MA3002 Generell Topologi — Vår 2014

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12 Tuesday 11th February

12.1 Further examples of the number of distinct connected components as an invariant

Example 12.1.1. Let K be the subset of $\mathbb{R}^2$ given by the union of

$$\left\{(0,y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\right\}$$

and

$$\left\{(x,y) \in \mathbb{R}^2 \mid x = y \text{ and } -1 \leq y \leq 1\right\}.$$

Let $\mathcal{O}_{\mathsf{K}}$ be the subspace topology on K with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ be as in Example 11.5.2.

Suppose that

$$\mathsf{K} \xrightarrow{f} \mathsf{T}$$

is a homeomorphism. Let x be the point $(0,0)$ of K.

Let $\mathcal{O}_{\mathsf{K}\setminus\{x\}}$ be the subspace topology on $\mathsf{K} \setminus \{x\}$ with respect to $(\mathsf{K}, \mathcal{O}_{\mathsf{K}})$. Then

$$\left(\mathsf{K} \setminus \{x\}, \mathcal{O}_{\mathsf{K}\setminus\{x\}}\right)$$

has four distinct connected components.

Let $\mathcal{O}_{\mathsf{T}\setminus\{f(x)\}}$ be the subspace topology on $\mathsf{T} \setminus \{f(x)\}$ with respect to $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$. Suppose that $f(x)$ is $(0, 1)$.

Then

$$\left(\mathsf{T} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{T}\setminus\{f(x)\}}\right)$$

has three distinct connected components.

Suppose that $f(x) = (x', y')$. Suppose that $0 < |x'| < 1$.

Then

$$\left(\mathsf{T} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{T}\setminus\{f(x)\}}\right)$$

has two distinct connected components.

Suppose that $-1 < y' < 1$.

Then

$$\left(\mathsf{T} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{T} \setminus \{f(x)\}}\right)$$

has two distinct connected components.

Suppose that $f(x)$ is $(-1, 1)$, $(1, 1)$, or $(0, -1)$.

Then

$$\left(\mathsf{T} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{T} \setminus \{f(x)\}}\right)$$

is connected.

Thus

$$\left(\mathsf{T} \setminus \{f(x)\}\,, \mathcal{O}_{\mathsf{T}\setminus\{f(x)\}}\right)$$

has at most three distinct connected components. Since f is a homeomorphism, we have, by Task E7.1.20, that there is a homeomorphism

$$\mathsf{K} \setminus \{x\} \longrightarrow \mathsf{T} \setminus \{f(x)\}\,.$$

By Corollary E11.3.19, since

$$\left(\mathsf{K} \setminus \{x\}, \mathcal{O}_{\mathsf{K}\setminus\{x\}}\right)$$

has four distinct connected components, we deduce that

$$\left(\mathsf{T} \setminus \{f(x)\}\,, \mathcal{O}_{\mathsf{T}\setminus\{f(x)\}}\right)$$

has four distinct connected components. Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$\mathsf{K} \longrightarrow \mathsf{T}.$$

In other words, $(\mathsf{K}, \mathcal{O}_{\mathsf{K}})$ is not homeomorphic to $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$.

Remark 12.1.2. To fill in the details of the calculations of numbers of distinct connected components in Example 12.1.1 is the topic of Task **??**.

Example 12.1.3. Let $\mathsf{Ø}$ be the subset of $\mathbb{R}^2$ given by the union of S^1 and

$$\{(x, y) \mid -1 \leq x \leq 1 \text{ and } x = y\}\,.$$

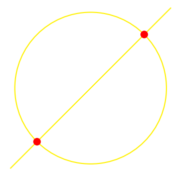
Let $\mathcal{O}_{\mathsf{Ø}}$ be the subspace topology on $\mathsf{Ø}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ be as in Example 11.5.2.

Suppose that

$$\varnothing \xrightarrow{f} \mathsf{I}$$

is a homeomorphism. Let x be the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ of $\varnothing$. Let y be the point $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ of $\varnothing$.



Let $\mathcal{O}_{\varnothing \setminus \{x,y\}}$ be the subspace topology on $\varnothing \setminus \{x,y\}$ with respect to $(\varnothing, \mathcal{O}_{\varnothing})$. Then

$$\left(\varnothing \setminus \{x,y\}, \mathcal{O}_{\varnothing \setminus \{x,y\}}\right)$$

has five distinct connected components.

Suppose that neither $f(x)$ nor $f(y)$ is $(0,0)$ or $(0,1)$.

Then

$$\left(\mathsf{I} \setminus \{f(x), f(y)\}, \mathcal{O}_{\mathsf{I} \setminus \{f(x),f(y)\}}\right)$$

has three distinct connected components.

Suppose that one of $f(x)$ or $f(y)$ is $(0,0)$ or $(0,1)$, and that the other is neither $(0,0)$ nor $(0,1)$.

Then

$$\left(\mathsf{I} \setminus \{f(x), f(y)\}, \mathcal{O}_{\mathsf{I}\setminus\{f(x),f(y)\}}\right)$$

has two distinct connected components.

Suppose that one of $f(x)$ or $f(y)$ is $(0,0)$, and that the other is $(0,1)$.

Then

$$\left(\mathsf{I} \setminus \{f(x), f(y)\}, \mathcal{O}_{\mathsf{I} \setminus \{f(x), f(y)\}}\right)$$

is connected.

Thus

$$\left(\mathsf{I} \setminus \{f(x), f(y)\}, \mathcal{O}_{\mathsf{I} \setminus \{f(x), f(y)\}}\right)$$

has at most three distinct connected components. Since f is a homeomorphism, we have by Task E7.1.20 that there is a homeomorphism

$$\emptyset \setminus \{x, y\} \longrightarrow \mathsf{I} \setminus \{f(x), f(y)\}.$$

By Corollary E11.3.19, since

$$\left(\emptyset \setminus \{x, y\}, \mathcal{O}_{\emptyset \setminus \{x,y\}}\right)$$

has five distinct connected components, we deduce that

$$\left(\mathsf{I} \setminus \{f(x), f(y)\}, \mathcal{O}_{\mathsf{I} \setminus \{f(x), f(y)\}}\right)$$

has five distinct connected components. Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$\emptyset \longrightarrow \mathsf{I}.$$

In other words, $(\emptyset, \mathcal{O}_{\emptyset})$ is not homeomorphic to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$.

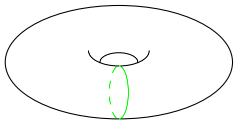
Remark 12.1.4. We cannot distinguish $(\emptyset, \mathcal{O}_{\mathsf{O}})$ from $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ by removing just one point from each topological space and counting the resulting numbers of distinct connected components. To check that you understand why is the topic of Task E12.2.2.

Remark 12.1.5. To fill in the details of the calculations of numbers of distinct connected components in Example 12.1.3 is the topic of Task E12.2.3.

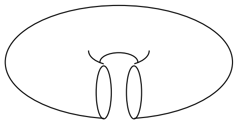
12.2 Can we take our technique further?

Remark 12.2.1. In all of our examples of distinguishing a pair of topological spaces by means of connectedness, at least one of the two has been 'one dimensional': built out of lines. Can our technique distinguish between 'higher dimensional' topological spaces?

Remark 12.2.2. Let us try to distinguish $(T^2, \mathcal{O}_{T^2})$ from $(S^2, \mathcal{O}_{S^2})$. Let X be a subset of T^2 such that $(X, \mathcal{O}_X)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$, where $\mathcal{O}_X$ is the subspace topology on X with respect to $(T^2, \mathcal{O}_{T^2})$.

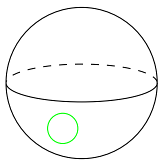


For the X depicted above, we have that $T^2 \setminus X$ is as depicted below.

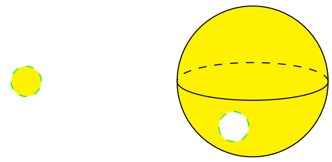


Let $\mathcal{O}_{T^2 \setminus X}$ be the subspace topology on $T^2 \setminus X$ with respect to $(T^2, \mathcal{O}_{T^2})$. We have that $(T^2 \setminus X, \mathcal{O}_{T^2 \setminus X})$ is homeomorphic to a cylinder.

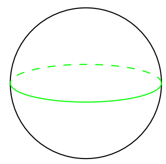
In particular, we have that $(T^2 \setminus X, \mathcal{O}_{T^2 \setminus X})$ is connected. Let Y be a subset of S^2 such that $(Y, \mathcal{O}_Y)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$, where $\mathcal{O}_Y$ is the subspace topology on Y with respect to $(S^2, \mathcal{O}_{S^2})$.



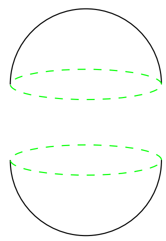
For any such Y, it seems intuitively that $S^2 \setminus Y$ has exactly two distinct connected components. In the example depicted above, we obtain the open disc enclosed by the circle, and the open subset of S^2 which remains after cutting out the closed disc enclosed by the circle.



Suppose that Y is the equator.



Then $S^2 \setminus Y$ consists of the northern hemisphere and the southern hemisphere.



Suppose that

$$T^2 \xrightarrow{\ f\ } S^2$$

is a homeomorphism. Let $(X, \mathcal{O}_X)$ be as above, with the property that $(T^2 \setminus X, \mathcal{O}_{T^2\setminus X})$ is connected. Let $S^2 \setminus f(X)$ be equipped with the subspace topology $\mathcal{O}_{S^2\setminus f(X)}$ with respect to $(S^2, \mathcal{O}_{S^2})$. Since f is a homeomorphism, we have, by Task E7.1.20, that there is a homeomorphism

$$T^2 \setminus \{X\} \longrightarrow S^2 \setminus f(X).$$

By Corollary 10.5.2, we deduce that $\big(S^2 \setminus f(X), \mathcal{O}_{S^2\setminus f(X)}\big)$ is connected. Let $\mathcal{O}_{f(X)}$ be the subspace topology on $f(X)$ with respect to $(S^2, \mathcal{O}_{S^2})$.

Since f is a homeomorphism, we have, by Task **??**, that $\big(f(X), \mathcal{O}_{f(X)}\big)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$. If our intuition is correct, we deduce that $\big(S^2 \setminus f(X), \mathcal{O}_{S^2\setminus f(X)}\big)$ has exactly two distinct connected components. Thus we have a contradiction. We deduce that there does not exist a homeomorphism

$$T^2 \longrightarrow S^2.$$

In other words, $(T^2, \mathcal{O}_{T^2})$ and $(S^2, \mathcal{O}_{S^2})$ are not homeomorphic.

Remark 12.2.3. This argument *does* prove that $(T^2, \mathcal{O}_{T^2})$ is not homeomorphic to $(S^2, \mathcal{O}_{S^2})$. However, we have to be very careful! We must rigorously prove that $(S^2 \setminus Y, \mathcal{O}_{S^2\setminus Y})$ has exactly two distinct connected components.

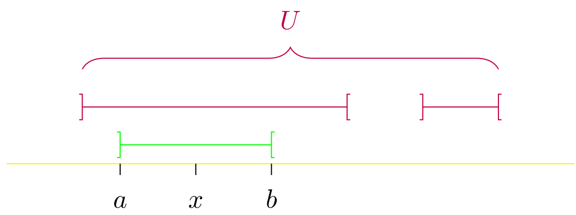
This is not at all an easy matter! Homeomorphism is a very flexible notion, and Y could be very wild. How do we know that the two examples we considered in Remark 12.2.2 are representative of all possible Y? We need to be sure that the requirement that we have a homeomorphism, as opposed to only a continuous surjection, excludes examples which are as wild as the Peano curve of Task **??**.

The fact that $(S^2 \setminus Y, \mathcal{O}_{S^2\setminus Y})$ has exactly two distinct connected components, for any $(Y, \mathcal{O}_Y)$ which is homeomorphic to $(S^1, \mathcal{O}_{S^1})$, is known as the *Jordan curve theorem*.

12.3 Locally connected topological spaces

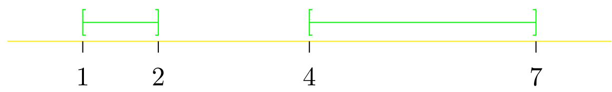
Definition 12.3.1. A topological space $(X, \mathcal{O}_X)$ is *locally connected* if, for every x which belongs to X, and every neighbourhood U of x in X with respect to $\mathcal{O}_X$, there is a neighbourhood W of x in X with respect to $\mathcal{O}_X$ which is both a connected subset of X with respect to $\mathcal{O}_X$, and a subset of U.

Example 12.3.2. Suppose that x belongs to $\mathbb{R}$. Let U be a neighbourhood of x in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. By definition of $\mathcal{O}_{\mathbb{R}}$, there is an open interval $]a, b[$ to which x belongs, and which is a subset of U.

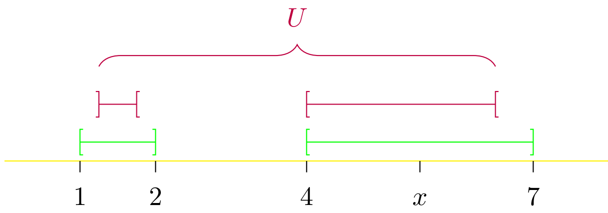


By Task E10.3.5, we have that $]a, b[$ is a connected subset of $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. We conclude that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally connected.

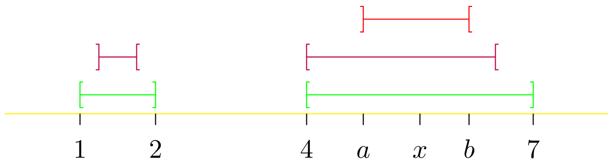
Example 12.3.3. Let $X = [1, 2] \cup [4, 7]$.



Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that x belongs to $[4, 7]$. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_X$.



By definition of $\mathcal{O}_X$ and $\mathcal{O}_{\mathbb{R}}$, there is an open interval $]a, b[$, to which x belongs, such that $X \cap]a, b[$ is a subset of U.



The following hold.

(1) By Task E1.3.5, we have that $[4, 7] \cap]a, b[$ is an interval. By Task E10.3.5, we deduce that $[4, 7] \cap]a, b[$ is a connected subset of $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

(2) By definition of $\mathcal{O}_X$, we have that $X \cap]a, b[$ belongs to $\mathcal{O}_X$. As was demonstrated in Example 9.6.2, we also have that $[4, 7]$ belongs to $\mathcal{O}_X$. Thus

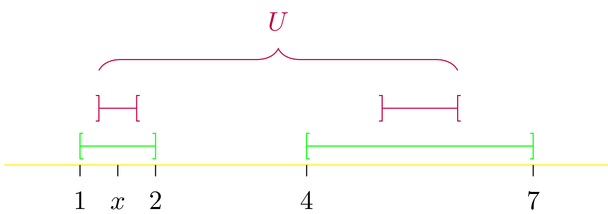
$$[4, 7] \cap]a, b[= [4, 7] \cap (X \cap]a, b[)$$

belongs to $\mathcal{O}_X$.

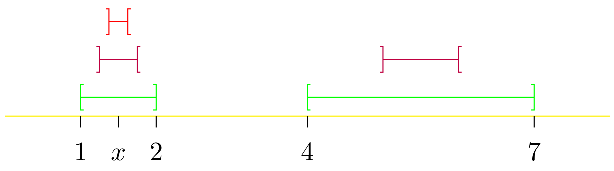
By Task E12.3.2, we deduce from (1) and (2) that $[4, 7] \cap]a, b[$ is a connected subset of X with respect to $\mathcal{O}_X$.

In addition, we have that x belongs to $[4, 7] \cap]a, b[$. By (2), we thus have that $[4, 7] \cap]a, b[$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$. Moreover, since $[4, 7]$ is a subset of X, and since $X \cap]a, b[$ is a subset of U.

Suppose now that x belongs to $[1, 2]$. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_X$.



By an analogous argument to that which we gave in the case that x belongs to $[4,7]$, there is an open interval $]a', b'[$ such that $[1,2] \cap]a', b'[$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$, is a connected subset of X with respect to $\mathcal{O}_X$, and is a subset of U.



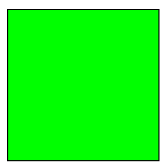
To fill in the details is the topic of Task E12.2.5. We conclude that $(X, \mathcal{O}_X)$ is locally connected.

Remark 12.3.4. The ingredients of this argument can be organised into a more general method for proving that a topological space is locally connected. By Task E2.3.1 and Task E12.3.9, both $[1,2]$ and $[4,7]$ are connected subsets of X with respect to $\mathcal{O}_X$. Moreover, as was demonstrated in Example 9.6.2, both $[1,2]$ and $[4,7]$ belong to $\mathcal{O}_X$. By Task E12.3.8, we conclude that $(X, \mathcal{O}_X)$ is locally connected.

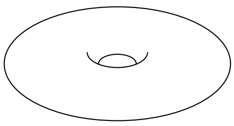
Example 12.3.5. By Example 12.3.2, we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally connected. By Task E12.1.7, we deduce that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is locally connected.

Example 12.3.6. By Task E12.3.9, we have that $(I, \mathcal{O}_I)$ is locally connected.

By Task E12.1.7, we deduce that $(I^2, \mathcal{O}_{I^2})$ is locally connected.



Example 12.3.7. By Example 12.3.6, we have that $(I^2, \mathcal{O}_{I^2})$ is locally connected. By Task E12.3.10, we deduce that $(T^2, \mathcal{O}_{T^2})$ is locally connected.

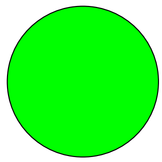


Remark 12.3.8. By a similar argument, $(M^2, \mathcal{O}_{M^2})$ and $(K^2, \mathcal{O}_{K^2})$ are locally connected. To check that you understand this is the topic of Task E12.2.6.

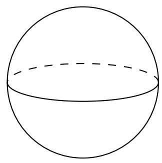
Example 12.3.9. By Example 12.3.6, we have that $(I^2, \mathcal{O}_{I^2})$ is locally connected. By Task E7.2.9, there is a homeomorphism

$$I^2 \longrightarrow D^2.$$

By Task E12.1.8, we deduce that $(D^2, \mathcal{O}_{D^2})$ is connected.



Example 12.3.10. By Example 12.3.9, we have that $(D^2, \mathcal{O}_{D^2})$ is locally connected. By Task E12.3.10, we deduce that $(S^2, \mathcal{O}_{S^2})$ is locally connected.



Example 12.3.11. Let $\mathcal{O}_{\mathbb{Q}}$ be the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that q belongs to $\mathbb{Q}$. By Example 11.4.4, we have that $\Gamma^q_{(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})}$ is $\{q\}$. In other words, $\{q\}$ is the only connected subset of $\mathbb{Q}$ to which q belongs. However, the set $\{q\}$ does not belong to $\mathcal{O}_{\mathbb{Q}}$. To check this is the topic of Task E12.2.4. Thus there is no neighbourhood of q in $\mathbb{Q}$ with respect to $\mathcal{O}_{\mathbb{Q}}$ which is a connected subset of $\mathbb{Q}$ with respect to $\mathcal{O}_{\mathbb{Q}}$. We conclude that $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not locally connected.

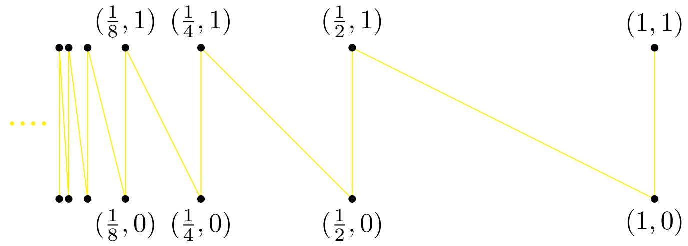
Remark 12.3.12. We could also argue as follows. By Example 11.4.4, we have that $\Gamma^q_{(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})}$ is $\{q\}$. The set $\{q\}$ does not belong to $\mathcal{O}_{\mathbb{Q}}$, as you are asked to check in Task E12.2.4. By Corollary E12.3.4, we deduce that $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not locally connected.

12.4 A topological space which is connected but not locally connected

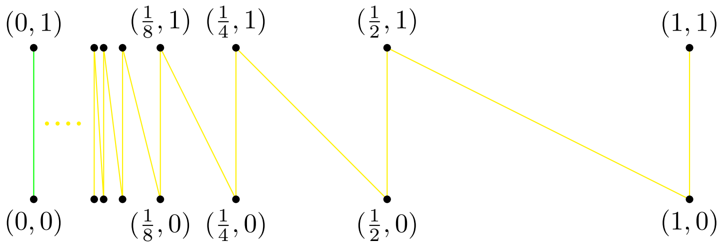
Example 12.4.1. Let A be the subset of $\mathbb{R}^2$ given by the union of the sets

$$\bigcup_{n \in \mathbb{N}} \left\{ \left(\frac{1}{2^{n-1}}, y \right) \mid y \in [0, 1] \right\}$$

and
$$\bigcup_{n \in \mathbb{N}} \left\{ (x, -2^n x + 2) \mid x \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \right\}.$$



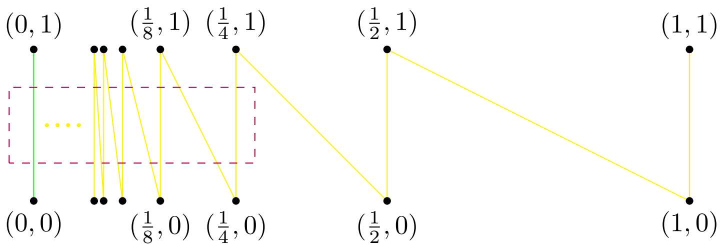
Let X be the closure of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. By Task E8.1.7, we have that X is the union of A and the line $\{0\} \times [0, 1]$.



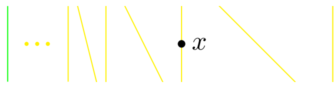
Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. By Task E10.1.6, we have that $(X, \mathcal{O}_X)$ is connected.

Let U be the neighbourhood of $\left(0, \frac{1}{2}\right)$ in X with respect to $\mathcal{O}_X$ given by
$$X \cap \left(\left]-1, \tfrac{9}{32}\right[\times \left]\tfrac{1}{4}, \tfrac{3}{4}\right[\right).$$

Let $\mathcal{O}_U$ be the subspace topology on U with respect to $(X, \mathcal{O}_X)$.



Suppose that (x, y) belongs to U. Let B be a subset of U to which both $(0, \frac{1}{2})$ and (x, y) belong. Let $\mathcal{O}_B$ be the subspace topology on B with respect to $(U, \mathcal{O}_U)$. Suppose that $x = \frac{1}{2^n}$, where n belongs to $\mathbb{N}$.



Let c be a real number with the property that $\frac{7}{2^{n+2}} < c < \frac{1}{2^n}$. Then B is the union of, for example,

$$B \cap \left(]-1, c[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right)$$

and

$$B \cap \left(]c, 2[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right),$$

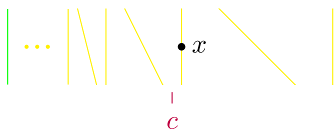
and this union is disjoint. The only significance in the choice of -1 and 2 is that $-1 < 0$, and $2 > 1$. Both

$$B \cap \left(]-1, c[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right)$$

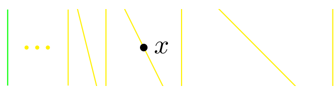
and

$$B \cap \left(]c, 2[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right)$$

belong to $\mathcal{O}_B$. Thus B is not a connected subset of U with respect to $\mathcal{O}_U$.



Suppose instead that $\frac{5}{2^{n+2}} < x < \frac{7}{2^{n+2}}$, where n belongs to $\mathbb{N}$.



Let c be a real number with the property that $\frac{1}{2^{n+1}} < c < \frac{5}{2^{n+2}}$. Then B is the union of, for example,

$$B \cap \left(]-1, c[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right)$$

and

$$B \cap \left(]c, 2[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right),$$

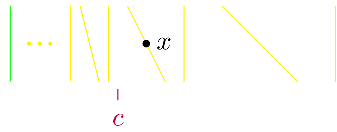
and this union is disjoint. Moreover, both

$$B \cap \left(]-1, c[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right)$$

and

$$B \cap \left(]c, 2[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right)$$

belong to $\mathcal{O}_B$. Thus B is not a connected subset of U with respect to $\mathcal{O}_U$.



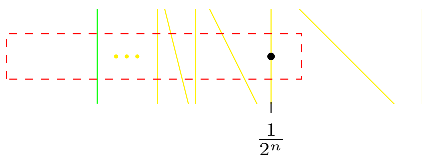
We have now demonstrated that if $x > 0$, then there is no connected subset of U with respect to $\mathcal{O}_U$ to which both $(0, \frac{1}{2})$ and (x, y) belong. Thus $\Gamma^{(0,\frac{1}{2})}_{(U,\mathcal{O}_U)}$ is a subset of $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$. We have that $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$ is a connected subset of U with respect to $\mathcal{O}_U$. To check this is the topic of Task E12.2.7. We conclude that $\Gamma^{(0,\frac{1}{2})}_{(U,\mathcal{O}_U)}$ is $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$.

Suppose that $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$ belongs to $\mathcal{O}_U$. By Task E2.3.1 and the definition of $\mathcal{O}_{\mathbb{R}^2}$, there are real numbers $a_0 < 0 < a_1$ and $\frac{1}{4} \leq b_0 < \frac{1}{2} < b_1 \leq \frac{3}{4}$ such that

$$U \cap (]a_0, a_1[\times]b_0, b_1[)$$

is a subset of $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$. Let n be a natural number such that $0 < \frac{1}{2^n} < a_1$. Then $(\frac{1}{2^n}, \frac{1}{2})$ belongs to

$$U \cap (]a_0, a_1[\times]b_0, b_1[) \,.$$



Since $(\frac{1}{2^n}, \frac{1}{2})$ does not belong to $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$, we have a contradiction. We conclude that $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$ does not belong to $\mathcal{O}_U$.

Putting everything together, we have demonstrated that $\Gamma^{(0,\frac{1}{2})}_{(U,\mathcal{O}_U)}$ does not belong to $\mathcal{O}_U$. By Task E12.3.3, we conclude that $(X, \mathcal{O}_X)$ is not locally connected.

Remark 12.4.2. The topological space $(X, \mathcal{O}_X)$ is a variant of a topological space known as the *topologist's sine curve*.

Remark 12.4.3. We could have proven that $(X, \mathcal{O}_X)$ is not locally connected by working with any $(0, y)$ such that $0 \leq y \leq 1$ in place of $(0, \frac{1}{2})$. To check that you understand this is the topic of Task E12.2.8.

Remark 12.4.4. In a nutshell, the reason that $(X, \mathcal{O}_X)$ is not locally connected is that, for any particular (x, y) which belongs to X with $x > 0$, there is a 'gap' between (x, y) and the y-axis, which is detected when we explore connectedness 'locally' around (x, y).

When we work 'globally', namely when we consider $(X, \mathcal{O}_X)$ as a whole, there is no 'gap' between the y-axis and the rest of X, because the intervals zig-zag infinitely closely towards the y-axis.

E12 Exercises for Lecture 12

E12.1 Exam questions

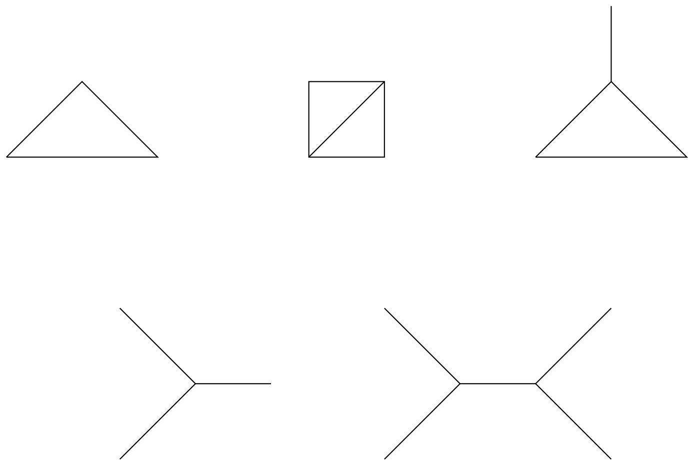
Task E12.1.1 (Continuation Exam, August 2013)**.** Let X be the subset of $\mathbb{R}^2$ depicted below.

Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let Y be the subset of $\mathbb{R}^2$ depicted below.

Let $\mathcal{O}_Y$ be the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Is $(X, \mathcal{O}_X)$ homeomorphic to $(Y, \mathcal{O}_Y)$?

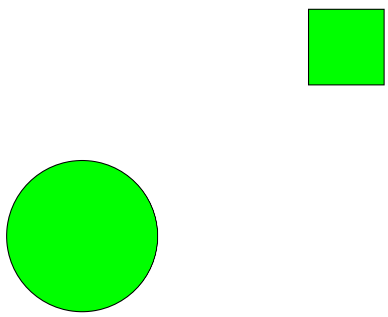
Task E12.1.2. View the letters B, C, D, E, F, G, H as subsets of $\mathbb{R}^2$. Let each be equipped with the subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Which of the letters are homeomorphic, and which are not?

Task E12.1.3. View each of the following shapes as a subset of $\mathbb{R}^2$. Each consists of intervals glued together. In particular, all of the shapes have no 'inside'.



Let each shape be equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Prove that no two of the shapes are homeomorphic.

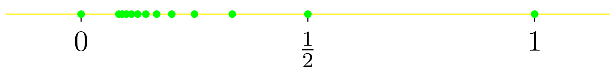
Task E12.1.4. Let X be the union of D^2 and $[3,4] \times [2,3]$.



Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Prove that $(X, \mathcal{O}_X)$ is locally connected. You may wish to appeal to Task E12.3.8.

Task E12.1.5. Let X be a set. Let $\mathcal{O}_X$ be the discrete topology on X. Prove that $(X, \mathcal{O}_X)$ is locally connected.

Task E12.1.6. Let X be the subset of $\mathbb{R}$ given by the union of $\{0\}$ and $\left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Prove that $(X, \mathcal{O}_X)$ is not locally connected.



Task E12.1.7. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally connected topological spaces. Prove that $(X \times Y, \mathcal{O}_{X \times Y})$ is locally connected.

Task E12.1.8. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that $(X, \mathcal{O}_X)$ is locally connected. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Prove that $(Y, \mathcal{O}_Y)$ is locally connected.

E12.2 In the lecture notes

Task E12.2.1. Prove carefully the assertions concerning numbers of distinct connected components in Example 12.1.1.

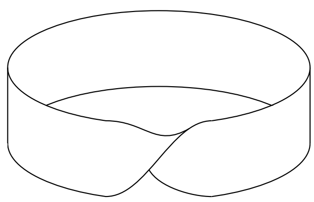
Task E12.2.2. How many distinct connected components can we obtain by removing one point from O? Explain why your answer means that we cannot distinguish $(\varnothing, \mathcal{O}_{\mathsf{O}})$ from $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ by removing just one point from each topological space, and counting the resultings numbers of distinct connected components.

Task E12.2.3. Prove carefully the assertions concerning numbers of distinct connected components in Example 12.1.3.

Task E12.2.4. Let $\mathcal{O}_{\mathbb{Q}}$ be the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that q belongs to $\mathbb{Q}$. Prove that $\{q\}$ does not belong to $\mathcal{O}_{\mathbb{Q}}$.

Task E12.2.5. Let $(X, \mathcal{O}_X)$ be as in Example 12.3.3. Suppose that x belongs to $[1,2]$. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_X$. Prove that there is an open interval $]a', b'[$ such that $[1,2] \cap]a', b'[$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$, is a connected subset of X with respect to $\mathcal{O}_X$, and is a subset of U.

Task E12.2.6. Prove that $(M^2, \mathcal{O}_{M^2})$ is locally connected.



Task E12.2.7. Let $(U, \mathcal{O}_U)$ be as in Example 12.4.1. Prove that $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$ is a connected subset of U with respect to $\mathcal{O}_U$. You may wish to proceed as follows.

(1) Let $\mathcal{O}_{\{0\} \times]\frac{1}{4}, \frac{3}{4}[}$ be the subspace topology on $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let $\mathcal{O}_{]\frac{1}{4}, \frac{3}{4}[}$ be the subspace topology on $\left]\frac{1}{4}, \frac{3}{4}\right[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Prove that $\left(\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[, \mathcal{O}_{\{0\} \times]\frac{1}{4}, \frac{3}{4}[}\right)$ is homeomorphic to $\left(\left]\frac{1}{4}, \frac{3}{4}\right[, \mathcal{O}_{]\frac{1}{4}, \frac{3}{4}[}\right)$. You may wish to look back at your argument for Task E7.1.8.

(2) Appeal to Task E2.3.1, and to Corollary 10.5.2.

Task E12.2.8. Let $(X, \mathcal{O}_X)$ be as in Example 12.4.1. Prove that $(X, \mathcal{O}_X)$ is not locally connected by working with $(0, 1)$ rather than $(0, \frac{1}{2})$. Can you furthermore see how to adapt the argument of Example 12.4.1 to any $(0, y)$ such that $0 \leq y \leq 1$?

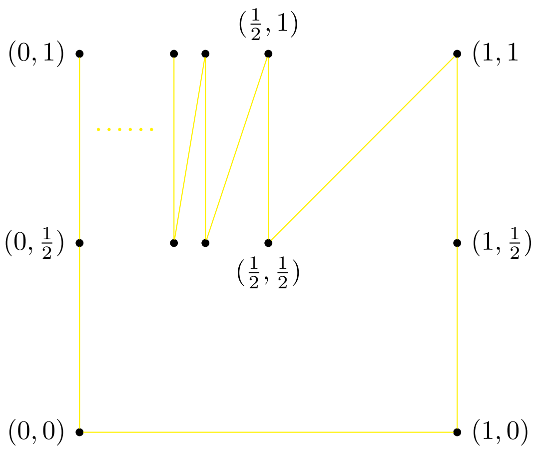
Task E12.2.9. Let X be the subset of $\mathbb{R}^2$ given by the union of the sets $\{0,1\} \times \left[0, \frac{1}{2}\right]$, $I \times \{0\}$,

$$\bigcup_{n \geq 0} \left\{ \left(\frac{1}{2^n}, y \right) \mid y \in \left[\frac{1}{2}, 1 \right] \right\},$$

and

$$\bigcup_{n \geq 0} \left\{ (x, 2^n x) \mid x \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n} \right] \right\},$$

where n is an integer.



Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Check that you understand Example 12.4.1 by proving that $(X, \mathcal{O}_X)$ is not locally connected.

Remark E12.2.10. The topological space $(X, \mathcal{O}_X)$ is a variant of a topological space known as the *Warsaw circle*.

E12.3 For a deeper understanding

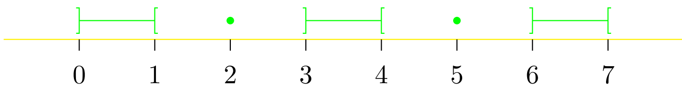
Task E12.3.1. Let X be the subset of $\mathbb{R}$ given by

$$]0,1[\cup \{2\} \cup]3,4[\cup \{5\} \cup]6,7[\cup \{8\} \cdots .$$

In other words, X is given by

$$\bigcup_{n \in \mathbb{N}}]3n-3, 3n-2[\cup \{3n-1\}.$$

Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



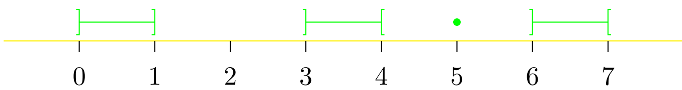
Let Y be the subset of $\mathbb{R}$ given by

$$]0,1] \cup]3,4[\cup \{5\} \cup]6,7[\cup \{8\} \cup \cdots .$$

In other words, Y is given by

$$]0,1] \cup \left(\bigcup_{n \in \mathbb{N}}]3n, 3n+1[\cup \{3n+2\} \right).$$

Let $\mathcal{O}_Y$ be the subspace topology on $\mathcal{O}_Y$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Prove that there is a continuous bijection

$$X \xrightarrow{f} Y$$

and a continuous bijection

$$Y \xrightarrow{g} X,$$

but that $(X, \mathcal{O}_X)$ is not homeomorphic to $(Y, \mathcal{O}_Y)$. You may wish to proceed to as follows.

(1) Let

$$X \xrightarrow{f} Y$$

be given by

$$f(x) = \begin{cases} x & \text{if } x \neq 2, \\ 1 & \text{if } x = 2. \end{cases}$$

Observe that f is a bijection. By Task E5.3.14 and Task E5.3.23 (1), observe that f is continuous.

(2) Let

$$Y \xrightarrow{g} X$$

be given by

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \in \left]0,1\right], \\ \frac{x-2}{2} & \text{if } x \in \left]3,4\right[, \\ x-3 & \text{otherwise.} \end{cases}$$

Observe that g is a bijection. By Task E5.3.14 and Task E5.3.23 (1), observe that g is continuous.

(3) Suppose that y belongs to $]0,1]$. Demonstrate that Γ_y is $]0,1]$.

(4) Suppose that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are homeomorphic. Let $\mathcal{O}_{]0,1]}$ be the subspace topology on $]0,1]$ with respect to $(Y, \mathcal{O}_Y)$. By Task E11.3.17, deduce from (3) that there is an x which belongs to X with the property that $\left(]0,1], \mathcal{O}_{]0,1]}\right)$ is homeomorphic to $(\Gamma_x, \mathcal{O}_{\Gamma_x})$, where $\mathcal{O}_{\Gamma_x}$ is the subspace topology on Γ_x with respect to $(X, \mathcal{O}_X)$.

(5) Suppose that n belongs to $\mathbb{N}$. Demonstrate that if x belongs to $]3n-3, 3n-2[$, then Γ_x is $]3n-3, 3n-2[$. Demonstrate that if x is $3n-1$, then Γ_x is $\{3n-1\}$.

(6) Let $\mathcal{O}_{]3n-3,3n-2[}$ be the subspace topology on $]3n-3, 3n-2[$ with respect to $(X, \mathcal{O}_X)$. By Task E2.3.1 and Task E11.1.3 we have that $\left(]0,1], \mathcal{O}_{]0,1]}\right)$ is not homeomorphic to $\left(]3n-3, 3n-2[, \mathcal{O}_{]3n-3,3n-2[}\right)$.

(7) Let $\mathcal{O}_{\{3n-1\}}$ be the subspace topology on $\{3n-1\}$ with respect to $(X, \mathcal{O}_X)$. Observe that $\left(]0,1], \mathcal{O}_{]0,1]}\right)$ is not homeomorphic to $\left(\{3n-1\}, \mathcal{O}_{\{3n-1\}}\right)$, since there cannot be a bijection between a set with one element and $]0,1]$. To check that you understand this was the topic of Task E7.2.2.

(8) Observe that (6) and (7) together contradict (4) and (5). Conclude that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are not homeomorphic.

Task E12.3.2. Let $(X, \mathcal{O}_X)$ be a topological space. Let U be a subset of X which belongs to $\mathcal{O}_X$. Let $\mathcal{O}_U$ be the subspace topology on U with respect to $(X, \mathcal{O}_X)$. Let A be a connected subset of X with respect to $\mathcal{O}_X$. Suppose that A is a subset of U. Prove that A is a connected subset of U with respect to $\mathcal{O}_U$. You may wish to proceed as follows.

(1) Let U_0 and U_1 be subsets of A such that $A = U_0 \sqcup U_1$, and such that both U_0 and U_1 belong to $\mathcal{O}_U$. By Task E2.3.3 (1), observe that U_0 and U_1 belong to $\mathcal{O}_X$.

(2) Since A is a connected subset of X with respect to $\mathcal{O}_X$, deduce that at least one of U_0 and U_1 is empty. Conclude that A is a connected subset of U with respect to $\mathcal{O}_U$.

Task E12.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Prove that $(X, \mathcal{O}_X)$ is locally connected if and only if, for every subset U of X which belongs to $\mathcal{O}_X$, we have that $\Gamma^x_{(U,\mathcal{O}_U)}$ belongs to $\mathcal{O}_X$, where $\mathcal{O}_U$ is the subspace topology on U with respect to $(X, \mathcal{O}_X)$. You may wish to proceed as follows.

(1) Suppose that $(X, \mathcal{O}_X)$ is locally connected. Let U be a subset of X which belongs to $\mathcal{O}_X$. Let $\mathcal{O}_U$ be the subspace topology on U with respect to $(X, \mathcal{O}_X)$. Suppose that x belongs to U. Since $(X, \mathcal{O}_X)$ is locally connected, there is a neighbourhood W of x in X with respect to $\mathcal{O}_X$ such that W is a connected subset of X with respect to $\mathcal{O}_X$, and such that W is a subset of U. By Task E12.3.2, we have that W is a connected subset of U with respect to $\mathcal{O}_U$. Deduce that W is a subset of $\Gamma^x_{(U,\mathcal{O}_U)}$.

(2) By Task E8.3.1, deduce that $\Gamma^x_{(U,\mathcal{O}_U)}$ belongs to $\mathcal{O}_X$.

(3) Conversely, suppose that, for every subset U of X which belongs to $\mathcal{O}_X$, we have that $\Gamma^x_{(U,\mathcal{O}_U)}$ belongs to $\mathcal{O}_X$. Suppose that x belongs to X. Let U_x be a neighbourhood of x in X with respect to $\mathcal{O}_X$. Then $\Gamma^x_{(U_x,\mathcal{O}_{U_x})}$ is a connected subset of U_x with respect to $\mathcal{O}_{U_x}$. By assumption, we have that $\Gamma^x_{(U_x,\mathcal{O}_{U_x})}$ belongs to $\mathcal{O}_X$. Conclude that $(X, \mathcal{O}_X)$ is locally connected.

Corollary E12.3.4. Let $(X, \mathcal{O}_X)$ be a locally connected topological space. Suppose that x belongs to $(X, \mathcal{O}_X)$. Then $\Gamma^x_{(X,\mathcal{O}_X)}$ belongs to $\mathcal{O}_X$.

Proof. Follows immediately from Task E12.3.3, since X belongs to $\mathcal{O}_X$. □

Task E12.3.5. Let $(X, \mathcal{O}_X)$ be a locally connected topological space. Let U be a subset of X which belongs to $\mathcal{O}_X$. Let $\mathcal{O}_U$ be the subspace topology on U with respect to $(X, \mathcal{O}_X)$. Prove that $(U, \mathcal{O}_U)$ is locally connected. You may wish to appeal to Task E12.3.3, Task E2.3.3 (1), and Task E2.3.1.

Task E12.3.6. Prove that a topological space $(X, \mathcal{O}_X)$ is both totally disconnected and locally connected if and only if $\mathcal{O}_X$ is the discrete topology on X. You may wish to appeal to Task E12.1.5 and to Corollary E12.3.4.

Task E12.3.7. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that X is finite. Prove that $(X, \mathcal{O}_X)$ is locally connected. You may wish to appeal to Task E11.3.15 and to Task E12.3.3.

Task E12.3.8. Let $(X, \mathcal{O}_X)$ be a topological space. Let X_0 and X_1 be subsets of X which belong to $\mathcal{O}_X$. Suppose that $X = X_0 \sqcup X_1$. Let $\mathcal{O}_{X_0}$ be the subspace topology on X_0 with respect to $(X, \mathcal{O}_X)$, and let $\mathcal{O}_{X_1}$ be the subspace topology on X_1 with respect to $(X, \mathcal{O}_X)$. Prove that $(X, \mathcal{O}_X)$ is locally connected if and only if both $(X_0, \mathcal{O}_{X_0})$ and $(X_1, \mathcal{O}_{X_1})$ are locally connected. You may wish to proceed as follows.

(1) Suppose that $(X, \mathcal{O}_X)$ is locally connected. By Task E12.3.5, deduce that $(X_0, \mathcal{O}_{X_0})$ and $(X_1, \mathcal{O}_{X_1})$ are locally connected.

(2) Suppose that $(X_0, \mathcal{O}_{X_0})$ and $(X_1, \mathcal{O}_{X_1})$ are locally connected. Suppose that x belongs to X. Since $X = X_0 \sqcup X_1$, observe that either x belongs to X_0 or that x belongs to X_1.

(3) Suppose that x belongs to X_0. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_X$. Then $X_0 \cap U$ is a neighbourhood of x in X_0 with respect to $\mathcal{O}_{X_0}$. Since $(X_0, \mathcal{O}_{X_0})$ is locally connected, deduce that there is a neighbourhood U_x of x in X_0 with respect to $\mathcal{O}_{X_0}$ such that U_x is both a connected subset of X_0 with respect to $\mathcal{O}_{X_0}$, and a subset of $X_0 \cap U$.

(4) Since $X_0 \cap U$ is a subset of U, observe that U_x is a subset of U.

(5) By Task E2.3.3 (1), since X_0 belongs to $\mathcal{O}_X$ and U_x is a neighbourhood of x in X_0 with respect to $\mathcal{O}_{X_0}$, deduce that U_x is a neighbourhood of x in X with respect to $\mathcal{O}_X$.

(6) By Task E2.3.1, since U_x is a connected subset of X_0 with respect to $\mathcal{O}_{X_0}$, deduce that U_x is a connected subset of X with respect to $\mathcal{O}_X$.

(7) By an analogous argument, observe that if x belongs to X_1, then there is a neighbourhood U_x of x in X with respect to $\mathcal{O}_X$ which is both a connected subset of X with respect to $\mathcal{O}_X$, and a subset of U.

(8) Conclude from (4) – (7) that $(X, \mathcal{O}_X)$ is locally connected.

Task E12.3.9. Let X be an interval. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Prove that $(X, \mathcal{O}_X)$ is locally connected. You may wish to appeal to Task E1.3.5 and to Task E10.3.5.

Task E12.3.10. Let $(X, \mathcal{O}_X)$ be a locally connected topological space. Let $\sim$ be an equivalence relation on X. Prove that $(X/\sim, \mathcal{O}_{X/\sim})$ is locally connected. You may wish to proceed as follows.

(1) Suppose that $[x]$ belongs to $X/\sim$. Let U be a neighbourhood of $[x]$ in $X/\sim$ with respect to $\mathcal{O}_{X/\sim}$. Let

$$X \xrightarrow{\pi} X/\sim$$

be the quotient map. By Remark 6.1.9, we have that π is continuous. Thus, observe that $\pi^{-1}(U)$ belongs to $\mathcal{O}_X$.

(2) Let $\mathcal{O}_{\pi^{-1}(U)}$ be the subspace topology on $\pi^{-1}(U)$ with respect to $(X, \mathcal{O}_X)$. By Corolllary E12.3.4, observe that

$$\Gamma^x_{\left(\pi^{-1}(U), \mathcal{O}_{\pi^{-1}(U)}\right)}$$

is a neighbourhood of x in X with respect $\mathcal{O}_X$, and is a connected subset of X with respect to $\mathcal{O}_X$.

(3) Since

$$\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}$$

is a connected subset of X with respect to $\mathcal{O}_X$, deduce, by Task E10.3.2, that

$$\pi\left(\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}\right)$$

is a connected subset of $X/\sim$ with respect to $\mathcal{O}_{X/\sim}$.

(4) Let $\mathcal{O}_U$ be the subspace topology on U with respect to $(X/\sim, \mathcal{O}_{X/\sim})$. By definition of $\Gamma^{[x]}_{(U,\mathcal{O}_U)}$, deduce that

$$\pi\left(\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}\right)$$

is a subset of $\Gamma^{[x]}_{(U,\mathcal{O}_U)}$.

(5) Deduce that

$$\pi^{-1}\left(\pi\left(\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}\right)\right)$$

is a subset of $\pi^{-1}\left(\Gamma^{[x]}_{(U,\mathcal{O}_U)}\right)$.

(6) We have that

$$\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}$$

is a subset of

$$\pi^{-1}\left(\pi\left(\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}\right)\right).$$

Deduce that

$$\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}$$

is a subset of

$$\pi^{-1}\left(\Gamma^{[x]}_{(U,\mathcal{O}_U)}\right).$$

(7) As observed in (2), we have that

$$\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}$$

is a neighbourhood of x in X with respect to $\mathcal{O}_X$. By Task E8.3.1, deduce that

$$\pi^{-1}\left(\Gamma^{[x]}_{(U,\mathcal{O}_U)}\right)$$

belongs to $\mathcal{O}_X$. Thus $\Gamma^{[x]}_{(U,\mathcal{O}_U)}$ belongs to $\mathcal{O}_{X/\sim}$.

(8) By Task E12.3.3, conclude that $\left(X/\sim, \mathcal{O}_{X/\sim}\right)$ is locally connected.

Task E12.3.11. Let $\mathcal{O}_{\mathbb{N}}$ be the discrete topology on $\mathbb{N}$. Let $(X, \mathcal{O}_X)$ be the topological space of Task E12.1.6. Let

$$\mathbb{N} \xrightarrow{f} X$$

be the map given by

$$n \mapsto \begin{cases} 0 & \text{if } n = 1, \\ \frac{1}{n-1} & \text{if } n > 1. \end{cases}$$

Prove that f is a continuous surjection. You may wish to appeal to Task E5.1.14.

Task E12.3.12. Let $\mathcal{O}_{]0,1]}$ be the subspace topology on $]0, 1]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let $(X, \mathcal{O}_X)$ be the Warsaw circle. Construct a continuous surjection

$$]0,1] \longrightarrow X.$$

You may wish to appeal to (2) of Task E5.3.23.

Remark E12.3.13. Let $\mathcal{O}_{\mathbb{N}}$ be the discrete topology on $\mathbb{N}$. By Task E12.1.5, we have that $\mathbb{N}$ is locally connected. By Task E12.1.6, the topological space $(X, \mathcal{O}_X)$ of Task E12.3.11 is not locally connected. Thus Task E12.3.11 demonstrates that an analogue of Proposition 10.5.1 does not hold for locally connected topological spaces. It is for this reason that the proof needed for Task E12.3.10 is more involved than the proof of Corollary 10.5.3.

Let $\mathcal{O}_{]0,1]}$ be the subspace topology on $]0, 1]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Task E12.3.9, we have that $\left(]0,1], \mathcal{O}_{]0,1]}\right)$ is locally connected. By Task E12.2.9, the Warsaw circle is not locally connected. Thus Task E12.3.12 gives a a second demonstration that an analogue of Proposition 10.5.1 does not hold for locally connected topological spaces.