a,b[}$, and $]a,b]$ be the subspace topologies with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ on $[a,b]$, $]a,b[$, $[a,b[$, and $]a,b]$ respectively. Assembling Example 11.2.1, Example 11.2.3, Task E11.1.2, Task E11.1.3, and Task E11.1.4, we have proven that no two of $\left([a,b]\,,\mathcal{O}_{[a,b]}\right)$, $\left(]a,b[\,,\mathcal{O}_{]a,b[}\right)$, $\left([a,b[\,,\mathcal{O}_{[a,b[}\right)$, and $\left(]a,b]\,,\mathcal{O}_{]a,b]}\right)$ are homeomorphic.

Task E11.1.6. Let X be a figure of eight, viewed as a subset of $\mathbb{R}^2$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Prove that $(X, \mathcal{O}_X)$ is not homeomorphic to $(S^1, \mathcal{O}_{S^1})$.

Can you find an argument which does not involve removing the junction point of the figure of eight, depicted below?

Task E11.1.7. Let $(X, \mathcal{O}_X)$ be the figure of eight of Task E11.1.6. Prove that $(X, \mathcal{O}_X)$ is not homeomorphic to the unit interval $(I, \mathcal{O}_I)$.

You may wish to appeal to Task E11.3.17.

Task E11.1.8. Let $X = \{a, b, c\}$ be a set with three elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\} .$$

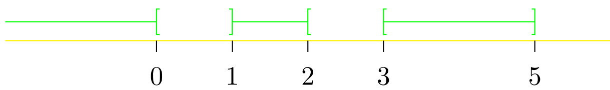
List all the subset of X, and determine whether each is a connected subset of X with respect to $\mathcal{O}_X$. For each which is not, explain why not. Find the connected component in $(X, \mathcal{O}_X)$ of each element of X.

Task E11.1.9. Let $X = \{a, b, c, d, e\}$ be a set with five elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{b\}, \{e\}, \{a, b\}, \{b, e\}, \{c, d\}, \{a, b, e\}, \{b, c, d\}, \{c, d, e\}, \{a, b, c, d\}, \{b, c, d, e\}, X\} .$$

Find the distinct connected components of $(X, \mathcal{O}_X)$. To save yourself a little work, you may wish to glance at Corollary E11.3.15 before proceeding.

Task E11.1.10. Let $X =]-\infty, 0[\cup]1, 2[\cup [3, 5]$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_\mathbb{R})$.

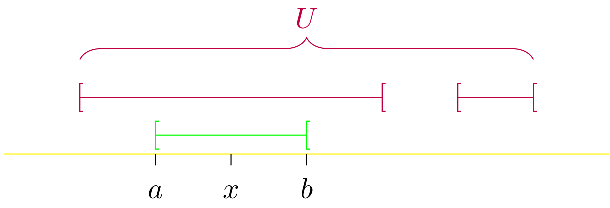


Prove that $(X, \mathcal{O}_X)$ has three distinct connected components.

Task E11.1.11. Let $X = I^2 \cup ([3, 4] \times [0, 1])$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let $\sim$ be the equivalence relation on X which you defined in Task E6.1.8. Prove that $(X/{\sim}, \mathcal{O}_{X/{\sim}})$ has two distinct connected components.

Task E11.1.12. Let $\mathcal{O}_{\mathsf{Sorg}}$ be the set of subsets U of $\mathbb{R}$ such that if x belongs to U, then there is a half open interval $[a, b[$ such that x belongs to $[a, b[$, and such that $[a, b[$ is a subset of U.



Check that $\mathcal{O}_{\mathsf{Sorg}}$ defines a topology on $\mathbb{R}$. Suppose that x belongs to $\mathbb{R}$. Prove that the connected component of x in $(\mathbb{R}, \mathcal{O}_{\mathsf{Sorg}})$ is $\{x\}$.

Remark E11.1.13. The topological space $(\mathbb{R}, \mathcal{O}_{\mathsf{Sorg}})$ is known as the *Sorgenfrey line*. The topology $\mathcal{O}_{\mathsf{Sorg}}$ is also known as the *lower limit topology* on $\mathbb{R}$.

Task E11.1.14. Let $(X, \mathcal{O}_X)$ be a topological space. Let A_0 and A_1 be connected subsets of X with respect to $\mathcal{O}_X$. Is it necessarily the case that $A_0 \cap A_1$ is a connected subset of X? You may find it helpful to take $(X, \mathcal{O}_X)$ to be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

E11.2 In the lecture notes

Task E11.2.1. In the notation of Example 11.1.2, define a map

$$[0,5] \xrightarrow{f} X$$

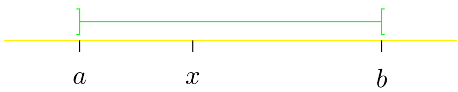
which captures the idea of 'snapping off' $]2,5]$ and 'moving it' to $]4,7]$. Prove that f is a bijection. Prove that f is not continuous.

Task E11.2.2. In the notation of Example 11.2.3, prove that if (I) holds and $f(a) > f(b)$, then

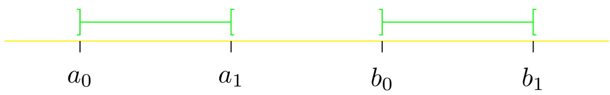
$$\left([a,b[\setminus \{f(a), f(b)\}, \mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

is not connected.

Task E11.2.3. Suppose that $a < x < b$ belong to $\mathbb{R}$. Let $\mathcal{O}_{]a,b[}$ denote the subspace topology on $]a,b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $\mathcal{O}_{]a,b[\setminus\{x\}}$ denote the subspace topology on $]a,b[\setminus \{x\}$ with respect to $\left(]a,b[, \mathcal{O}_{]a,b[}\right)$.



Suppose that $a_0 < a_1 < b_0 < b_1$ belong to $\mathbb{R}$. Let X be the union of $]a_0, a_1[$ and $]b_0, b_1[$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Prove that $\big(]a,b[\setminus \{x\}, \mathcal{O}_{]a,b[\setminus\{x\}}\big)$ is homeomorphic to $(X, \mathcal{O}_X)$. You may wish to proceed as follows.

(1) Let $\mathcal{O}_{]a,x[}$ denote the subspace topology on $]a,x[$ with respect to

$$\big(]a,b[\setminus \{x\}, \mathcal{O}_{]a,b[\setminus\{x\}}\big)\,.$$

Let $\mathcal{O}_{]a_0,a_1[}$ denote the subspace topology on $]a_0,a_1[$ with respect to $(X, \mathcal{O}_X)$.

By Task E2.3.1 and Example 7.3.3, observe that there is a homeomorphism

$$]a,x[\xrightarrow{f_0}]a_0,a_1[\,.$$

(2) Let $\mathcal{O}_{]x,b[}$ denote the subspace topology on $]x,b[$ with respect to

$$\big(]a,b[\setminus \{x\}, \mathcal{O}_{]a,b[\setminus\{x\}}\big)\,.$$

Let $\mathcal{O}_{]b_0,b_1[}$ denote the subspace topology on $]b_0,b_1[$ with respect to $(X, \mathcal{O}_X)$.

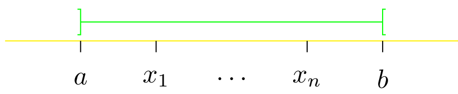
By Task E2.3.1 and Example 7.3.4, observe that there is a homeomorphism

$$]x,b[\xrightarrow{f_1}]b_0,b_1[\,.$$

(3) By Task E7.3.5, deduce from (1) and (2) that there is a homeomorphism

$$]a, b[\setminus \{x\} \longrightarrow X.$$

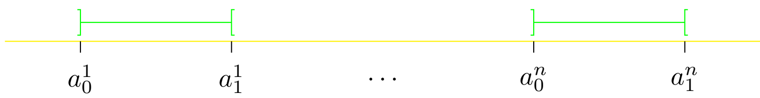
Task E11.2.4. Suppose that $a < x_1 < \ldots < x_n < b$ belong to $\mathbb{R}$. Let $\mathcal{O}_{]a,b[}$ denote the subspace topology on $]a, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $\mathcal{O}_{]a,b[\setminus\{x_1,\ldots,x_n\}}$ denote the subspace topology on $]a, b[\setminus \{x_1, \ldots, x_n\}$ with respect to $\big(]a, b[\,, \mathcal{O}_{]a,b[}\big)$.



Suppose that $a_0^1 < a_1^1 < \ldots < a_0^n < a_1^n$ belong to $\mathbb{R}$. Let X be

$$\bigcup_{1 \leq i \leq n}]a_0^i, a_1^i[\,.$$

Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Prove that $\big(]a, b[\setminus \{x_1, \ldots x_n\}, \mathcal{O}_{]a,b[\setminus\{x_1,\ldots,x_n\}}\big)$ is homeomorphic to $(X, \mathcal{O}_X)$. You may wish to proceed by induction, appealing to Task E11.2.3 and to Task E7.3.5.

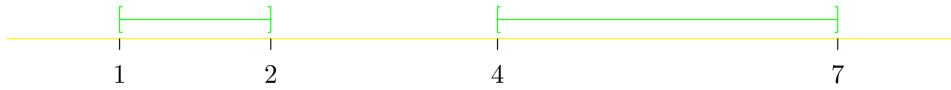
Task E11.2.5. Let X be a subset of $\mathbb{R}$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that x belongs to $\mathbb{R}$. Let $\mathcal{O}_{X\setminus\{x\}}$ be the subspace topology on $X \setminus \{x\}$ with respect to $(X, \mathcal{O}_X)$. Prove that $(X \setminus \{x\}, \mathcal{O}_{X\setminus\{x\}})$ is not connected.

Task E11.2.6. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Let A_0 and A_1 be connected subsets of X with respect to $\mathcal{O}_X$ which both satisfy (1) and (2) of Terminology 11.3.2. Prove that $A_0 = A_1$.

Task E11.2.7. Let $X = \{x\}$ be a set with one element. As discussed in Example **??**, the unique topology $\mathcal{O}_X$ on X is given by $\{\emptyset, X\}$. Then $(X, \mathcal{O}_X)$ is connected. Check that you understand why!

Task E11.2.8. Let X be a set. Let $\mathcal{O}_X$ be the discrete topology on X. Let A be a subset of X. Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Prove that $\mathcal{O}_A$ is the discrete topology on A.

Task E11.2.9. Let $(X, \mathcal{O}_X)$ be as in Example 11.4.5.



Prove that if x belongs to $[4, 7]$, then $\Gamma^x_{(X,\mathcal{O}_X)} = [4, 7]$.

Task E11.2.10. Prove carefully the three assertions concerning numbers of connected components in Example 11.5.2.

Task E11.2.11. In the notation of Example 11.2.5, prove that $\left(I \setminus \{t\}, \mathcal{O}_{I\setminus\{t\}}\right)$ has two distinct connected components, and that $\left(S^1 \setminus \{f(t)\}, \mathcal{O}_{S^1\setminus\{f(t)\}}\right)$ is connected.

E11.3 For a deeper understanding

Task E11.3.1. Suppose that n belongs to $\mathbb{N}$, and that $n > 1$. Suppose that x belongs to $\mathbb{R}^n$. Let $\mathcal{O}_{\mathbb{R}^n\setminus\{x\}}$ be the subspace topology on $\mathbb{R}^n \setminus \{x\}$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. Prove that $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n\setminus\{x\}})$ is connected. You may wish to proceed as follows.

(1) Observe that $\mathbb{R}^n \setminus \{x\}$ is the union of $]-\infty, x[\times\mathbb{R}^{n-1}$, $]x, \infty[\times\mathbb{R}^{n-1}$, $\mathbb{R}^{n-1}\times]x, \infty[$, and $\mathbb{R}^{n-1} \times]-\infty, x[$.

(2) By Task E10.3.5 and Proposition 10.7.1, observe each of these four sets is a connected subset of $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.

(3) By Task E2.3.1, deduce that each is a connected subset of $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n\setminus\{x\}})$.

(4) By Task E10.3.9, deduce that $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n\setminus\{x\}})$ is connected.

Task E11.3.2. Let $\mathcal{O}_{[0,1[}$ be the subspace topology on $[0, 1[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

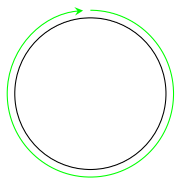
In Task E7.3.7, you were asked to prove that the map

$$[0, 1[\xrightarrow{f} S^1$$

given by $t \mapsto \phi(t)$, where

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is the map of Task E5.3.27. Prove that f is not a homeomorphism.



Task E11.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x_0 and x_1 belong to X. Prove that either $\Gamma^{x_0}_{(X,\mathcal{O}_X)} = \Gamma^{x_1}_{(X,\mathcal{O}_X)}$, or that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} \cap \Gamma^{x_1}_{(X,\mathcal{O}_X)}$ is empty. You may wish to proceed as follows.

(1) Suppose that x_0 and x_1 belong to X, and that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} \cap \Gamma^{x_1}_{(X,\mathcal{O}_X)}$ is not empty. By Proposition 11.3.7, we have that $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$ and $\Gamma^{x_1}_{(X,\mathcal{O}_X)}$ are connected subsets of X with respect to $\mathcal{O}_X$. By Task E10.3.9, deduce that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} \cup \Gamma^{x_1}_{(X,\mathcal{O}_X)}$ is a connected subset of X with respect to $\mathcal{O}_X$.

(2) By Remark 11.3.6, we have that x_0 belongs to $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$. Thus x_0 belongs to $\Gamma^{x_0}_{(X,\mathcal{O}_X)} \cup \Gamma^{x_1}_{(X,\mathcal{O}_X)}$. By (1) and the definition of $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$, deduce that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} \cup \Gamma^{x_1}_{(X,\mathcal{O}_X)}$ is a subset of $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$.

(3) Deduce that $\Gamma^{x_1}_{(X,\mathcal{O}_X)}$ is a subset of $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$.

(4) Arguing as in (2) and (3), demonstrate that $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$ is a subset of $\Gamma^{x_1}_{(X,\mathcal{O}_X)}$.

(5) Conclude that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} = \Gamma^{x_1}_{(X,\mathcal{O}_X)}$.

Terminology E11.3.4. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x_0 and x_1 belong to X. Then $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$ and $\Gamma^{x_1}_{(X,\mathcal{O}_X)}$ are *distinct* if $\Gamma^{x_0}_{(X,\mathcal{O}_X)} \cap \Gamma^{x_1}_{(X,\mathcal{O}_X)}$ is empty.

Remark E11.3.5. Let $(X, \mathcal{O}_X)$ be a topological space. By Remark 11.3.6, we have that x belongs to Γ_x. Thus $X = \bigcup_{x \in X} \Gamma^{x}_{(X,\mathcal{O}_X)}$.

Terminology E11.3.6. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that n belongs to $\mathbb{N}$. Then $(X, \mathcal{O}_X)$ has n *distinct connected components* if there is a set $\{x_j\}_{1 \leq j \leq n}$ of elements of X such that the following hold.

(1) We have that $X = \bigcup_{1 \leq j \leq n} \Gamma^{x_j}_{(X,\mathcal{O}_X)}$.

(2) For every $1 \leq j < k \leq n$, we have that $\Gamma^{x_j}_{(X,\mathcal{O}_X)}$ and $\Gamma^{x_k}_{(X,\mathcal{O}_X)}$ are distinct.

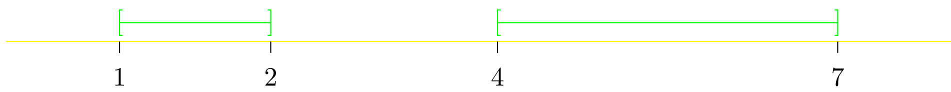
Remark E11.3.7. By Task E11.3.3, a topological space $(X, \mathcal{O}_X)$ has n distinct connected components if the set $\left\{\Gamma^{x}_{(X,\mathcal{O}_X)}\right\}_{x \in X}$ has exactly n elements (remember that all equal elements of a set count as one!).

Remark E11.3.8. In particular, there is at most one n such that $(X, \mathcal{O}_X)$ has n distinct connected components.

Terminology E11.3.9. Let $(X, \mathcal{O}_X)$ be a topological space. Then $(X, \mathcal{O}_X)$ has *finitely many distinct connected components* if there is an $n \in \mathbb{N}$ such that $(X, \mathcal{O}_X)$ has n distinct connected components.

Example E11.3.10. Suppose that $(X, \mathcal{O}_X)$ is connected. By Example 11.4.1, we then have that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} = \Gamma^{x_1}_{(X,\mathcal{O}_X)}$ for all x_0 and x_1 which belong to X. Thus $(X, \mathcal{O}_X)$ has one distinct connected component.

Example E11.3.11. Let $X = [1, 2] \cup [4, 7]$.



Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By 11.4.5, $(X, \mathcal{O}_X)$ has two distinct connected components.

Example E11.3.12. Let $\mathbb{Q}$ be the set of rational numbers. Let $\mathcal{O}_{\mathbb{Q}}$ be the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Example 11.4.4, $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ has infinitely many distinct connected components.

Task E11.3.13. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Prove that $\Gamma^{x}_{(X,\mathcal{O}_X)}$ is closed with respect to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) By Proposition 11.3.7, we have that $\Gamma^{x}_{(X,\mathcal{O}_X)}$ is a connected subset of X with respect to $\mathcal{O}_X$. By Corollary E10.3.4, deduce that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is a connected subset of X with respect to $\mathcal{O}_X$.

(2) By Remark 11.3.6, we have that x belongs to $\Gamma^{x}_{(X,\mathcal{O}_X)}$. By Remark 8.3.3, deduce that x belongs to $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$. By (1) and the definition of $\Gamma^{x}_{(X,\mathcal{O}_X)}$, deduce that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is a subset of $\Gamma^{x}_{(X,\mathcal{O}_X)}$.

(3) By Remark 8.5.4, we have that $\Gamma^{x}_{(X,\mathcal{O}_X)}$ is a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$. Deduce that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is equal to $\Gamma^{x}_{(X,\mathcal{O}_X)}$.

(4) By Proposition 9.1.1, conclude that $\Gamma^{x}_{(X,\mathcal{O}_X)}$ is closed in X with respect to $\mathcal{O}_X$.

Task E11.3.14. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that $(X, \mathcal{O}_X)$ has finitely many distinct connected components. Prove that every connected component belongs to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) Since $(X, \mathcal{O}_X)$ has finitely many distinct connected components, there is an $n \in \mathbb{N}$, and a set $\{x_j\}_{1 \leq j \leq n}$, such that $X = \bigcup_{1 \leq j \leq n} \Gamma^{x_j}_{(X,\mathcal{O}_X)}$, and $\Gamma^{x_j}_{(X,\mathcal{O}_X)}$ and $\Gamma^{x_k}_{(X,\mathcal{O}_X)}$ are distinct for every $1 \leq j < k \leq n$.

(2) Suppose that x belongs to X. By (1) and Task E11.3.3, observe that $\Gamma^{x}_{(X,\mathcal{O}_X)} = \Gamma^{x_k}_{(X,\mathcal{O}_X)}$ for some $1 \leq k \leq n$.

(3) By Task E11.3.13, we have that $\Gamma^{x_j}_{(X,\mathcal{O}_X)}$ is closed in X with respect to $\mathcal{O}_X$ for every $1 \leq j \in n$ such that $j \neq k$. By Remark E1.3.2, deduce that $X \setminus \Gamma^{x}_{(X,\mathcal{O}_X)} = \bigcup_{1 \leq j \leq n \text{ and } j \neq k} \Gamma^{x_j}_{(X,\mathcal{O}_X)}$ is closed in X with respect to $\mathcal{O}_X$.

(4) Conclude that $\Gamma^{x}_{(X,\mathcal{O}_X)}$ belongs to $\mathcal{O}_X$.

Corollary E11.3.15. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that X is finite. Then every connected component of $(X, \mathcal{O}_X)$ belongs to $\mathcal{O}_X$.

Proof. If X is finite, then X has only finitely many subsets. Thus $(X, \mathcal{O}_X)$ has only finitely many distinct connected components. By Task E11.3.14, we deduce that every connected component of $(X, \mathcal{O}_X)$ belongs to $\mathcal{O}_X$. □

Task E11.3.16. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a continuous map. Suppose that x belongs to X. Prove that $f\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is a subset of $\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$. You may wish to proceed as follows.

(1) By Proposition 11.3.7, we have that $\Gamma^{x}_{(X,\mathcal{O}_X)}$ is a connected subset of X with respect to $\mathcal{O}_X$. By Task E10.3.2, deduce that $f\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is a connected subset of Y with respect to $\mathcal{O}_Y$.

(2) We have that $f(x)$ belongs to $f\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$. By definition of $\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$, deduce that $f\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is a subset of $\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$.

Task E11.3.17. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Suppose that x belongs to X. Let $\mathcal{O}_{\Gamma^{x}_{(X,\mathcal{O}_X)}}$ be the subspace topology on $\Gamma^{x}_{(X,\mathcal{O}_X)}$ with respect to $(X, \mathcal{O}_X)$. Let $\mathcal{O}_{\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}}$ be the subspace topology on $\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$ with respect to $(Y, \mathcal{O}_Y)$. Prove that $\left(\Gamma^{x}_{(X,\mathcal{O}_X)}, \mathcal{O}_{\Gamma^{x}_{(X,\mathcal{O}_X)}}\right)$ is homeomorphic to $\left(\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}, \mathcal{O}_{\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}}\right)$. You may wish to proceed as follows.

(1) By Task E11.3.16, we have that $f\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is a subset of $\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$. Since f is continuous, deduce by Task E5.1.8 and Task E5.1.9 that the map

$$\Gamma^{x}_{(X,\mathcal{O}_X)} \xrightarrow{f'} \Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$$

given by $y \mapsto f(y)$ is continuous.

(2) Since f is a homeomorphism, there is a continuous map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$. By Task E11.3.16, we have that $g\left(\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}\right)$ is a subset of $\Gamma^{g(f(x))}_{(X,\mathcal{O}_X)}$. Since $g \circ f = id_X$, we have that $g\left(f(x)\right) = x$. Thus $g\left(\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}\right)$ is a subset of $\Gamma^{x}_{(X,\mathcal{O}_X)}$. Since g is continuous, deduce by Task E5.1.9 and Task E5.1.9 that the map

$$\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)} \xrightarrow{g'} \Gamma^{x}_{(X,\mathcal{O}_X)}$$

given by $y \mapsto g(y)$ is continuous.

(3) Observe that $g' \circ f' = id_{\Gamma^{x}_{(X,\mathcal{O}_X)}}$, and that $f' \circ g' = id_{\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}}$. Conclude that f' is a homeomorphism.

Task E11.3.18. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homeomorphic topological spaces. Prove that there is a bijection between the set $\Gamma_{(X,\mathcal{O}_X)} = \left\{\Gamma^{x}_{(X,\mathcal{O}_X)}\right\}_{x \in X}$ and the set $\Gamma_{(Y,\mathcal{O}_Y)} = \left\{\Gamma^{y}_{(Y,\mathcal{O}_Y)}\right\}_{y \in Y}$. You may wish to proceed as follows.

(1) Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Suppose that x_0 and x_1 belong to X, and that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} = \Gamma^{x_1}_{(X;\mathcal{O}_X)}$. As a corollary of (3) of Task E11.3.17, we have that $\Gamma^{f(x_0)}_{(Y,\mathcal{O}_Y)} = f\left(\Gamma^{x_0}_{(X,\mathcal{O}_X)}\right)$, and that $\Gamma^{f(x_1)}_{(Y,\mathcal{O}_Y)} = f\left(\Gamma^{x_1}_{(X,\mathcal{O}_X)}\right)$. Since $\Gamma^{x_0}_{(X,\mathcal{O}_X)} = \Gamma^{x_1}_{(X,\mathcal{O}_X)}$, we have that $f\left(\Gamma^{x_0}_{(X,\mathcal{O}_X)}\right) = f\left(\Gamma^{x_1}_{(X,\mathcal{O}_X)}\right)$. Deduce that $\Gamma^{f(x_0)}_{(Y,\mathcal{O}_Y)} = \Gamma^{f(x_1)}_{(Y,\mathcal{O}_Y)}$.

(2) By Task E7.3.2, there is a homeomorphism

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$. Suppose that y_0 and y_1 belong to Y, and that $\Gamma^{y_0}_{(Y,\mathcal{O}_Y)} = \Gamma^{y_1}_{(Y,\mathcal{O}_Y)}$. Arguing as in (1), demonstrate that $\Gamma^{g(y_0)}_{(X,\mathcal{O}_X)} = \Gamma^{g(y_1)}_{(X,\mathcal{O}_X)}$.

(3) Let

$$\Gamma_X \xrightarrow{f'} \Gamma_Y$$

be the map given by $\Gamma^{x}_{(X,\mathcal{O}_X)} \mapsto \Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$. By (1), this map is well-defined. Let

$$\Gamma_Y \xrightarrow{g'} \Gamma_X$$

be the map given by $\Gamma^{y}_{(Y,\mathcal{O}_Y)} \mapsto \Gamma^{g(y)}_{(X,\mathcal{O}_X)}$. By (2), this map is well-defined. Observe that $g' \circ f' = id_{\Gamma_{(X,\mathcal{O}_X)}}$, and that $f' \circ g' = id_{\Gamma_{(Y,\mathcal{O}_Y)}}$,

Corollary E11.3.19. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homeomorphic topological spaces. Suppose that there is an $n \in \mathbb{N}$ such that $(X, \mathcal{O}_X)$ has n distinct connected components. Then $(Y, \mathcal{O}_Y)$ has n distinct connected components.

Proof. Follows immediately from Task E11.3.18 and Remark E11.3.7. □

Notation E11.3.20. Let J be a set. For every j which belongs to J, let X_j be a set. Let $\bigsqcup_{j \in J} X_j$ be the corresponding coproduct, in the sense of Definition A.3.3. We denote by

$$X_j \xrightarrow{i_j} \bigsqcup_{j \in J} X_j$$

the map given by $x \mapsto (x, j)$.

Task E11.3.21. Let J be a set. For every j which belongs to J, let $(X_j, \mathcal{O}_{X_j})$ be a topological space. Let $\mathcal{O}_{\bigsqcup_{j \in J} X_j}$ be the set of subsets U of the coproduct $\bigsqcup_{j \in J} X_j$ such that $i_j^{-1}(U)$ belongs to $\mathcal{O}_{X_j}$. Prove that $\left(\bigsqcup_{j \in J} X_j, \mathcal{O}_{\bigsqcup_{j \in J} X_j} \right)$ is a topological space. You may wish to look back at the proof of Proposition 6.1.5.

Terminology E11.3.22. We refer to $\mathcal{O}_{\bigsqcup_{j \in J} X_j}$ as the *coproduct topology* on $\bigsqcup_{j \in J} X_j$.

Task E11.3.23. Let J be a set. For every j which belongs to J, let $(X_j, \mathcal{O}_{X_j})$ be a topological space. Let $\bigsqcup_{j \in J} X_j$ be equipped with the coproduct topology $\mathcal{O}_{\bigsqcup_{j \in J} X_j}$. Observe that

$$X_j \xrightarrow{i_j} \bigsqcup_{j \in J} X_j$$

is continuous, for every j which belongs to J.

Task E11.3.24. For every pair of integers j and n such that $0 \leq j \leq n$, let $(X_j, \mathcal{O}_{X_j})$ be a topological space. How many connected components does $\left(\bigsqcup_{0 \leq j \leq n} X_j, \mathcal{O}_{\bigsqcup_{0 \leq j \leq n} X_j}\right)$ have? Prove that your guess holds!

E11.4 Exploration — bijections

Task E11.4.1. Suppose that $a < b$ and $a_0 < a_1 < b_0 < b_1$ belong to $\mathbb{R}$. Prove that there is a bijection

$$]a, b[\longrightarrow]a_0, a_1[\cup]b_0, b_1[.$$

You may wish to proceed as follows.

(1) A homeomorphism is in particular a bijection. By Example 7.3.4, we thus have that there is a bijection

$$]a, b[\xrightarrow{f}]a_0, a_1[.$$

By Task E7.2.1, deduce that f is an injection.

(2) As observed in Remark A.2.3, the inclusion map

$$]a_0, a_1[\xrightarrow{i}]a_0, a_1[\cup]b_0, b_1[$$

is an injection. By Proposition A.2.2, deduce that the map

$$]a, b[\xrightarrow{f \circ i}]a_0, a_1[\cup]b_0, b_1[$$

is an injection.

(3) By Example 7.3.10 and Task E7.3.2, there is a bijection

$$\mathbb{R} \xrightarrow{g}]a, b[.$$

By Task E7.2.1, deduce that f is an injection.

(4) The inclusion map

$$]a_0, a_1[\, \cup \,]b_0, b_1[\xrightarrow{\;\; j \;\;} \mathbb{R}$$

is an injection. By Proposition A.2.2, deduce that the map

$$]a_0, a_1[\, \cup \,]b_0, b_1[\xrightarrow{\; g \circ j \;}]a, b[$$

is an injection.

(5) By (2), (4), and Proposition A.2.5, conclude that there is a bijection

$$]a, b[\longrightarrow]a_0, a_1[\, \cup \,]b_0, b_1[\,.$$

Task E11.4.2. Find a bijection

$$[1, 2] \cup [4, 7] \xrightarrow{\;\; f \;\;} [1, 5]$$

You may wish to proceed as follows.

(1) Let f be the identity on $[1, 2]$.

(2) Send 4 to 3, and send 7 to 5.

(3) Appealing to Task E11.4.1, let f map $]4, 7[$ bijectively to $]2, 3[\, \cup \,]3, 5[$.

Task E11.4.3. Find a bijection between I and S^1. You may wish to proceed as follows.

(1) Map 0 to $(0, 1)$, and map 1 to $(0, -1)$.

(2) By Task E11.4.1, observe that there is a bijection from $]0, 1[$ to $\left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[$.

(3) Use the bijection of (2) and the map ϕ of Task E5.3.27to map $]0, 1[$ bijectively to the union of

$$\left\{(x, y) \in S^1 \mid y > 0\right\}$$

and

$$\left\{(x, y) \in S^1 \mid y < 0\right\}.$$

Task E11.4.4. Find a bijection between $\mathbb{R}$ and $\mathbb{R}^n$. You may wish to proceed as follows.

(1) Observe that the map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}^n$$

given by $x \mapsto (x, 0, \ldots, 0)$ is an injection.

(2) By Example 7.3.10 and Task E7.3.2, there is a bijection

$$\mathbb{R} \xrightarrow{g_1} \left]1, \tfrac{3}{2}\right[.$$

By Task E7.2.1, we have that g_1 is an injection. Let

$$\mathbb{R} \xrightarrow{g_n} \left]n, n + \tfrac{1}{2}\right[$$

be the map given by $x \mapsto g_1(x) + n - 1$. Since g_1 is an injection, deduce that g_n is an injection.

(3) Deduce from (2) that the map

$$\mathbb{R}^n \longrightarrow \mathbb{R}$$

given by $(x_1, \ldots, x_n) \mapsto (g_1(x_1), \ldots, g_n(x_n))$ is an injection.

(4) By (1), (3), and Proposition A.2.5, deduce that there is a bijection between $\mathbb{R}$ and $\mathbb{R}^n$.

E11.5 Exploration — totally disconnected topological spaces

Definition E11.5.1. A topological space $(X, \mathcal{O}_X)$ is *totally disconnected* if, for every x which belongs to X, the connected component of x in $(X, \mathcal{O}_X)$ is $\{x\}$.

Example E11.5.2. Let X be a set. Let $\mathcal{O}_X$ be the discrete topology on X. By Example 11.4.2, we have that $(X, \mathcal{O}_X)$ is totally disconnected.

Example E11.5.3. Let $\mathbb{Q}$ be the set of rational numbers. Let $\mathcal{O}_{\mathbb{Q}}$ be the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Example 11.4.4, we have that $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is totally disconnected.

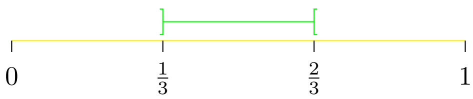
Example E11.5.4. By Task E11.1.12, the Sorgenfrey line $(\mathbb{R}, \mathcal{O}_{\mathsf{Sorg}})$ is totally disconnected.

Notation E11.5.5. Let Cantor be the subset of I given by

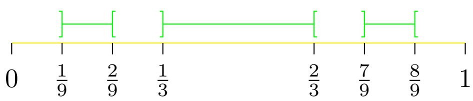
$$I \setminus \left(\bigcup_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N},\ 1 \leq n \leq 3^{m-1}} \left] \frac{3(n-1)+1}{3^m}, \frac{3(n-1)+2}{3^m} \right[\right).$$

Remark E11.5.6. In other words, Cantor is obtained as follows.

(1) Delete $\left]\frac{1}{3}, \frac{2}{3}\right[$ from I.



(2) Delete $\left]\frac{1}{9}, \frac{2}{9}\right[$ and $\left]\frac{7}{9}, \frac{8}{9}\right[$ from I.



(3) Delete $\left]\frac{1}{27}, \frac{2}{27}\right[$, $\left]\frac{7}{27}, \frac{8}{27}\right[$, $\left]\frac{19}{27}, \frac{20}{27}\right[$, and $\left]\frac{25}{27}, \frac{26}{27}\right[$ from I.

(4) Continue this pattern of deletions of open intervals for all 3^n, where n belongs to $\mathbb{N}$.

Terminology E11.5.7. We refer to Cantor as the *Cantor set*.

Task E11.5.8. Let $\mathcal{O}_{\mathsf{Cantor}}$ be the subspace topology on Cantor with respect to $(I, \mathcal{O}_I)$. Prove that $(\mathsf{Cantor}, \mathcal{O}_{\mathsf{Cantor}})$ is totally disconnected.