MA3002 Generell Topologi — Vår 2014

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3 Monday 13th January

3.1 Product topologies

Remark 3.1.1. In this section, we discuss our second 'canonical way' to construct topological spaces.

Definition 3.1.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\mathcal{O}_{X \times Y}$ denote the set of subsets U of $X \times Y$ with the property that, for every $(x, y) \in U$, there is a subset U_X of X and a subset U_Y of Y with the following properties.

(1) We have that $x \in U_X$, and that U_X belongs to $\mathcal{O}_X$.

(2) We have that $y \in U_Y$, and that U_Y belongs to $\mathcal{O}_Y$.

(3) We have that $U_X \times U_Y \subset U$.

Proposition 3.1.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then $(X \times Y, \mathcal{O}_{X \times Y})$ is a topological space.

Proof. We verify that each of the conditions of Definition 1.1.1 holds.

(1) The empty set $\emptyset$ belongs to $\mathcal{O}_{X \times Y}$, since the required property vacuously holds.

(2) Let $(x, y) \in X \times Y$. We have the following.

   (a) Since $\mathcal{O}_X$ is a topology on X, we have that X belongs to $\mathcal{O}_X$. Evidently, $x \in X$.

   (b) Since $\mathcal{O}_Y$ is a topology on Y, we have that Y belongs to $\mathcal{O}_Y$. Evidently, $y \in Y$.

   (c) We have that $X \times Y \subset X \times Y$.

   Taking U_X to be X, and taking U_Y to be Y, we deduce that $X \times Y$ belongs to $\mathcal{O}_{X \times Y}$.

(3) Let $\{U_j\}_{j \in J}$ be a set of subsets of $X \times Y$ which belong to $\mathcal{O}_{X \times Y}$. Let $(x, y) \in \bigcup_{j \in J} U_j$. By definition of $\bigcup_{j \in J} U_j$, there is a $j \in J$ such that $(x, y) \in U_j$.

   By definition of $\mathcal{O}_{X \times Y}$, there is a subset U_X of X and a subset U_Y of Y with the following properties.

   (a) We have that $x \in U_X$, and that U_X belongs to $\mathcal{O}_X$.

   (b) We have that $y \in U_Y$, and that U_Y belongs to $\mathcal{O}_Y$.

(c) We have that $U_X \times U_Y \subset U_j$.

We have that $U_j \subset \bigcup_{j \in J} U_j$. By (c), we deduce that $U_X \times U_Y \subset \bigcup_{j \in J} U_j$. We conclude from the latter, (a), and (b), that $\bigcup_{j \in J} U_j$ belongs to $\mathcal{O}_{X \times Y}$.

(4) Let U_0 and U_1 be subsets of $X \times Y$ which belong to $\mathcal{O}_{X \times Y}$. Let $(x, y) \in U_0 \cap U_1$. By definition of $\mathcal{O}_{X \times Y}$, there is a subset U_0^X of X and a subset U_0^Y of Y with the following properties.

(a) We have that $x \in U_0^X$, and that U_0^X belongs to $\mathcal{O}_X$.

(b) We have that $y \in U_0^Y$, and that U_0^Y belongs to $\mathcal{O}_Y$.

(c) We have that $U_0^X \times U_0^Y \subset U_0$.

Moreover, by definition of $\mathcal{O}_{X \times Y}$, there is a subset U_1^X of X and a subset U_1^Y of Y with the following properties.

(d) We have that $x \in U_1^X$, and that U_1^X belongs to $\mathcal{O}_X$.

(e) We have that $y \in U_1^Y$, and that U_1^Y belongs to $\mathcal{O}_Y$.

(f) We have that $U_1^X \times U_1^Y \subset U_1$.

We deduce the following.

(i) By (a) and (d), we have that $x \in U_0^X \cap U_1^X$. Moreover, since $\mathcal{O}_X$ defines a topology on X, we have by (a) and (d) that $U_0^X \cap U_1^X$ belongs to $\mathcal{O}_X$.

(ii) By (b) and (e), we have that $y \in U_0^Y \cap U_1^Y$. Moreover, since $\mathcal{O}_Y$ defines a topology on Y, we have by (b) and (e) that $U_0^Y \cap U_1^Y$ belongs to $\mathcal{O}_Y$.

(iii) We have that

$$\left(U_0^X \cap U_1^X\right) \times \left(U_0^Y \cap U_1^Y\right) = \left(U_0^X \times U_0^Y\right) \cap \left(U_1^X \times U_1^Y\right).$$

By (c) and (f), we have that

$$\left(U_0^X \times U_0^Y\right) \cap \left(U_1^X \times U_1^Y\right) \subset U_0 \cap U_1.$$

Hence

$$\left(U_0^X \cap U_1^X\right) \times \left(U_0^Y \cap U_1^Y\right) \subset U_0 \cap U_1.$$

Taking U_X to be $U_0^X \cap U_1^X$, and taking U_Y to be $U_0^Y \cap U_1^Y$, we conclude that $U_0 \cap U_1$ belongs to $\mathcal{O}_{X \times Y}$.

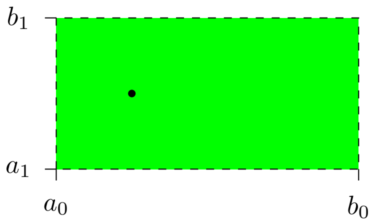
□

Remark 3.1.4. This proof has much in common with the proof of Proposition 2.1.9 and the proof of Proposition 2.2.3. Perhaps you can begin to see how to approach a proof of this kind? Again, it is a very good idea to work on the proof until you thoroughly understand it. This is the topic of Task E3.2.1.

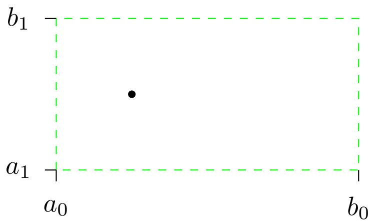
3.2 The product topology on $\mathbb{R}^2$

Definition 3.2.1. Let $\mathcal{O}_{\mathbb{R}^2}$ denote the product topology on $\mathbb{R}^2$ with respect to two copies of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Example 3.2.2. Let $U_0 =]a_0, b_0[$, and let $U_1 =]a_1, b_1[$ be open intervals. Let $(x, y) \in U_0 \times U_1$.



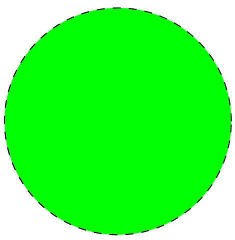
By Example 1.6.3, both U_0 and U_1 belong to $\mathcal{O}_{\mathbb{R}}$. We deduce that $U_0 \times U_1$ belongs to $\mathcal{O}_{\mathbb{R}^2}$, since we can take U_X to be U_0, and can take U_Y to be U_1.



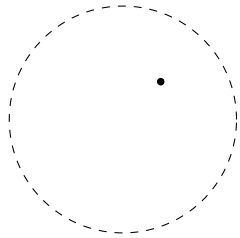
In the figures, the dashed boundary does *not* belong to $U_0 \times U_1$. We shall adopt the same convention in all our figures.

Example 3.2.3. Let U denote the disc

$$\left\{ (x, y) \in \mathbb{R}^2 \mid \|(x, y)\| < 1 \right\}.$$



Let (x, y) be a point of U.



Let ϵ be a real number such that

$$0 < \epsilon < 1 - \|(x, y)\|.$$

Let U_X denote the open interval

$$\left] x - \tfrac{\epsilon\sqrt{2}}{2}, x + \tfrac{\epsilon\sqrt{2}}{2} \right[.$$

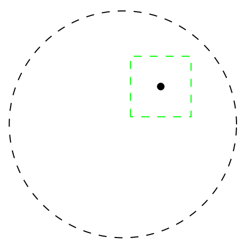
Let U_Y denote the open interval

$$\left] y - \tfrac{\epsilon\sqrt{2}}{2}, y + \tfrac{\epsilon\sqrt{2}}{2} \right[.$$

We have that $x \in U_X$, and that $y \in U_Y$. Let (x', y') be a point of $U_X \times U_Y$. Then

$$\begin{aligned} \|(x', y')\| &= \|(|x'|, |y'|)\| \\ &< \left\| \left(|x| + \tfrac{\epsilon\sqrt{2}}{2}, |y| + \tfrac{\epsilon\sqrt{2}}{2} \right) \right\| \\ &\leq \|(|x|, |y|)\| + \left\| \left(\tfrac{\epsilon\sqrt{2}}{2}, \tfrac{\epsilon\sqrt{2}}{2} \right) \right\| \\ &= \|(x, y)\| + \epsilon \\ &< \|(x, y)\| + (1 - \|(x, y)\|) \\ &= 1. \end{aligned}$$

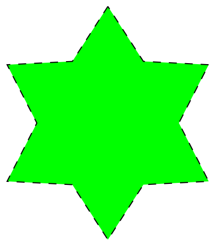
Thus $U_X \times U_Y \subset U$.



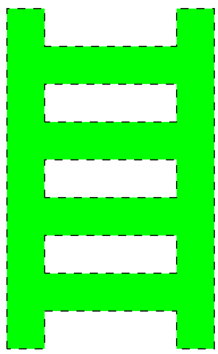
By Example 1.6.3, both U_0 and U_1 belong to $\mathcal{O}_{\mathbb{R}}$. We conclude that U belongs to $\mathcal{O}_{\mathbb{R}^2}$.

Remark 3.2.4. There are very many subsets U of $\mathbb{R}^2$ which belong to $\mathcal{O}_{\mathbb{R}^2}$. We just have to be able to find a small enough 'open rectangle' around every point of U which is contained in U.

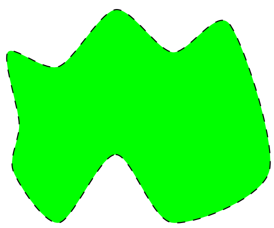
Example 3.2.5. An 'open star' belongs to $\mathcal{O}_{\mathbb{R}^2}$.



Example 3.2.6. An 'open ladder' belongs to $\mathcal{O}_{\mathbb{R}^2}$.



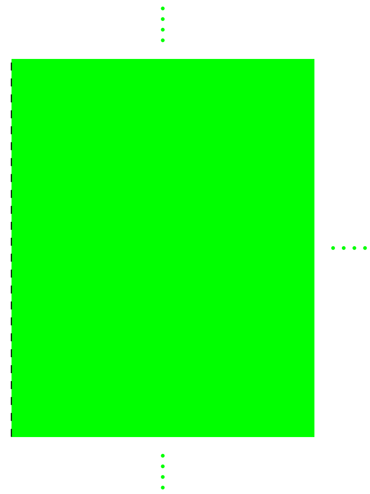
Example 3.2.7. An 'open blob' belongs to $\mathcal{O}_{\mathbb{R}^2}$.



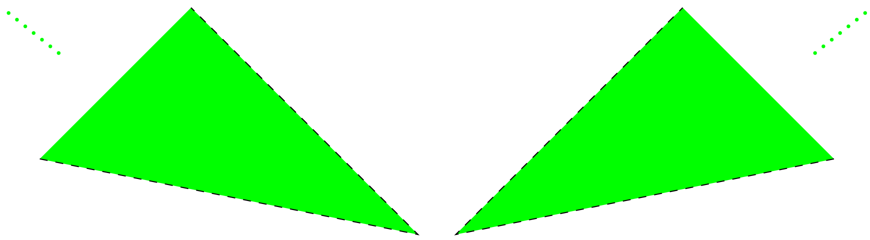
Example 3.2.8. The open half plane given by

$$\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$$

belongs to $\mathcal{O}_{\mathbb{R}^2}$.



Example 3.2.9. The union of two 'open infinite wedges' belongs to $\mathcal{O}_{\mathbb{R}^2}$.

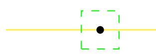


Example 3.2.10. Let X denote the subset of $\mathbb{R}^2$ given by

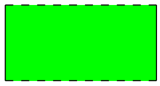
$$\left\{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1 \text{ and } y = 0\right\}.$$

Let x be any point of X.

No matter how small a rectangle we take around x, there will always be a point of $\mathbb{R}^2$ inside this rectangle which does not belong to X. Thus X does not belong to $\mathcal{O}_{\mathbb{R}^2}$.



Example 3.2.11. Let X denote the 'half open strip' given by $[0,1[\times]0,1[$.



The solid part of the boundary of this figure belongs to X. Let (x,y) belong to either of the vertical boundary lines. For example, we can take (x,y) to be $(0,\frac{1}{2})$.

No matter how small a rectangle we take around (x,y), there will always be a point inside this rectangle which does not belong to X. For example, if (x,y) is $(0,\frac{1}{2})$, there will always be a point (x',y') inside this rectangle such that $x'<0$. Thus X does not belong to $\mathcal{O}_{\mathbb{R}^2}$.

E3 Exercises for Lecture 3

E3.1 Exam questions

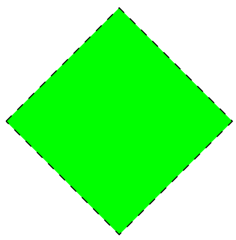
Task E3.1.1. Are the following subsets of $\mathbb{R}^2$ open, closed, both, or neither with respect to the topology $\mathcal{O}_{\mathbb{R}^2}$?

(1) The union of the set

$$\left\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x < 1 \text{ and } |y| < 1-x\right\}$$

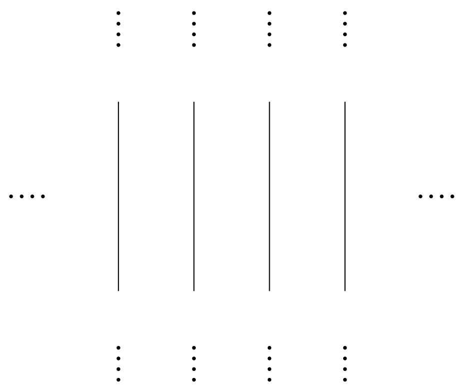
and the set

$$\left\{(x,y) \in \mathbb{R}^2 \mid -1 < x \leq 0 \text{ and } |y| < x+1\right\}.$$

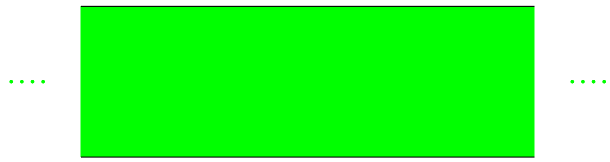


(2) $\bigcup_{n \in \mathbb{Z}} X_n$, where

$$X_n = \left\{(n,y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\right\}.$$



(3) $\mathbb{R} \times [0, 1]$.



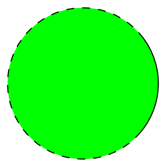
(4) The set consisting of the single point $\{(123, \pi)\}$.

(5) The union of the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid -1 < x < \tfrac{3}{4} \text{ and } \|(x, y)\| < 1\right\}$$

and the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid \tfrac{3}{4} \leq x < 1 \text{ and } \|(x, y)\| \leq 1\right\}.$$



(6) The union of the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid y > 0 \text{ and } \|(x, y)\| < 1\right\}$$

and the set $[3, 5] \times [0, 1]$.

(7) $\bigcup_{n \in \mathbb{N}} \left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| \leq 3 - \frac{1}{n}\right\}$.

(8) The set

$$\left\{(x, 0) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\right\}.$$

E3.2 In the lecture notes

Task E3.2.1. Do the same as in Task E2.2.2 for the proof of Proposition 3.1.3.

E3.3 For a deeper understanding

Task E3.3.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let V_X be a subset of X which is closed with respect to $\mathcal{O}_X$, and let V_Y be a subset of Y which is closed with respect to $(Y, \mathcal{O}_Y)$. Prove that $V_X \times V_Y$ is closed with respect to $(X \times Y, \mathcal{O}_{X \times Y})$.

Task E3.3.2. Let $(X_0, \mathcal{O}_{X_0})$ and $(X_1, \mathcal{O}_{X_1})$ be topological spaces. Let Y_0 be a subset of X_0, and let Y_1 be a subset of X_1.

Let $\mathcal{O}_{Y_0}$ denote the subspace topology on Y_0 with respect to $(X_0, \mathcal{O}_{X_0})$. Let $\mathcal{O}_{Y_1}$ denote the subspace topology on Y_1 with respect to $(X_1, \mathcal{O}_{X_1})$.

Let $\mathcal{O}_{Y_0 \times Y_1}$ denote the product topology on $Y_0 \times Y_1$ with respect to $(Y_0, \mathcal{O}_{Y_0})$ and $(Y_1, \mathcal{O}_{Y_1})$. Let $\mathcal{O}'_{Y_0 \times Y_1}$ denote the subspace topology on $Y_0 \times Y_1$ with respect to $(X_0 \times X_1, \mathcal{O}_{X_0 \times X_1})$.

Prove that $\mathcal{O}_{Y_0 \times Y_1} = \mathcal{O}'_{Y_0 \times Y_1}$.

Task E3.3.3. Let $(X_0, \mathcal{O}_{X_0})$, $(X_1, \mathcal{O}_{X_1})$, and $(X_2, \mathcal{O}_{X_2})$ be topological spaces. Let $\mathcal{O}_{X_0 \times (X_1 \times X_2)}$ denote the product topology on $X_0 \times X_1 \times X_2$ with respect to $(X_0, \mathcal{O}_{X_0})$ and $(X_1 \times X_2, \mathcal{O}_{X_1 \times X_2})$. Let $\mathcal{O}_{(X_0 \times X_1) \times X_2}$ denote the product topology on $X_0 \times X_1 \times X_2$ with respect to $(X_0 \times X_1, \mathcal{O}_{X_0 \times X_1})$ and $(X_2, \mathcal{O}_{X_2})$.

Prove that $\mathcal{O}_{X_0 \times (X_1 \times X_2)} = \mathcal{O}_{(X_0 \times X_1) \times X_2}$.

Notation E3.3.4. We shall denote the topology $\mathcal{O}_{X_0 \times (X_1 \times X_2)} = \mathcal{O}_{(X_0 \times X_1) \times X_2}$ on $X_0 \times X_1 \times X_2$ by $\mathcal{O}_{X_0 \times X_1 \times X_2}$.

Notation E3.3.5. We shall denote by $\mathcal{O}_{\mathbb{R}^3}$ the topology $\mathcal{O}_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}}$ on $\mathbb{R}^3$.

Remark E3.3.6. Let $n \in \mathbb{N}$. For every $1 \leq i \leq n$, let $(X_i, \mathcal{O}_{X_i})$ be a topological space. By induction, it follows from Task E3.3.3 that all the possible ways of equipping $X_1 \times \ldots \times X_n$ with a topology, using only the topologies $\mathcal{O}_{X_i}$, for $1 \leq i \leq n$, and product topologies built from these, coincide.

Notation E3.3.7. We shall denote this topology on $X_1 \times \ldots X_n$ by $\mathcal{O}_{X_1 \times \ldots \times X_n}$.

Notation E3.3.8. We shall denote by $\mathcal{O}_{\mathbb{R}^n}$ the topology $\mathcal{O}_{\underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n}}$ on $\mathbb{R}^n$.

E3.4 Exploration — metric spaces

Remark E3.4.1. Some of you may have met the notion of a metric before, for instance in TMA4145 Lineære Metoder. Don't worry if not, all material on metric spaces below, and in future exercises, will not be examined. Certainly I recommend to focus on the topics covered in the lectures, before looking into any of the exercises on metric spaces.

Nevertheless, those of you who are comfortable with the lectures may find the exercises on metric spaces interesting, and useful in future courses. Though it will not be necessary, you are welcome to make use of any of the exercises on metric spaces in the exam wherever there is an opportunity for this.

Definition E3.4.2. Let X be a set. A *metric* on X is a map

$$X \times X \xrightarrow{d} [0, \infty[$$

such that the following hold.

(1) For every x which belongs to X, we have that $d(x, x) = 0$.

(2) For all x_0, x_1, and x_2 which belong to X, we have that

$$d(x_0, x_1) + d(x_1, x_2) \geq d(x_0, x_2).$$

Remark E3.4.3. The condition of Definition E3.4.2 is known as the *triangle inequality.*

Remark E3.4.4. If you have seen the definition of a metric in a previous course, a couple of additional conditions were probably required to be satisfied. For many purposes, these are not needed. In particular, we shall not need them in this section.

Definition E3.4.5. A *metric space* is a pair (X, d) of a set X and a metric d on X.

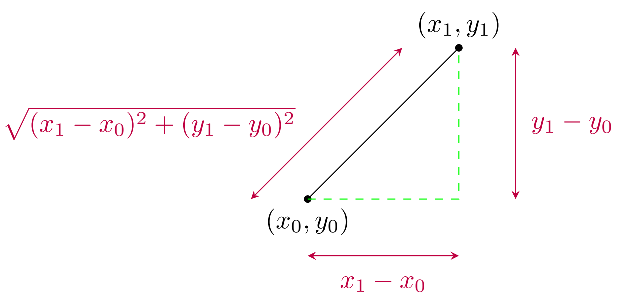
Notation E3.4.6. Let

$$\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{d_{\mathbb{R}^n}} \mathbb{R}^n$$

denote the map given by

$$((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto \sqrt{(y_1 - x_1)^2 + \ldots + (y_n - x_n)^2}.$$

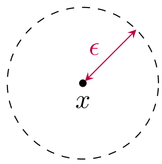
In other words, $d_{\mathbb{R}^n}$ is the usual notion of distance between a pair of points in $\mathbb{R}^n$.



Task E3.4.7. Prove that $d_{\mathbb{R}^n}$ defines a metric on $\mathbb{R}^n$, or look up a proof from an earlier course.

Definition E3.4.8. Let (X, d) be a metric space. Let $x \in X$, and let $\epsilon > 0$ be a real number. The *open ball of radius ϵ around x* is the set $B_\epsilon(x)$ given by

$$\left\{ x' \in X \mid d(x, x') < \epsilon \right\}.$$



Task E3.4.9. Let (X, d) be a metric space. Let $\mathcal{O}_d$ denote the set of subsets U of X with the property that, for every $x \in U$, there is a real number $\epsilon > 0$ such that $B_\epsilon(x)$ is a subset of U. Prove that $\mathcal{O}_d$ defines a topology on X.

Remark E3.4.10. We shall take the point of view that a metric is a way to construct a topology. Once we have constructed this topology, we can forget about the metric from whence it came!

All the topological spaces that we shall be interested in can be constructed without using a metric. For this reason, metrics will never appear in the lectures.

A characteristic feature of topology, as opposed to *geometry*, is that we shall often be manipulating topological spaces in ways which change the distance between pairs of points: squashing and stretching, for instance.

Nevertheless, there are many important areas of mathematics, such as *differential geometry*, which merge both topological and geometrical ideas. Here one sometimes emphasises a construction which relies on a metric, sometimes emphasises a purely topological construction, and often investigates the interplay between both worlds. The courses TMA4190 Mangfoldigheter and MA8402 Lie-Grupper og Lie-Algebraer can lead in this direction.

Remark E3.4.11. Though metrics will never appear in the lectures, many of the concepts that we shall look at for arbitrary topological spaces can be thought of in other, equivalent, ways for topologies coming from a metric. We shall explore this in future exercises.

Task E3.4.12. Let $n \geq 1$. Prove that $\mathcal{O}_{d_{\mathbb{R}^n}} = \mathcal{O}_{\mathbb{R}^n}$.