MA3002 Generell Topologi — Vår 2014

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14 Tuesday 18th February

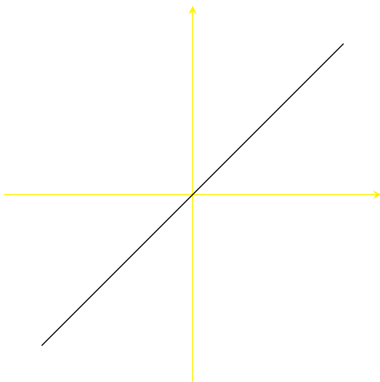
14.1 Characterisation of Hausdorff topological spaces

Notation 14.1.1. Let X be a set. We denote the subset

$$\{(x, x) \in X \times X \mid x \in X\}$$

of $X \times X$ by $\Delta(X)$.

Example 14.1.2. Let X be $\mathbb{R}$. Then $\Delta(X)$ is the line in $\mathbb{R}^2$ defined by $y = x$.



Proposition 14.1.3. A topological space $(X, \mathcal{O}_X)$ is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$.

Proof. We consider the following assertions.

(1) We have that $\Delta(X)$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$.

(2) Every limit point of $\Delta(X)$ in $X \times X$ with respect to $\mathcal{O}_{X \times X}$ belongs to $\Delta(X)$.

(3) For every (x_0, x_1) which belongs to $X \times X$, there is a neighbourhood W of (x_0, x_1) in $X \times X$ with respect to $\mathcal{O}_{X \times X}$ such that $W \cap \Delta(X)$ is empty.

(4) For every (x_0, x_1) which belongs to $X \times X$, there is a neighbourhood U_0 of x_0 in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1 of x_1 in X with respect to $\mathcal{O}_X$, such that $(U_0 \times U_1) \cap \Delta(X)$ is empty.

(5) There is a neighbourhood U_0 of x_0 in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1 of x_1 in X with respect to $\mathcal{O}_X$, such that $U_0 \cap U_1$ is empty.

By Proposition 9.1.1, we have that (1) holds if and only if (2) holds. By definition of a limit point of $\Delta(X)$ in $X \times X$ with respect to $\mathcal{O}_{X \times X}$, we have that (2) holds if and only if (3) holds. By Task E14.2.1, we have that (3) holds if and only if (4) holds. By Task E14.2.2, we have that (4) holds if and only if (5) holds. We conclude that (1) holds if and only if (5) holds, as required.

□

14.2 A necessary condition for a quotient of a Hausdorff topological space to be Hausdorff

Remark 14.2.1. Let X be a set. As discussed in §A.4, a relation on X is formally a subset R of $X \times X$. When we write that $x_0 \sim x_1$, we formally mean that (x_0, x_1) belongs to R.

By extension, when we write that $\sim$ is a relation on X, this is shorthand for: we have a subset R of $X \times X$, and shall write $x_0 \sim x_1$ when (x_0, x_1) belongs to R. When we adopt this shorthand, we shall denote R by $R_\sim$. Tautologically, we thus have that

$$R_\sim = \{(x_0, x_1) \in X \times X \mid x_0 \sim x_1\}.$$

Proposition 14.2.2. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let $\sim$ be an equivalence relation on X. Suppose that $(X/\sim, \mathcal{O}_{X/\sim})$ is a Hausdorff topological space. Then $R_\sim$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$.

Proof. Let

$$X \xrightarrow{\pi} X/\sim$$

be the quotient map. Let

$$X \times X \xrightarrow{\pi \times \pi} (X/\sim) \times (X/\sim)$$

be the map given by $(x_0, x_1) \mapsto (\pi(x_0), \pi(x_1))$. By Remark 6.1.9 we have that π is continuous. By Task E5.3.17, we deduce that $\pi \times \pi$ is continuous.

Since $(X/\sim, \mathcal{O}_{X/\sim})$ is a Hausdorff topological space, we have, by Proposition 14.1.3, that $\Delta(X/\sim)$ is closed in $(X/\sim) \times (X/\sim)$ with respect to $\mathcal{O}_{(X/\sim)\times(X/\sim)}$. Since $\pi \times \pi$ is continuous, we deduce, by Task E5.1.13, that $(\pi \times \pi)^{-1}(\Delta(X/\sim))$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$.

We have that $R_\sim = (\pi \times \pi)^{-1}(\Delta(X/\sim))$. To verify this is the topic of Task E14.2.3. We conclude that $R_\sim$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$. □

Example 14.2.3. Let X and $\sim$ be as in Example 13.4.1. Then $R_\sim$ is the union of the following four sets.

(1) $\Delta((\mathbb{R} \setminus \{0\}) \times \{0\})$.

(2) $\Delta((\mathbb{R} \setminus \{0\}) \times \{1\})$.

(3) $((\mathbb{R} \setminus \{0\}) \times \{0\}) \times ((\mathbb{R} \setminus \{0\}) \times \{1\})$.

(4) $((\mathbb{R} \setminus \{0\}) \times \{1\}) \times ((\mathbb{R} \setminus \{0\}) \times \{0\})$.

By Task E14.2.4. we have that $((0,0),(0,0))$ is a limit point of $R_\sim$ in $X \times X$ with respect to $\mathcal{O}_{X \times X}$. Since $((0,0),(0,0))$ does not belong to $R_\sim$, we deduce, by Proposition 9.1.1, that $R_\sim$ is not closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$. By Proposition 14.2.2, we conclude that $(X/\sim, \mathcal{O}_{X/\sim})$ is not Hausdorff, as we demonstrated directly in Example 13.4.1.

Remark 14.2.4. In general, that $R_\sim$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$ is not sufficient to ensure that $(X/\sim, \mathcal{O}_{X/\sim})$ is Hausdorff. An example is discussed in Task E14.3.1 – Task E14.3.5.

14.3 Compact topological spaces

Definition 14.3.1. Let $(X, \mathcal{O}_X)$ be a topological space. An *open covering* of X with respect to $\mathcal{O}_X$ is a set $\{U_j\}_{j \in J}$ of subsets of X such that the following hold.

(1) We have that U_j belongs to $\mathcal{O}_X$ for every j which belongs to J.

(2) We have that $X = \bigcup_{j \in J} U_j$.

Definition 14.3.2. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\mathcal{U} = \{U_j\}_{j \in J}$ be an open covering of X with respect to $\mathcal{O}_X$. Let K be a subset of J. Then $\{U_k\}_{k \in K}$ is a *finite subcovering* of $\mathcal{U}$ if the following hld.

(1) We have that $\{U_k\}_{k \in K}$ is finite.

(2) We have that $X = \bigcup_{k \in K} U_k$.

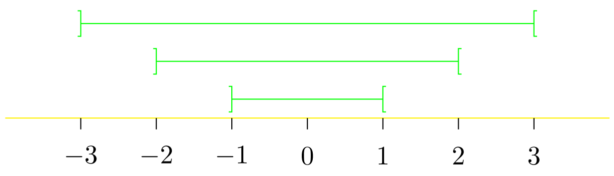
Definition 14.3.3. A topological space $(X, \mathcal{O}_X)$ is *compact* if, for every open covering $\mathcal{U} = \{U_j\}_{j \in J}$ of X with respect to $\mathcal{O}_X$, there is a subset K of J such that $\{U_k\}_{k \in K}$ is a finite subcovering of $\mathcal{U}$.

Example 14.3.4. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that $\mathcal{O}_X$ is finite. Then every set $\{U_j\}_{j \in J}$ such that U_j belongs to $\mathcal{O}_X$ for all j which belong to J is finite. Thus $(X, \mathcal{O}_X)$ is compact.

Remark 14.3.5. In particular, if X is finite, then $(X, \mathcal{O}_X)$ is compact.

14.4 Examples of topological spaces which are not compact

Example 14.4.1. The set $\mathcal{U} = \{]-n, n[\}_{n \in \mathbb{N}}$ is an open covering of $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.



Let K be a subset of $\mathbb{N}$ such that $\{]-n, n[\}_{n \in K}$ is finite. This is the same as to say that K is a finite subset of $\mathbb{N}$. Then

$$\bigcup_{n \in K}]-n, n[=]-m, m[\,,$$

where $m = \max K$. In particular, we do not have that

$$\bigcup_{n \in K}]-n, n[= \mathbb{R}.$$

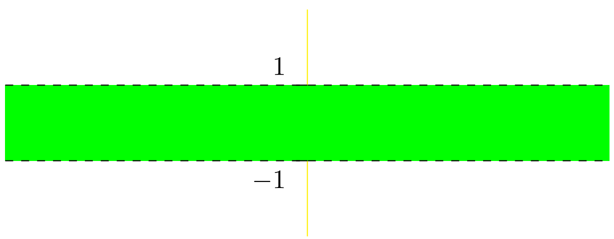
Thus $\{]-n, n[\}_{n \in K}$ is not a finite subcovering of $\mathcal{U}$.

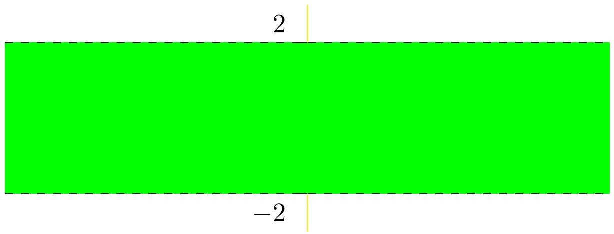
This demonstrates that $\mathcal{U}$ does not admit a finite subcovering. We conclude that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not compact.

Example 14.4.2. The set

$$\mathcal{U} = \{\mathbb{R} \times]-n, n[\}_{n \in \mathbb{N}}$$

is an open covering of $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.





Let K be a subset of $\mathbb{N}$ such that $\{\mathbb{R} \times]-n, n[\}_{n \in K}$ is finite. This is the same as to say that K is a finite subset of $\mathbb{N}$. Then

$$\bigcup_{n \in K} (\mathbb{R} \times]-n, n[) = \mathbb{R} \times]-m, m[\,,$$

where $m = \max K$. In particular, we do not have that

$$\bigcup_{n \in K} (\mathbb{R} \times]-n, n[) = \mathbb{R}^2.$$

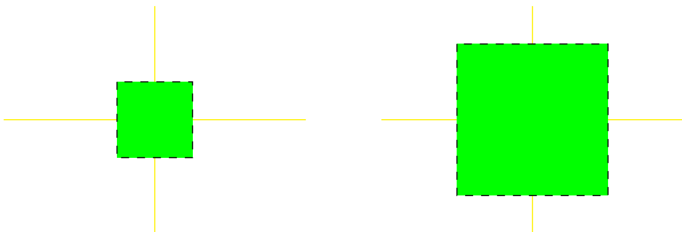
Thus $\{\mathbb{R} \times]-n, n[\}_{n \in K}$ is not a finite subcovering of $\mathcal{U}$.

This demonstrates that $\mathcal{U}$ does not admit a finite subcovering. We conclude that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is not compact.

Example 14.4.3. The set

$$\mathcal{U} = \{]-n, n[\times]-n, n[\}_{n \in \mathbb{N}}$$

is an open covering of $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.



Let K be a subset of $\mathbb{N}$ such that $\{]-n, n[\times]-n, n[\}_{n \in K}$ is finite. This is the same as to say that K is a finite subset of $\mathbb{N}$. Then

$$\bigcup_{n \in K} (]-n, n[\times]-n, n[) =]-.m, m[\times]-m, m[\,,$$

where $m = \max K$. In particular, we do not have that

$$\bigcup_{n \in K} (]-n, n[\times]-n, n[) = \mathbb{R}^2.$$

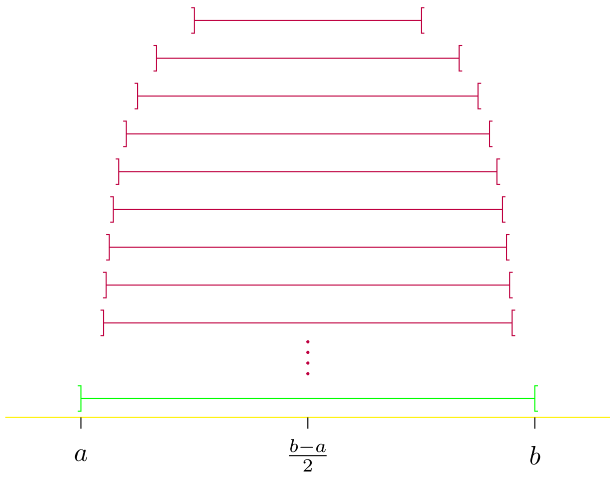
Thus $\{]-n, n[\times]-n, n[\}_{n \in K}$ is not a finite subcovering of $\mathcal{U}$.

This demonstrates that $\mathcal{U}$ does not admit a finite subcovering. Thereby it gives a second proof that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is not compact.

Example 14.4.4. Suppose that a and b belong to $\mathbb{R}$. Let $\mathcal{O}_{]a,b[}$ be the subspace topology on $]a, b[$ with respect to $\mathcal{O}_{]a,b[}$. The set

$$\mathcal{U} = \left\{ \left]a + \tfrac{1}{n}, b - \tfrac{1}{n}\right[\right\}_{n \in \mathbb{N} \text{ and } \frac{1}{n} < \frac{b-a}{2}}$$

is an open covering of $]a,b[$ with respect to $\mathcal{O}_{]a,b[}$.



Let K be a subset of $\{n \in \mathbb{N} \mid \frac{1}{n} < \frac{b-a}{2}\}$ such that $\{]a+\frac{1}{n}, b-\frac{1}{n}[\}_{n \in K}$ is finite. This is the same as to say that K is a finite subset of $\{n \in \mathbb{N} \mid \frac{1}{n} < \frac{b-a}{2}\}$. Then

$$\bigcup_{n \in K} \left]a+\tfrac{1}{n}, b-\tfrac{1}{n}\right[= \left]a+\tfrac{1}{m}, b-\tfrac{1}{m}\right[,$$

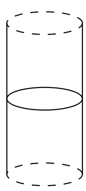
where $m = \max K$. In particular, we do not have that

$$\bigcup_{n \in K} \left]a+\tfrac{1}{n}, b-\tfrac{1}{n}\right[=]a,b[.$$

Thus $\{]a+\frac{1}{n}, b-\frac{1}{n}[\}_{n \in K}$ is not a finite subcovering of $\mathcal{U}$.

This demonstrates that $\mathcal{U}$ does not admit a finite subcovering. We conclude that $(]a,b[, \mathcal{O}_{]a,b[})$ is not compact.

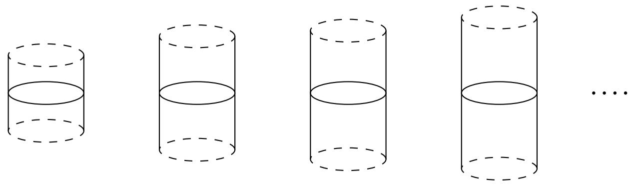
Example 14.4.5. Let us think of $S^1 \times]0,1[$ as a cylinder with the two circles at its ends removed.



The set

$$\{S^1 \times]\tfrac{1}{n}, 1-\tfrac{1}{n}[\}_{n \in \mathbb{N} \text{ and } n > 2}$$

is an open covering of $S^1 \times]0,1[$ with respect to $\mathcal{O}_{S^1 \times]0,1[}$.



Let K be a subset of $\{n \in \mathbb{N} \mid n > 2\}$ such that $\left\{S^1 \times \left]\frac{1}{n}, 1 - \frac{1}{n}\right[\right\}_{n \in K}$ is finite. This is the same as to say that K is a finite subset of $\{n \in \mathbb{N} \mid n > 2\}$. Then

$$\bigcup_{n \in K} \left(S^1 \times \left]\tfrac{1}{n}, 1 - \tfrac{1}{n}\right[\right) = S^1 \times \left]\tfrac{1}{m}, 1 - \tfrac{1}{m}\right[,$$

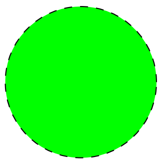
where $m = \max K$. In particular, we do not have that

$$\bigcup_{n \in K} \left(S^1 \times \left]\tfrac{1}{n}, 1 - \tfrac{1}{n}\right[\right) = S^1 \times]0, 1[.$$

Thus $\left\{S^1 \times \left]\frac{1}{n}, 1 - \frac{1}{n}\right[\right\}_{n \in K}$ is not a finite subcovering of $\mathcal{U}$.

This demonstrates that $\mathcal{U}$ does not admit a finite subcovering. We conclude that $\left(S^1 \times]0,1[, \mathcal{O}_{S^1 \times]0,1[}\right)$ is not compact.

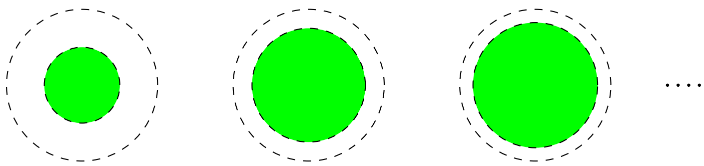
Example 14.4.6. Let $\mathcal{O}_{D^2 \setminus S^1}$ be the subspace topology on $D^2 \setminus S^1$ with respect to $(D^2, \mathcal{O}_{D^2})$.



Let U_n be the subset of $D^2 \setminus S^1$ given by

$$\left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| < 1 - \tfrac{1}{n}\right\}.$$

The set $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ is an open covering of $D^2 \setminus S^1$ with respect to $\mathcal{O}_{D^2 \setminus S^1}$.



Let K be a subset of $\mathbb{N}$ such that $\{U_n\}_{n \in K}$ is finite. This is the same as to say that K is a finite subset of $\mathbb{N}$. Then

$$\bigcup_{n \in K} U_n = U_m,$$

where $m = \max K$. In particular, we do not have that

$$\bigcup_{n \in K} U_n = D^2 \setminus S^1.$$

Thus $\{U_n\}_{n \in K}$ is not a finite subcovering of $\mathcal{U}$.

This demonstrates that $\mathcal{U}$ does not admit a finite subcovering. We conclude that $\left(D^2 \setminus S^1, \mathcal{O}_{D^2 \setminus S^1}\right)$ is not compact.

E14 Exercises for Lecture 14

E14.1 Exam questions

Task E14.1.1. Give a counterexample to the following assertion: the set

$$\{(x, x) \in \mathbb{R}^2 \mid x \in X\}$$

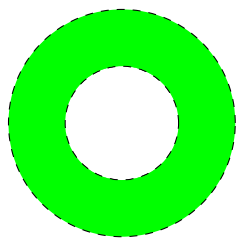
is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ for every subset X of $\mathbb{R}$. Give an example of a topological property which can be imposed upon X to ensure that the assertion correct. Justify your answer.

Task E14.1.2. Let $\mathcal{O}_{[0,1[}$ be the subspace topology on $[0, 1[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Is $\left([0, 1[\, , \mathcal{O}_{[0,1[}\right)$ compact?

Task E14.1.3. Find an open covering of $I^2 \times \mathbb{R}$ with respect to $\mathcal{O}_{I^2\times\mathbb{R}}$ which does not admit a finite subcovering. Conclude that $(I^2 \times \mathbb{R}, \mathcal{O}_{I^2\times\mathbb{R}})$ is not compact.

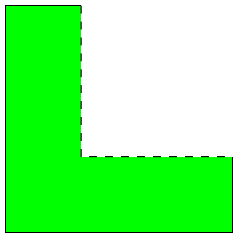
Task E14.1.4. Let X be the 'open annulus' given by

$$\left\{(x, y) \in \mathbb{R}^2 \mid \tfrac{1}{2} < \|(x, y)\| < 1\right\}.$$



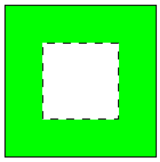
Let $\mathcal{O}_X$ the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Give an example of an open covering of $(X, \mathcal{O}_X)$ which does not admit a finite subcovering. Deduce that $(X, \mathcal{O}_X)$ is not compact.

Task E14.1.5. Let X be the union of $[0,1[\times [0,3]$ and $[0,3] \times [0,1[$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Find an open covering of X which does not admit a finite subcovering. Conclude that $(X, \mathcal{O}_X)$ is not compact.

Task E14.1.6. Let X be the set given by $I^2 \setminus \left(\left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{4}, \frac{3}{4}\right]\right)$.



Let

$$I^2 \xrightarrow{\pi} T^2$$

be the quotient map. Let $\mathcal{O}_{\pi(X)}$ be the subspace topology on $\pi(X)$ with respect to $(T^2, \mathcal{O}_{T^2})$. Demonstrate that $\left(\pi(X), \mathcal{O}_{\pi(X)}\right)$ is not compact.

Task E14.1.7. Prove that the Sorgenfrey line of Task E11.1.12 is not compact.

Task E14.1.8. Give an example of an equivalence relation $\sim$ on $\mathbb{R}$ such that $(\mathbb{R}/\sim, \mathcal{O}_{\mathbb{R}/\sim})$ is compact.

E14.2 In the lecture notes

Task E14.2.1. Let $(X_0, \mathcal{O}_{X_0})$ and $(X_1, \mathcal{O}_{X_1})$ be topological spaces. Let A be a subset of $X_0 \times X_1$. Suppose that (x_0, x_1) belongs to $X_0 \times X_1$. Prove that the following assertions are equivalent.

(1) There is a neighbourhood W of (x_0, x_1) in $X \times X$ with respect to $\mathcal{O}_{X \times X}$ such that $W \cap A$ is empty.

(2) There is a neighbourhood U_0 of x_0 in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1 of x_1 in X with respect to $\mathcal{O}_X$ such that $(U_0 \times U_1) \cap A$ is empty.

Task E14.2.2. Let X_0 and X_1 be sets. Let A be a subset of $X_0 \times X_1$. Let U_0 be a subset of X_0, and let U_1 be a subset of X_1. Prove that $(U_0 \times U_1) \cap A = U_0 \cap U_1$.

Task E14.2.3. Let X be a set, and let $\sim$ be an equivalence relation on X. Let

$$X \xrightarrow{\pi} X/\sim$$

be the quotient map. Let

$$X \times X \xrightarrow{\pi \times \pi} (X/\sim) \times (X/\sim)$$

be the map given by $(x_0, x_1) \mapsto (\pi(x_0), \pi(x_1))$. Prove that $R_\sim = (\pi \times \pi)^{-1} (\Delta (X/\sim))$.

Task E14.2.4. Let X and $\sim$ be as in Example 13.4.1. Prove that $((0,0),(0,0))$ is a limit point of $R_\sim$ in $X \times X$ with respect to $\mathcal{O}_X$.

E14.3 For a deeper understanding

Task E14.3.1. Let Σ be the set given by

$$\left\{ \tfrac{1}{n} \mid n \in \mathbb{N} \right\}.$$

Let $\mathcal{O}^K$ be the set of subsets U with the property that, for every x which belongs to U, there are real numbers a and b such that one of the following holds.

(1) We have that x belongs to $]a,b[$, and that $]a,b[$ is a subset of U.

(2) We have that x belongs to $]a,b[\setminus (]a,b[\cap \Sigma)$, and that $]a,b[\setminus (]a,b[\cap \Sigma)$ is a subset of U.

Prove that $\mathcal{O}^K$ defines a topology on $\mathbb{R}$.

Terminology E14.3.2. The topology $\mathcal{O}^K$ is known as the *K-topology* on $\mathbb{R}$.

Task E14.3.3. Prove that $(\mathbb{R}, \mathcal{O}^K)$ is Hausdorff. You may wish to proceed as follows.

(1) Observe that $\mathcal{O}_{\mathbb{R}}$ is a subset of $\mathcal{O}^K$.

(2) Appeal to Example 13.2.1 and to Task E13.3.1.

Task E14.3.4. Let $\sim$ be the equivalence relation on $\mathbb{R}$ generated by $1 \sim \frac{1}{n}$ for every n which belongs to $\mathbb{N}$. Prove that $(\mathbb{R}/\sim, \mathcal{O}_{\mathbb{R}/\sim})$ is not Hausdorff. You may wish to proceed as follows.

(1) Let

$$\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\sim$$

be the quotient map. Let U_0 be a neighbourhood of $\pi(0)$ in $\mathbb{R}/\sim$ with respect to $\mathcal{O}_{\mathbb{R}/\sim}$. Let U_1 be a neighbourhood of $\pi(1)$ in $\mathbb{R}/\sim$ with respect to $\mathcal{O}_{\mathbb{R}/\sim}$. By Remark 6.1.9, we have that π is continuous. Deduce that $\pi^{-1}(U_0)$ and $\pi^{-1}(U_1)$ belong to $\mathcal{O}_{\mathbb{R}}$.

(2) Since $\pi(1)$ belongs to U_1, observe that, by definition of $\sim$, the set Σ is a subset of $\pi^{-1}(U_1)$.

(3) Suppose that n belongs to $\mathbb{N}$. Since $\frac{1}{n}$ belongs to $\pi^{-1}(U_1)$, and since $\pi^{-1}(U_1)$ belongs to $\mathcal{O}^K$, observe that, by definition of $\mathcal{O}^K$ and the fact that $\frac{1}{n}$ belongs to Σ, there are real numbers a_n and b_n such that $a_n < \frac{1}{n} < b_n$, and such that $]a_n, b_n[$ is a subset of $\pi^{-1}(U_1)$.

(4) Since $\pi^{-1}(U_0)$ belongs to $\mathcal{O}^K$, we have, by definition of $\mathcal{O}^K$, that there are real numbers a and b such that one of the following holds.

(I) We have that 0 belongs to $]a, b[$, and that $]a, b[$ is a subset of $\pi^{-1}(U_0)$.

(II) We have that 0 belongs to $]a, b[\setminus (]a, b[\cap \Sigma)$, and that $]a, b[\setminus (]a, b[\cap \Sigma)$ is a subset of $\pi^{-1}(U_0)$.

In either case, let n be a natural number such that $\frac{1}{n} < b$. Let x be a real number which does not belong to Σ, and which has the property that $a_n < x < \frac{1}{n}$ and that $0 < x$. Observe that x belongs to both $\pi^{-1}(U_0)$ and to $\pi^{-1}(U_1)$.

(5) Deduce from (4) that $\pi(x)$ belongs to both U_0 and U_1. In other words, $U_0 \cap U_1$ is not empty.

(6) Conclude that $(\mathbb{R}/\sim, \mathcal{O}_{\mathbb{R}/\sim})$ is not Hausdorff.

Task E14.3.5. Let $\sim$ be the equivalence relation on $\mathbb{R}$ of Task E14.3.4. Let $\mathcal{O}^{K^2}$ be the product topology on $\mathbb{R}^2$ with respect to two copies of $(\mathbb{R}, \mathcal{O}^K)$. Prove that $R_\sim$ is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}^{K^2}$. You may wish to proceed as follows.

(1) Suppose that x is a limit point of Σ in $\mathbb{R}$ with respect to $\mathcal{O}^K$. By Task E8.3.10, deduce that x is a limit point of Σ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

(2) Demonstrate that the only limit point of Σ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$ is 0.

(3) Suppose that a and b belong to $\mathbb{R}$, and that $a < 0 < b$. Observe $]a, b[\setminus \Sigma$ is a neighbourhood of 0 in $\mathbb{R}$ with respect to $\mathcal{O}^K$. Since $\Sigma \cap (]a, b[\setminus \Sigma)$ is empty, deduce that 0 is not a limit point of Σ in $\mathbb{R}$ with respect to $\mathcal{O}^K$.

(4) Deduce from (1)–(3) that Σ is closed in $\mathbb{R}$ with respect to $\mathcal{O}^K$.

(5) By Task E3.3.1, deduce from (4) that $\Sigma \times \Sigma$ is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}^{K^2}$.

(6) Observe that $R_\sim$ is $\Sigma \times \Sigma$.

(7) Conclude that $R_\sim$ is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}^{K^2}$.

Task E14.3.6. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that either $(X, \mathcal{O}_X)$ or $(Y, \mathcal{O}_Y)$ is not compact. Prove that $(X \times Y, \mathcal{O}_{X \times Y})$ is not compact. You may wish to glance back at Example 14.4.2 and Example 14.4.5.