MA3002 Generell Topologi — Vår 2014

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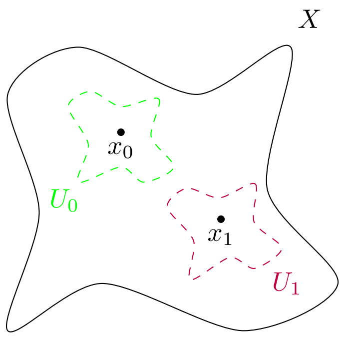
13.1 Hausdorff topological spaces

Definition 13.1.1. A topological space $(X, \mathcal{O}_X)$ is *Hausdorff* if, for all x_0 and x_1 which belong to X such that $x_0 \neq x_1$, there is a neighbourhood U_0 of x_0 in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1 of x_1 in X with respect to $\mathcal{O}_X$, such that $U_0 \cap U_1$ is empty.



13.2 Examples and non-examples of Hausdorff topological spaces

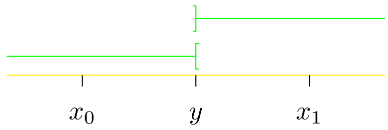
Example 13.2.1. Suppose that x_0 and x_1 belong to $\mathbb{R}$, and that $x_0 \neq x_1$. Relabelling x_0 and x_1 if necessary, we may assume that $x_0 < x_1$.

Let$y$ be a real number such that $x_0 < y < x_1$. The following hold.

(1) We have that x_0 belongs to $]-\infty, y[$.

(2) We have that x_1 belongs to $]y, \infty[$.

(3) We have that $]-\infty, y[\cap]y, \infty[$ is empty.



Both $]-\infty, y[$ and $]y, \infty[$ belong to $\mathcal{O}_{\mathbb{R}}$. To verify this is the topic of Task E13.2.1. We conclude that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff.

Example 13.2.2. Let X be a set. Let $\mathcal{O}_X^{\mathsf{indisc}}$ be the indiscrete topology on X. Suppose that x_0 and x_1 belong to X, and that $x_0 \neq x_1$. The only neighbourhood of x_0 in X with respect to $\mathcal{O}_X^{\text{indisc}}$ is X, and x_1 belongs to X. Thus there is no neighbourhood of x_0 in X with respect to $\mathcal{O}_X^{\mathsf{indisc}}$ which does not contain x_1. In particular, $(X, \mathcal{O}_X^{\mathsf{indisc}})$ is not Hausdorff.

Example 13.2.3. Let X be a set. Let $\mathcal{O}_X^{\mathsf{disc}}$ be the discrete topology on X. Suppose that x_0 and x_1 belong to X, and that $x_0 \neq x_1$. The following hold.

(1) We have that $\{x_0\}$ belongs to $\mathcal{O}_X^{\mathsf{disc}}$.

(2) We have that $\{x_1\}$ belongs to $\mathcal{O}_X^{\mathsf{disc}}$.

(3) We have that $\{x_0\} \cap \{x_1\}$ is empty.

We conclude that $(X, \mathcal{O}_X^{\mathsf{disc}})$ is Hausdorff.

Example 13.2.4. Let X be the set $\{a, b, c\}$. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{a, b\}, \{a, c\} X\}.$$

Every neighbourhood of b in X with respect to $\mathcal{O}_X$ also contains c. Thus $(X, \mathcal{O}_X)$ is not Hausdorff.

Example 13.2.5. Let X be the set $\{a, b, c\}$. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}.$$

Every neighbourhood of b in X with respect to $\mathcal{O}_X$ also contains a. Thus $(X, \mathcal{O}_X)$ is not Hausdorff.

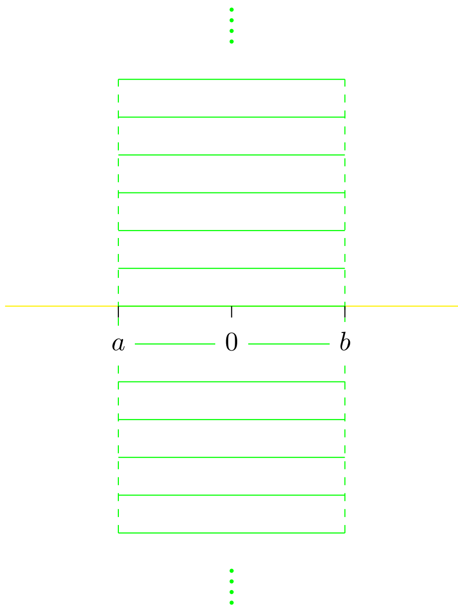
Remark 13.2.6. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that X is finite, or more generally that $\mathcal{O}_X$ is finite. Then $(X, \mathcal{O}_X)$ is Hausdorff if and only if $\mathcal{O}_X$ is the discrete topology. This is Corollary E13.3.7.

Example 13.2.7. Let $\mathcal{O}$ be the topology on $\mathbb{R}^2$ given by

$$\{U \times \mathbb{R} \mid U \text{ belongs to } \mathcal{O}_{\mathbb{R}}\}.$$

To verify that $\mathcal{O}$ defines a topology is Task E13.2.2. Suppose that x_0 and x_1 belong to $\mathbb{R}$, and that $x_0 \neq x_1$. Let W be a neighbourhood of $(0, x_0)$ in X with respect to $\mathcal{O}$.

By definition of $\mathcal{O}$, there is a neighbourhood U of 0 in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$ such that $W = U \times \mathbb{R}$. By definition of $\mathcal{O}_{\mathbb{R}}$, there is an open interval $]a, b[$ with $a < 0 < b$ such that $]a, b[$ is a subset of U. Thus $]a, b[\times \mathbb{R}$ is a subset of W.



We have that $(0, x_1)$ belongs to $]a, b[\times \mathbb{R}$. Thus $(0, x_1)$ belongs to W.

We have demonstrated that every neighbourhood of $(0, x_0)$ in $\mathbb{R}^2$ with respect to $\mathcal{O}$ contains $(0, x_1)$. We conclude that $(\mathbb{R}^2, \mathcal{O})$ is not Hausdorff.

13.3 Canonical methods to prove that a topological space is Hausdorff

Proposition 13.3.1. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let A be a subset of X. Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Then $(A, \mathcal{O}_A)$ is Hausdorff.

Proof. Suppose that a_0 and a_1 belong to A, and that $a_0 \neq a_1$. Since $(X, \mathcal{O}_X)$ is Hausdorff, there is a neighbourhood U_0 of a_0 in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1 of a_1 in X with respect to $\mathcal{O}_X$, such that $U_0 \cap U_1$ is empty. The following hold.

(1) By definition of $\mathcal{O}_A$, we have that $A \cap U_0$ belongs to $\mathcal{O}_A$. Thus $A \cap U_0$ is a neighbourhood of a_0 in A with respect to $\mathcal{O}_A$.

(2) By definition of $\mathcal{O}_A$, we have that $A \cap U_1$ belongs to $\mathcal{O}_A$. Thus $A \cap U_1$ is a neighbourhood of a_1 in A with respect to $\mathcal{O}_A$.

(3) We have that $(A \cap U_0) \cap (A \cap U_1)$ is a subset of $U_0 \cap U_1$. Since $U_0 \cap U_1$ is empty, we deduce that $(A \cap U_0) \cap (A \cap U_1)$ is empty.

We conclude that $(A, \mathcal{O}_A)$ is Hausdorff.

□

Example 13.3.2. By Example 13.2.1, we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. By Proposition 13.3.1, we deduce that $(I, \mathcal{O}_I)$ is Hausdorff.

Proposition 13.3.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be Hausdorff topological spaces. Then $(X \times Y, \mathcal{O}_{X \times Y})$ is Hausdorff.

Proof. Suppose that (x_0, y_0) and (x_1, y_1) belong to $X \times Y$, and that $(x_0, y_0) \neq (x_1, y_1)$. Then either $x_0 \neq x_1$ or $y_0 \neq y_1$, or both $x_0 \neq x_1$ and $y_0 \neq y_1$.

Suppose that $x_0 \neq x_1$. Since $(X, \mathcal{O}_X)$ is Hausdorff, there is a neighbourhood U_0^X of x_0 in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1^X of x_1 in X with respect to $\mathcal{O}_X$, such that $U_0^X \cap U_1^X$ is empty. The following hold.

(1) We have that $U_0^X \times Y$ belongs to $\mathcal{O}_{X \times Y}$. Thus $U_0^X \times Y$ is a neighbourhood of (x_0, y_0) in $X \times Y$ with respect to $\mathcal{O}_{X \times Y}$.

(2) We have that $U_1^X \times Y$ belongs to $\mathcal{O}_{X \times Y}$. Thus $U_1^X \times Y$ is a neighbourhood of (x_1, y_1) in $X \times Y$ with respect to $\mathcal{O}_{X \times Y}$.

(3) We have that $(U_0^X \times Y) \cap (U_1^X \times Y) = (U_0^X \cap U_1^X) \times Y$. Since $U_0^X \cap U_1^X$ is empty, we deduce that $(U_0^X \times Y) \cap (U_1^X \times Y)$ is empty.

Suppose instead that $y_0 \neq y_1$. By an analogous argument, there is a neighbourhood U_0^Y of y_0 in Y with respect to $\mathcal{O}_Y$, and a neighbourhood U_1^Y of y_1 in Y with respect to $\mathcal{O}_Y$, such that the following hold.

(1 bis) We have that $X \times U_0^Y$ is a neighbourhood of (x_0, y_0) in $X \times Y$ with respect to $\mathcal{O}_{X \times Y}$.

(2 bis) We have that $X \times U_1^Y$ is a neighbourhood of (x_1, y_1) in $X \times Y$ with respect to $\mathcal{O}_{X \times Y}$.

(3 bis) We have that $(X \times U_0^Y) \cap (X \times U_1^Y)$ is empty.

We conclude that $(X \times Y, \mathcal{O}_{X \times Y})$ is Hausdorff. □

Example 13.3.4. By Example 13.2.1, we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. By Proposition 13.3.3, we deduce that $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is Hausdorff, for any $n \geq 1$.

Example 13.3.5. By Example 13.3.2, we have that $(I, \mathcal{O}_I)$ is Hausdorff. By Proposition 13.3.3, we deduce that $(I^2, \mathcal{O}_{I^2})$ is Hausdorff.

Alternatively, by Example 13.3.4 we have that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff. We can deduce from this that $(I^2, \mathcal{O}_{I^2})$ is Hausdorff by Proposition 13.3.1.

Example 13.3.6. By Example 13.3.4 we have that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff. By Proposition 13.3.1, we deduce that $(S^1, \mathcal{O}_{S^1})$ is Hausdorff.

Proposition 13.3.7. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that $(X, \mathcal{O}_X)$ is Hausdorff. Let

$$X \xrightarrow{f} Y$$

be a bijection. Suppose that f is open, in the sense of Definition E7.1.15. Then $(Y, \mathcal{O}_Y)$ is Hausdorff.

Proof. Since f is a bijection, there is a map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$. Suppose that y_0 and y_1 belong to Y, and that $y_0 \neq y_1$. Since $y_0 \neq y_1$, we have that $g(y_0) \neq g(y_1)$. To check that you understand this is the topic of Task E13.2.3 (1).

Since $(X, \mathcal{O}_X)$ is Hausdorff, there is a neighbourhood U_0 of $g(y_0)$ in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1 of $g(y_1)$ in X with respect to $\mathcal{O}_X$, such that $U_0 \cap U_1$ is empty. The following hold.

(1) Since $U_0 \cap U_1$ is empty, we have that $f(U_0) \cap f(U_1)$ is empty. To verify this is the topic of Task E13.2.3 (2).

(2) Since f is open, we have that $f(U_0)$ belongs to $\mathcal{O}_Y$. Since $f \circ g = id_Y$, we have that $f(g(y_0)) = y_0$. Thus we have that $f(U_0)$ is a neighbourhood of y_0 in Y with respect to $\mathcal{O}_Y$.

(3) Since f is open, we have that $f(U_1)$ belongs to $\mathcal{O}_Y$. Since $f \circ g = id_Y$, we have that $f(g(y_1)) = y_1$. Thus we have that $f(U_1)$ is a neighbourhood of y_1 in Y with respect to $\mathcal{O}_Y$.

We conclude that $(Y, \mathcal{O}_Y)$ is Hausdorff.

□

Corollary 13.3.8. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that $(X, \mathcal{O}_X)$ is Hausdorff. Suppose that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are homeomorphic. Then $(Y, \mathcal{O}_Y)$ is Hausdorff.

Proof. Follows immediately from Proposition 13.3.7 since, by Task E7.3.1, a homeomorphism is in particular bijective and open. □

Example 13.3.9. By Example 13.3.5, we have that $(I^2, \mathcal{O}_{I^2})$ is Hausdorff. By Task E7.2.9, there is a homeomorphism

$$I^2 \longrightarrow D^2.$$

By Corollary 13.3.8, we deduce that $(D^2, \mathcal{O}_{D^2})$ is Hausdorff.

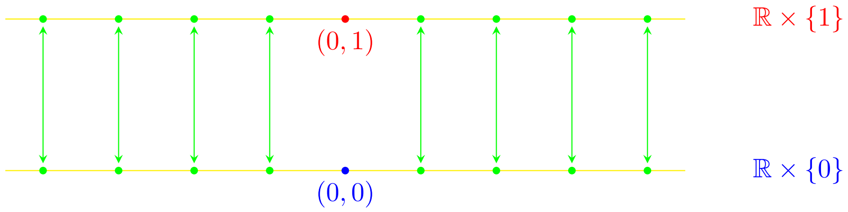
Alternatively, by Example 13.3.4 we have that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff. We can thus deduce from Proposition 13.3.1 that $(D^2, \mathcal{O}_{D^2})$ is Hausdorff.

13.4 Example of a quotient of a Hausdorff topological space which is not Hausdorff

Example 13.4.1. Let X be the subset of $\mathbb{R}^2$ given by the union of $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$.

Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. By Example 13.3.4, we have that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff. By Proposition 13.3.1, we deduce that $(X, \mathcal{O}_X)$ is Hausdorff.

Let $\sim$ be the equivalence relation on X generated by $(x, 0) \sim (x, 1)$, for all $x \in \mathbb{R}$ such that $x \neq 0$.

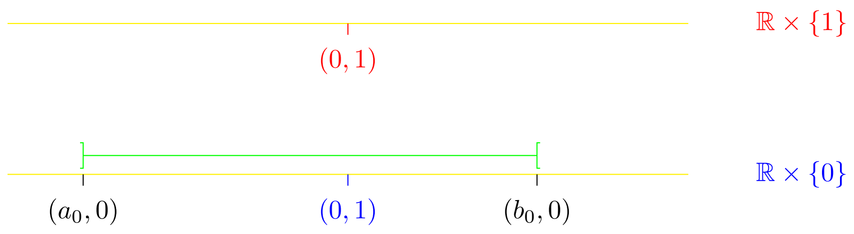


We shall demonstrate that $(X/\sim, \mathcal{O}_{X/\sim})$ is not Hausdorff. Let

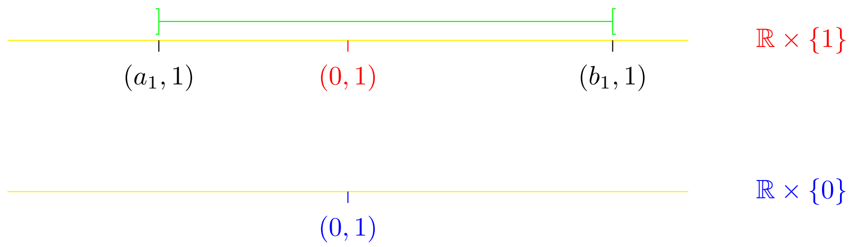
$$X \xrightarrow{\pi} X/\sim$$

be the quotient map. Let U_0 be a neighbourhood of $\pi((0,0))$ in $X/\sim$ with respect to $\mathcal{O}_{X/\sim}$. Let U_1 be a neighbourhood of $\pi((0,1))$ in $X/\sim$ with respect to $\mathcal{O}_{X/\sim}$.

By definition of $\mathcal{O}_{X/\sim}$, we have that $\pi^{-1}(U_0)$ belongs to $\mathcal{O}_X$. By definition of $\mathcal{O}_X$ and $\mathcal{O}_{\mathbb{R}^2}$, we deduce that there is an open interval $]a_0, b_0[$, with $a_0 < 0 < b_0$, such that $]a_0, b_0[\times \{0\}$ is a subset of $\pi^{-1}(U_0)$. To check that you understand this is the topic of Task E13.2.4.



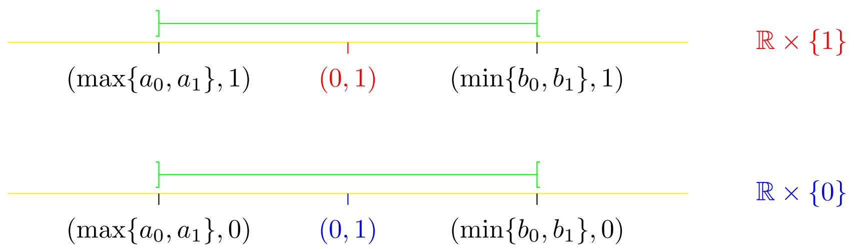
By an analogous argument, there is an open interval $]a_1, b_1[$, with $a_1 < 0 < b_1$, such that $]a_1, b_1[\times \{1\}$ is a subset of $\pi^{-1}(U_1)$.



The following hold.

(1) We have that $]\max\{a_0, a_1\}, \min\{b_0, b_1\}[\times \{0\}$ is a subset of $\pi^{-1}(U_0)$.

(2) We have that $]\max\{a_0, a_1\}, \min\{b_0, b_1\}[\times \{1\}$ is a subset of $\pi^{-1}(U_1)$.



We deduce that

$$\pi\left((]\max(a_0, a_1), \min(b_0, b_1)[\setminus \{0\}) \times \{0\} \right)$$

is a subset of both U_0 and U_1. In particular, $U_0 \cap U_1$ is not empty. We conclude that $(X/\sim, \mathcal{O}_{X/\sim})$ is not Hausdorff.

Remark 13.4.2. The topological space $(X/\sim, \mathcal{O}_{X/\sim})$ is sometimes known as the *real line with two origins*.

Remark 13.4.3. Example 13.4.1 demonstrates that a quotient of a Hausdorff topological space is not necessarily Hausdorff. Thus we do not yet have a 'canonical method' to prove that $(M^2, \mathcal{O}_{M^2})$, $(K^2, \mathcal{O}_{K^2})$, and our other examples of quotients of topological spaces, are Hausdorff.

We shall see later that if $(X, \mathcal{O}_X)$ and $\sim$ satisfy certain conditions, then $(X/\sim, \mathcal{O}_{X/\sim})$ can be proven by a 'canonical method' to be Hausdorff.

Remark 13.4.4. We can intuitively believe that a quotient of a Hausdorff topological space might not be Hausdorff. In a Hausdorff topological space, every two points can be 'separated' by subsets belonging to the topology: the points are 'not too close together'.

When we take a quotient, however, we may identify many points. Thus points which were not 'close together' before taking the quotient may be 'close together' afterwards. So much so that we may no longer be able to 'separate' every two points.

E13 Exercises for Lecture 13

E13.1 Exam questions

Task E13.1.1. Let $X = \{a, b, c, d\}$ be a set with four elements. Let $\mathcal{O}_X$ be the topology on X given by

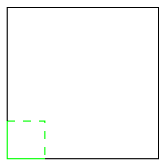
$$\{\emptyset, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}.$$

Demonstrate that $(X, \mathcal{O}_X)$ is not Hausdorff.

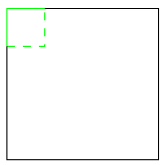
Task E13.1.2. Prove that the Sorgenfrey line of Task E11.1.12 is Hausdorff.

Task E13.1.3. Let $\mathcal{O}$ be the topology on I^2 given by the set of subsets U of I^2 such that, for every x which belongs to U, we have either that $x = 0$, or else that one of the following holds.

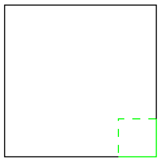
(1) We have that x belongs to $[0, y[\times [0, y[$ for some $0 < y < \frac{1}{2}$, and this set is a subset of U.



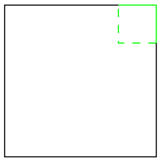
(2) We have that x belongs to $[0, y[\times]1 - y, 1]$ for some $0 < y < \frac{1}{2}$, and this set is a subset of U.



(3) We have that x belongs to $]1 - y, 1] \times [0, y[$ for some $0 < y < \frac{1}{2}$, and this set is a subset of U.

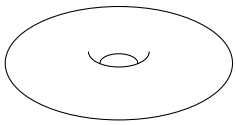


(4) We have that x belongs to $]1-y,1] \times]1-y,1]$ for some $0 < y < \frac{1}{2}$, and this set is a subset of U.



Is $(I^2, \mathcal{O})$ homeomorphic to $(I^2, \mathcal{O}_{I^2})$?

Task E13.1.4. Prove that $(T^2, \mathcal{O}_{T^2})$ is Hausdorff.



Remark E13.1.1. The intention in Task E13.1.4 is for you to give a proof from first principles. In a later lecture, we shall see how to prove that $(T^2, \mathcal{O}_{T^2})$ is Hausdorff by a 'canonical method'.

It is also possible to give a proof by appealing to Corollary 13.3.8 and the fact, discussed in Example 8.1.4, that $(T^2, \mathcal{O}_{T^2})$ is homeomorphic to $(S^1 \times S^1, \mathcal{O}_{S^1 \times S^1})$. Since $(S^1, \mathcal{O}_{S^1})$ is Hausdorff by Example 13.3.6, we have that $(S^1 \times S^1, \mathcal{O}_{S^1 \times S^1})$ is Hausdorff by Proposition 13.3.3.

E13.2 In the lecture notes

Task E13.2.1. Suppose that x belongs to $\mathbb{R}$. Prove that $]-\infty, x[$ and $]x, \infty[$ belong to $\mathcal{O}_{\mathbb{R}}$.

Task E13.2.2. Prove that the set $\mathcal{O}$ of Example 13.2.7 defines a topology on $\mathbb{R}^2$.

Task E13.2.3. Let X and Y be sets, and let

$$X \xrightarrow{f} Y$$

be a bijection. Thus there is a map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$.

(1) Suppose that y_0 and y_1 belong to Y, and that $y_0 \neq y_1$. Prove that $g(y_0) \neq g(y_1)$. You may wish to appeal to the fact that $f \circ g = id_Y$.

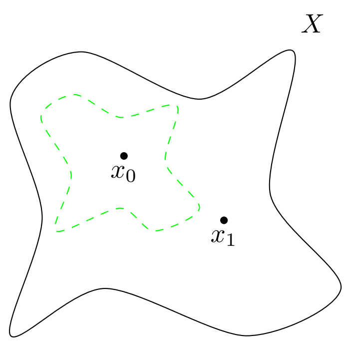
(2) Suppose that U_0 and U_1 are subsets of X, and that $U_0 \cap U_1$ is empty. Prove that $f(U_0) \cap f(U_1)$ is empty. You may wish to appeal to the fact that $g \circ f = id_X$.

Task E13.2.4. In the notation of Example 13.4.1, prove that, for any neighbourhood U of $\pi\left((0,0)\right)$ in $X/\sim$ with respect to $\mathcal{O}_{X/\sim}$, there is an open interval $]a,b[$ with $a < 0 < b$ such that $]a,b[\times \{0\}$ is a subset of $\pi^{-1}(U)$.

E13.3 For a deeper understanding

Task E13.3.1. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let $\mathcal{O}'_X$ be a topology on X such that $\mathcal{O}_X$ is a subset of $\mathcal{O}'_X$. Prove that $(X, \mathcal{O}'_X)$ is Hausdorff.

Definition E13.3.2. A topological space $(X, \mathcal{O}_X)$ is T1 if, for every ordered pair (x_0, x_1) such that x_0 and x_1 belong to X and $x_0 \neq x_1$, there is a neighbourhood of x_0 in X with respect to $\mathcal{O}_X$ which does not contain x_1.



Remark E13.3.3. . Suppose that $(X, \mathcal{O}_X)$ is a Hausdorff topological space. Then $(X, \mathcal{O}_X)$ is a T1 topological space.

Task E13.3.4. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Prove that $\{x\}$ is closed in X with respect to $\mathcal{O}_X$ if and only if $(X, \mathcal{O}_X)$ is a T1 topological space. You may wish to proceed as follows.

(1) Suppose that $(X, \mathcal{O}_X)$ is a T1 topological space. Suppose that y belongs to X, and that $x \neq y$. Since $(X, \mathcal{O}_X)$ is a T1 topological space, there is a neighbourhood U_y of y in X with respect to $\mathcal{O}_X$ such that x does not belong to U_y. Deduce that y is not a limit point of $\{x\}$ in X with respect to $\mathcal{O}_X$.

(2) By Proposition 9.1.1, deduce from (1) that $\{x\}$ is closed in X with respect to $\mathcal{O}_X$..

(3) Suppose instead that $\{x\}$ is closed in X with respect to $\mathcal{O}_X$ for every x which belongs to X. Suppose that x_0 and x_1 belong to X, and that $x_0 \neq x_1$. Since $\{x_0\}$ is closed in X with respect to $\mathcal{O}_X$, observe have that $X \setminus \{x_1\}$ belongs to $\mathcal{O}_X$.

(4) Moreover, observe that x_0 belongs to $X \setminus \{x_1\}$. Conclude that $(X, \mathcal{O}_X)$ is T1.

Corollary E13.3.5. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Suppose that x belongs to X. Then $\{x\}$ is closed in X with respect to $\mathcal{O}_X$.

Proof. Follows immediately from Task E13.3.4 and Remark E13.3.3. □

Task E13.3.6. Let $(X, \mathcal{O}_X)$ be a T1 topological space. Suppose that $\mathcal{O}_X$ is finite. Prove that $\mathcal{O}_X$ is the discrete topology on X. You may wish to proceed as follows.

(1) Suppose that x belongs to X. Since $(X, \mathcal{O}_X)$ is T1, there is, for every y which belongs to X such that $x \neq y$, a neighbourhood U_y of x in X with respect to $\mathcal{O}_X$ such that y does not belong to U_y. Observe that

$$\bigcap_{y \in Y \setminus \{x\}} U_y$$

is $\{x\}$.

(2) Since $\mathcal{O}_X$ is finite, observe that

$$\bigcap_{y \in Y \setminus \{x\}} U_y$$

belongs to $\mathcal{O}_X$.

(3) Deduce that $\{x\}$ belongs to $\mathcal{O}_X$. Conclude that $\mathcal{O}_X$ is the discrete topology on X.

Corollary E13.3.7. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Suppose that $\mathcal{O}_X$ is finite. Then $\mathcal{O}_X$ is the discrete topology on X.

Proof. Follows immediately from Task E13.3.6 and Remark E13.3.3. □

Task E13.3.8. Let $(X/\sim, \mathcal{O}_{X/\sim})$ be the real line with two origins of Example 13.4.1. Prove that $(X/\sim, \mathcal{O}_{X/\sim})$ is T1. You may wish to appeal to the fact that for any open interval $]a, b[$ such that $a < 0 < b$, we have that $\pi\left(]a, b[\times \{0\}\right)$ belongs to $\mathcal{O}_{X/\sim}$, but does not contain $\pi\left((0, 1)\right)$.

Remark E13.3.9. Example 13.4.1 and Task E13.3.8 demonstrate that a T1 topological space is not necessarily Hausdorff.

E13.4 Exploration — Hausdorffness for metric spaces

Definition E13.4.1. Let X be a set. A metric d on X is *separating* if, for any x_0 and x_1 which belong to X with the property that $d(x_0, x_1) = 0$, we have that $x_0 = x_1$.

Definition E13.4.2. A metric space (X, d) is *separated* if d is separating.

Task E13.4.3. Let (X, d) be a separated, symmetric metric space. Let $\mathcal{O}_d$ be the topology on X corresponding to d of Task E3.4.9. Prove that $(X, \mathcal{O}_d)$ is Hausdorff. You may wish to proceed as follows.

(1) Suppose that x_0 and x_1 belong to X, and that $x_0 \neq x_1$. Since (X, d) is separated, deduce that $d(x_0, x_1) > 0$.

(2) Let $\epsilon = \frac{d(x_0,x_1)}{2}$. Appealing to Task E4.3.2, observe that $B_\epsilon(x_0)$ is a neighbourhood of x_0 in X with respect to $\mathcal{O}_d$, and that $B_\epsilon(x_1)$ is a neighbourhood of x_1 in X with respect to $\mathcal{O}_d$-

(3) Suppose that y belongs to $B_\epsilon(x_0)$. By definition of d, we have that

$$\begin{aligned} d(x_0, x_1) &\leq d(x_0, y) + d(y, x_1) \\ &< \tfrac{d(x_0,x_1)}{2} + d(y, x_1). \end{aligned}$$

Thus we have that

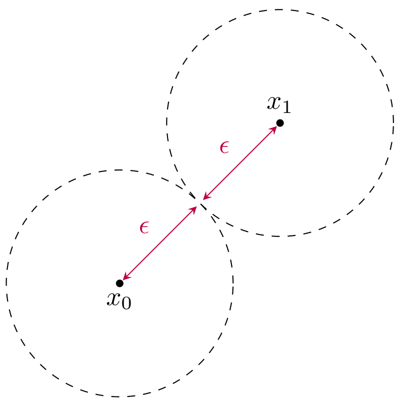
$$d(y, x_1) > \tfrac{d(x_0,x_1)}{2}.$$

Since (X, d) is symmetric, deduce that

$$d(x_1, y) > \tfrac{d(x_0,x_1)}{2}.$$

(4) Deduce from (3) that y does not belong to $B_\epsilon(x_1)$, and thus that $B_\epsilon(x_0) \cap B_\epsilon(x_1)$ is empty.

(5) Conclude from (2) and (4) that (X, d) is Hausdorff.



Definition E13.4.4. A topological space $(X, \mathcal{O}_X)$ is *perfectly normal* if, for every ordered pair of subsets A_0 and A_1 of subsets of X which are closed in X with respect to $\mathcal{O}_X$, which have the property that $A_0 \cap A_1$ is empty, and which are both not empty, there is a continuous map

$$X \xrightarrow{f} I$$

such that $f^{-1}(\{0\}) = A_0$ and $f^{-1}(\{1\}) = A_1$.

Task E13.4.5. Let $(X, \mathcal{O}_X)$ be a perfectly normal topological space. Prove that $(X, \mathcal{O}_X)$ is Hausdorff. You may wish to appeal to Corollary E13.3.5.

Task E13.4.6. Let (X, d) be a separated, symmetric metric space. Let $\mathcal{O}_d$ be the topology on X corresponding to d of Task E3.4.9. Prove that $(X, \mathcal{O}_d)$ is perfectly normal. You may wish to proceed as follows.

(1) Since $A_0 \cap A_1$ is empty, deduce, by Task E9.4.2, that $d(x, A_0) + d(x, A_1) > 0$ for every x which belongs to X.

(2) Since (X, d) is symmetric, we have by Task E4.3.8 that the map

$$X \xrightarrow{d(-, A_0)} \mathbb{R}$$

given by $x \mapsto d(x, A_0)$ is continuous, and that the map

$$X \xrightarrow{d(-, A_1)} \mathbb{R}$$

given by $x \mapsto d(x, A_1)$ is continuous. By (1), Task E5.3.6, Task E5.3.10, and Task E5.1.9, deduce that the map

$$X \xrightarrow{f} I$$

given by $x \mapsto \frac{d(x, A_0)}{d(x, A_0) + d(x, A_1)}$ is continuous.

(3) By Remark E4.3.1 and Task E9.4.2, observe that $f^{-1}(\{0\}) = A_0$, and that $f^{-1}(\{1\}) = A_1$.

(4) Conclude from (2) and (3) that (X, d) is perfectly normal.

Remark E13.4.1. Task E13.4.6 and Task E13.4.5 give a second proof that the topological space arising from every metric space is Hausdorff.