MA3002 Generell Topologi — Vår 2014

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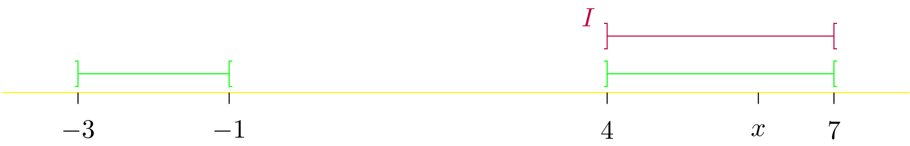
2.1 Standard topology on $\mathbb{R}$, continued

Example 2.1.1. Let U be a disjoint union of open intervals. For instance, the union of $]-3,-1[$ and $]4,7[$.

Then U belongs to $\mathcal{O}_{\mathbb{R}}$. There are two cases.

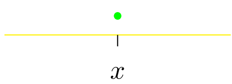
(1) If $-3 < x < -1$, we can, for instance, take I to be $]-3,-1[$

(2) If $4 < x < 7$, we can, for instance, take I to be $]4,7[$.



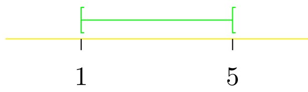
Remark 2.1.2. In fact, *every* subset of U which belongs to $\mathcal{O}_{\mathbb{R}}$ is a disjoint union of (possibly infinitely many) open intervals. To prove this is the topic of Task E2.3.7.

Example 2.1.3. Let $U = \{x\}$ be a subset of $\mathbb{R}$ consisting of a single $x \in \mathbb{R}$.

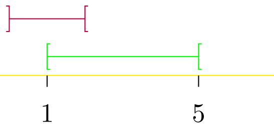


Then U does not belong to $\mathcal{O}_{\mathbb{R}}$. The only subset of $\{x\}$ to which x belongs is $\{x\}$ itself, and $\{x\}$ is not an open interval.

Example 2.1.4. Let U be the half open interval $[1, 5[$.



Then U does not belong to $\mathcal{O}_{\mathbb{R}}$, since there is no open interval I such that $1 \in I$ and $I \subset U$.



Lemma 2.1.5. Let $\{U_j\}_{j \in J}$ be a set of (possibly infinitely many) subsets of $\mathbb{R}$ such that $U_j \in \mathcal{O}_{\mathbb{R}}$ for all $j \in J$. Then $\bigcup_{j \in J} U_j$ belongs to $\mathcal{O}_{\mathbb{R}}$.

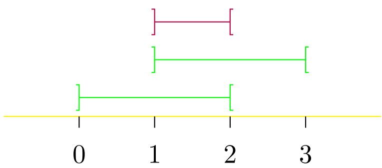
Proof. Let

$$x \in \bigcup_{j \in J} U_j.$$

By definition of $\bigcup_{j \in J} U_j$, we have that $x \in U_j$ for some $j \in J$. By definition of $\mathcal{O}_{\mathbb{R}}$, there is an open interval I such that $x \in I$ and $I \subset U_j \subset \bigcup_{j \in J} U_j$. $\square$

Observation 2.1.6. Let I and I' be open intervals. Then $I \cap I'$ is a (possibly empty) open interval. This is the topic of Task E2.2.1.

Example 2.1.7. The intersection of the open intervals $]0, 2[$ and $]1, 3[$ is the open interval $]1, 2[$.



The intersection of the open intervals $]-3, -1[$ and $]4, 7[$ is the empty set.

Lemma 2.1.8. Let U and U' be subsets of $\mathbb{R}$ which belong to $\mathcal{O}_{\mathbb{R}}$. Then $U \cap U'$ belongs to $\mathcal{O}_{\mathbb{R}}$.

Proof. Let $x \in U \cap U'$. By definition of $\mathcal{O}_{\mathbb{R}}$, we have the following.

(1) There is an open interval I_U such that $x \in I_U$ and $I_U \subset U$.

(2) There is an open interval $I_{U'}$ such that $x \in I_{U'}$ and $I_{U'} \subset U'$.

Then $x \in I_U \cap I_{U'}$ and $I_U \cap I_{U'} \subset U \cap U'$. By Observation 2.1.6, we have that $I_U \cap I_{U'}$ is an open interval. $\square$

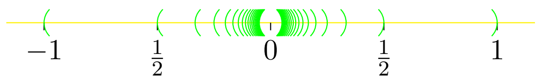
Proposition 2.1.9. The set $\mathcal{O}_{\mathbb{R}}$ defines a topology on $\mathbb{R}$.

Proof. This is exactly established by Observation 1.6.2, Lemma 2.1.5, and Lemma 2.1.8. $\square$

Terminology 2.1.10. We shall refer to $\mathcal{O}_{\mathbb{R}}$ as the *standard topology* on $\mathbb{R}$.

Remark 2.1.11. An infinite intersection of subsets of $\mathbb{R}$ which belong to $\mathcal{O}_{\mathbb{R}}$ does *not* necessarily belong to $\mathcal{O}_{\mathbb{R}}$. For instance, by Example 1.6.3, we have that $\left]-\frac{1}{n}, \frac{1}{n}\right[$ belongs to $\mathcal{O}_{\mathbb{R}}$ for every integer $n \geq 1$. However,

$$\bigcap_{n \in \mathbb{N}} \left]-\tfrac{1}{n}, \tfrac{1}{n}\right[= \{0\}.$$



By Example 2.1.3, the set $\{0\}$ does not belong to $\mathcal{O}_{\mathbb{R}}$.

Remark 2.1.12. The topological space $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is fundamental. We shall construct all our geometric examples of topological spaces in various 'canonical ways' from it.

A principal reason that we allow infinite unions in (3) of Definition 1.1.1, but only finite intersections in (4) of Definition 1.1.1, is that these properties hold for $\mathcal{O}_{\mathbb{R}}$.

Remark 2.1.13. An *Alexandroff topological space* is a topological space $(X, \mathcal{O})$ which, unlike $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ and the other geometric examples of topological spaces that we shall meet, has the property that if U is an intersection of (possibly infinitely many) subsets of X which belong to $\mathcal{O}$, then U belongs to $\mathcal{O}$. Alexandroff topological spaces are the topic of Exploration E1.4.

2.2 Subspace topologies

Remark 2.2.1. We shall explore several 'canonical ways' to construct topological spaces. In this section, we discuss the first of these.

Definition 2.2.2. Let $(Y, \mathcal{O}_Y)$ be a topological space, and let X be a subset of Y. Let $\mathcal{O}_X$ denote the set

$$\{X \cap U \mid U \in \mathcal{O}_Y\}.$$

Proposition 2.2.3. Let $(Y, \mathcal{O}_Y)$ be a topological space, and let X be a subset of Y. Then $(X, \mathcal{O}_X)$ is a topological space.

Proof. We verify that each of the conditions of Definition 1.1.1 holds.

(1) Since $\mathcal{O}_Y$ is a topology on Y, we have that $\emptyset$ belongs to $\mathcal{O}_Y$. We also have that $\emptyset = X \cap \emptyset$. Thus $\emptyset$ belongs to $\mathcal{O}_X$.

(2) Since $\mathcal{O}_Y$ is a topology on Y, we have that Y belongs to $\mathcal{O}_Y$. We also have that $X = X \cap Y$. Thus X belongs to $\mathcal{O}_X$.

(3) Let $\{U_j\}_{j \in J}$ be a set of subsets of X which belong to $\mathcal{O}_X$. By definition of $\mathcal{O}_X$, we have, for every $j \in J$, that

$$U_j = X \cap U'_j,$$

for a subset U'_j of Y which belongs to $\mathcal{O}_Y$. Now

$$\begin{aligned} \bigcup_{j \in J} U_j &= \bigcup_{j \in J} \left(X \cap U'_j\right) \\ &= X \cap \left(\bigcup_{j \in J} U'_j \right). \end{aligned}$$

Since $\mathcal{O}_Y$ is a topology on Y, we have that $\bigcup_{j \in J} U'_j$ belongs to $\mathcal{O}_Y$. We deduce that $\bigcup_{j \in J} U_j$ belongs to $\mathcal{O}_X$.

(4) Suppose that U_0 and U_1 are subsets of X which belong to $\mathcal{O}_X$. By definition of $\mathcal{O}_X$, we have that

$$U_0 = X \cap U'_0$$

and

$$U_1 = X \cap U'_1,$$

for a pair of subsets U'_0 and U'_1 of Y which belong to $\mathcal{O}_Y$. Now

$$\begin{aligned} U_0 \cap U_1 &= \left(X \cap U'_0\right) \cap \left(X \cap U'_1\right) \\ &= X \cap \left(U'_0 \cap U'_1\right). \end{aligned}$$

Since $\mathcal{O}_Y$ is a topology on Y, we have that $U'_0 \cap U'_1$ belongs to $\mathcal{O}_Y$. We deduce that $U_0 \cap U_1$ belongs to $\mathcal{O}_X$.

□

Remark 2.2.4. The flavour of this proof is very similar to many others in the early part of the course. It is a very good idea to work on it until you thoroughly understand it. This is the topic of Task E2.2.2.

Terminology 2.2.5. We refer to $\mathcal{O}_X$ as the *subspace topology* on X with respect to $(Y, \mathcal{O}_Y)$.

2.3 Example of a subspace topology — the unit interval

Definition 2.3.1. Let I denote the closed interval $[0,1]$. Let $\mathcal{O}_I$ denote the subspace topology on I with respect to $(\mathbb{R}, \mathcal{O}_\mathbb{R})$.

Terminology 2.3.2. We refer to $(I, \mathcal{O}_I)$ as the *unit interval*.

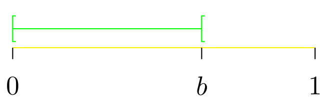
Example 2.3.3. Let $]a,b[$ be an open interval such that $0 < a < b < 1$.

As we observed in Example 1.6.3, the open interval $]a,b[$ belongs to $\mathcal{O}_\mathbb{R}$, We also have that

$$]a,b[= I \cap]a,b[.$$

Thus $]a,b[$ belongs to $\mathcal{O}_I$.

Example 2.3.4. Let $[0,b[$ be an half open interval such that $0 < b < 1$.



Let a be any real number such that $a < 0$. As we observed in Example 1.6.3, the open interval $]a,b[$ belongs to $\mathcal{O}_\mathbb{R}$, We have that

$$[0,b[= I \cap]a,b[.$$

Thus $[0,b[$ belongs to $\mathcal{O}_I$.

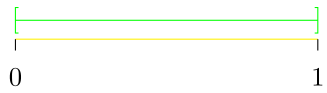
Example 2.3.5. Let $]a,1]$ be an half open interval such that $0 < a < 1$.

Let b be any real number such that $b > 1$. As we observed in Example 1.6.3, the open interval $]a, b[$ belongs to $\mathcal{O}_{\mathbb{R}}$, We have that

$$]a, 1] = I \cap]a, b[\,.$$

Thus $]a, 1]$ belongs to $\mathcal{O}_I$.

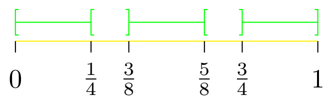
Example 2.3.6. As we proved in Proposition 2.2.3, the set I belongs to $\mathcal{O}_I$.



Example 2.3.7. Disjoint unions of subsets of I of the kind discussed in Example 2.3.3, Example 2.3.4, and Example 2.3.5, belong to $\mathcal{O}_I$. This is a consequence of Proposition 2.2.3, but could also be demonstrated directly. For instance, the set

$$\left[0, \tfrac{1}{4}\right[\cup \left]\tfrac{3}{8}, \tfrac{5}{8}\right[\cup \left]\tfrac{1}{4}, 1\right]$$

belongs to $\mathcal{O}_I$.



E2 Exercises for Lecture 2

E2.1 Exam questions

Task E2.1.1. Decide whether the following subsets of $\mathbb{R}$ are open, closed, both, or neither with respect to $\mathcal{O}_{\mathbb{R}}$.

(1) $]-23, 150[$

(2) $\mathbb{R}$

(3) $[2, 3]$

(4) $\bigcup_{n \in \mathbb{Z}} \left]n - \frac{1}{2}, n + \frac{1}{2}\right[$

(5) $]-\infty, 2]$.

(6) $\bigcup_{n \in \mathbb{N}} \left]\frac{1}{n}, 10\right[$.

(7) $]5, 8[\cup]47, 60]$

(8) $\bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, 1 - \frac{1}{n}\right[$

Task E2.1.2. Give an example to demonstrate that an infinite union of closed subsets of $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$ need not be closed.

Task E2.1.3. Let X be the subset $[1, 2] \cup [4, 5[$ of $\mathbb{R}$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. For each of the following, give an example of a subset U of X which has the required property, and which belongs to $\mathcal{O}_X$.

(1) We have that $U \cap [4, 5[= \emptyset$, and neither 1 nor 2 belongs to U.

(2) We have that $U \cap [1, 2] = \emptyset$, and 4 does not belong to U.

(3) We have that $U \cap [4, 5[= \emptyset$, and 1 belongs to U.

(4) We have that $U \cap [1, 2] = \emptyset$, and 4 belongs to U.

(5) Both 2 and 4 belong to U.

(6) We have that $U \cap [1, 2]$ is not empty, that $U \cap [4, 5[$ is not empty, and that neither 1, 2, nor 4 belongs to U.

E2.2 In the lecture notes

Task E2.2.1. Prove Observation 2.1.6.

Since this task is appealed to in the proof of Proposition 2.1.9, you are not permitted to use that $\mathcal{O}_{\mathbb{R}}$ is a topology on $\mathbb{R}$!

Task E2.2.2. Take a look at the proof of Proposition 2.2.3. Afterwards, cover it up, and try to prove Proposition 2.2.3 for yourself. There is esssentially only one way to do it. Keep working on this until you can manage it.

E2.3 For a deeper understanding

Task E2.3.1. Let $(Y, \mathcal{O}_Y)$ be a topological space. Let X be a subset of Y, and let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(Y, \mathcal{O}_Y)$. Let A be a subset of X. Let $\mathcal{O}_A^X$ denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Let $\mathcal{O}_A^Y$ denote the subspace topology on A with respect to $(Y, \mathcal{O}_Y)$. Prove that $\mathcal{O}_A^X = \mathcal{O}_A^Y$.

Task E2.3.2. Let $(Y, \mathcal{O}_Y)$ be a topological space. Let X be a subset of Y, and let $\mathcal{O}_X$ be the subspace topology on X with respect to $(Y, \mathcal{O}_Y)$. Prove that a subset V of X is closed with respect to $\mathcal{O}_X$ if and only if there is a subset V' of Y with the following properties.

(1) We have that V' is closed with respect to $(Y, \mathcal{O}_Y)$.

(2) We have that $V = X \cap V'$.

Task E2.3.3. Let $(Y, \mathcal{O}_Y)$ be a topological space. Let X be a subset of Y, and let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(Y, \mathcal{O}_Y)$.

(1) Suppose that X belongs to $\mathcal{O}_Y$. Prove that if U belongs to $\mathcal{O}_X$, then U belongs to $\mathcal{O}_Y$.

(2) Does the conclusion of (1) necessarily hold if X does not belong to $\mathcal{O}_Y$?

(3) Suppose that X is closed with respect to $\mathcal{O}_Y$. Let V be a subset of X which is closed with respect to $\mathcal{O}_X$. Prove that V, when viewed as a subset of Y, is closed with respect to $\mathcal{O}_Y$. You may wish to appeal to Task E2.3.2.

(4) Does the conclusion of (3) necessarily hold if X is not closed with respect to $\mathcal{O}_Y$?

Task E2.3.4. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{U_j\}_{j \in J}$ be a set of subsets of X with the property that $X = \bigcup_{j \in J} U_j$. For every $j \in J$, let $\mathcal{O}_{U_j}$ denote the subspace topology on U_j with respect to $(X, \mathcal{O}_X)$. Suppose that U_j belongs to $\mathcal{O}_X$ for every $j \in J$. Let U be a subset of X such that $U \cap U_j$ belongs to $\mathcal{O}_{U_j}$ for every $j \in J$. Prove that U belongs to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) Appealing to Task E2.3.3 (1), observe that $U \cap A_j$ belongs to $\mathcal{O}_X$.

(2) Prove that

$$U = \bigcup_{j \in J} U \cap A_j.$$

For this, you may wish to begin by observing that $U = U \cap X$, and then appeal to one of the assumptions.

Remark E2.3.5. There is an analogous result for closed sets, but an additional hypothesis is required. This is the topic of Task E8.3.8.

Task E2.3.6. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{U_j\}_{j \in J}$ be a set of subsets of X with the property that $X = \bigcup_{j \in J} U_j$. For every $j \in J$, let $\mathcal{O}_{U_j}$ denote the subspace topology on U_j with respect to $(X, \mathcal{O}_X)$. Suppose that U_j belongs to $\mathcal{O}_X$ for every $j \in J$. Let V be a subset of X such that $V \cap U_j$ is closed with respect to $\mathcal{O}_{U_j}$ for every $j \in J$. Prove that V is closed with respect to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) Observe that, since $V \cap U_j$ is closed with respect to $\mathcal{O}_{U_j}$, for every $j \in J$, we have that $U_j \setminus (V \cap U_j)$ belongs to $\mathcal{O}_{U_j}$, for every $j \in J$.

(2) Observe that $U_j \setminus (V \cap U_j) = U_j \cap (X \setminus V)$.

(3) By Task E2.3.4, deduce that $X \setminus V$ belongs to $\mathcal{O}_X$.

Task E2.3.7 (More difficult)**.** Prove that a subset of $\mathbb{R}$ belongs to $\mathcal{O}_{\mathbb{R}}$ if and only if it is a disjoint union of open intervals. For proving that if U belongs to $\mathcal{O}_{\mathbb{R}}$, then it is a disjoint union of open intervals, you may wish to proceed as follows.

(1) Define a relation $\sim$ on U by $a \sim b$ if

$$[\min\{a, b\}, \max\{a, b\}] \subset U.$$

Verify that $\sim$ defines an equivalence relation.

(2) Let

$$U \xrightarrow{q} U/\sim$$

denote the map given by $x \mapsto \langle x \rangle$, where $\langle x \rangle$ denotes the equivalence class of x with respect to $\sim$. By means of Task E1.3.3, prove that, for every $y \in U/\sim$, the subset $q^{-1}(y)$ of U is an interval.

(3) Moreover, appealing to the fact that U belongs to $\mathcal{O}_{\mathbb{R}}$, prove that, for every $y \in U/\sim$, the interval $q^{-1}(y)$ is open.

(4) Verify that, for distinct $y, y' \in U/{\sim}$, the set

$$q^{-1}(y) \cap q^{-1}(y')$$

is empty. Verify that

$$U = \bigcup_{y \in U/{\sim}} q^{-1}(y).$$

Remark E2.3.8. In fact, a subset of $\mathbb{R}$ is open in the standard topology on $\mathbb{R}$ if and only if it is a disjoint union of *countably many* open intervals. This will follow from Task E2.3.7 by a later task.