MA3002 Generell Topologi — Vår 2014

Richard Williamson

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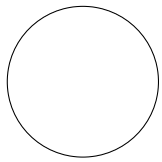
4.1 Examples of product and subspace topologies

Remark 4.1.1. We can combine our two 'canonical' ways of constructing new topological spaces from old ones to obtain many interesting examples of topological spaces.

Notation 4.1.2. We denote by S^1 the set

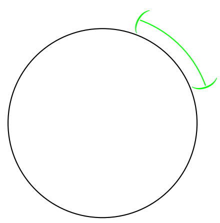
$$\left\{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\| = 1\right\}.$$



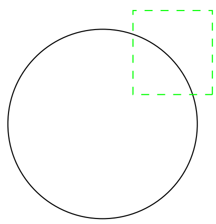
We denote by $\mathcal{O}_{S^1}$ the subspace topology on S^1 with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Terminology 4.1.3. We refer to S^1 as the *circle*.

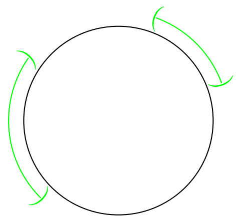
Example 4.1.4. By definition, a subset of S^1 belongs to $\mathcal{O}_{S^1}$ if and only if it is the intersection with S^1 of a subset of $\mathbb{R}^2$ which belongs to $\mathcal{O}_{\mathbb{R}^2}$. The generic example is an 'open arc'.



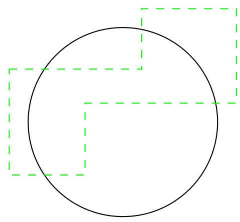
This is, for instance, the intersection with S^1 of an 'open rectangle' in $\mathbb{R}^2$, which belongs to $\mathcal{O}_{\mathbb{R}^2}$ by Example 3.2.2.



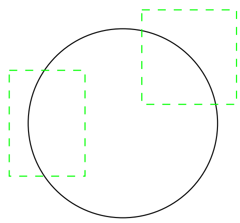
Since $\mathcal{O}_{S^1}$ defines a topology on S^1, we also have that disjoint unions of (possibly infinitely many) 'open arcs' belong to $\mathcal{O}_{S^1}$.



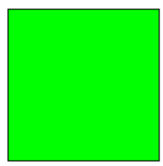
This can also be demonstrated directly. The subset of S^1 given by the two 'open arcs' in the previous picture is, for instance, the intersection with S^1 with the subset of $\mathbb{R}^2$ depicted below, which belongs to $\mathcal{O}_{\mathbb{R}^2}$.



Alternatively, it is the intersection with S^1 with the subset of $\mathbb{R}^2$ consisting of two disjoint 'open rectangles', depicted below.



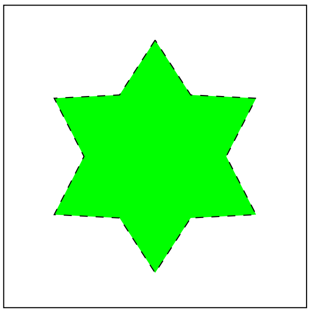
Notation 4.1.5. We denote by $\mathcal{O}_{I^2}$ the product topology on I^2 with respect to two copies of $(I, \mathcal{O}_I)$.



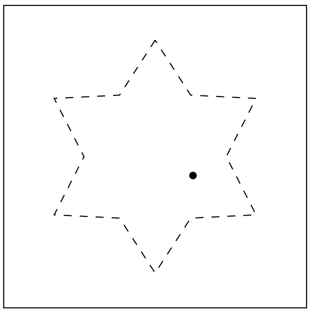
Terminology 4.1.6. We refer to I^2 as the *unit square*.

Remark 4.1.7. The topology $\mathcal{O}_{I^2}$ coincides with the subspace topology on I^2 with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. To prove this is the topic of Task E3.3.2.

Example 4.1.8. Any of the open sets pictured in Examples 3.2.2 – 3.2.3 and 3.2.5 – 3.2.7 which 'fit inside I^2' belong to I^2. For instance, an 'open star'.

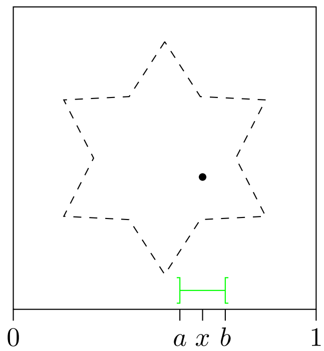


To see this, let (x, y) be a point of a subset U of I^2 of this kind.

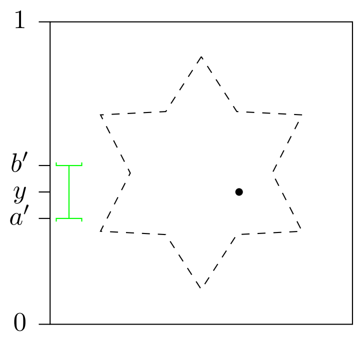


We have the following.

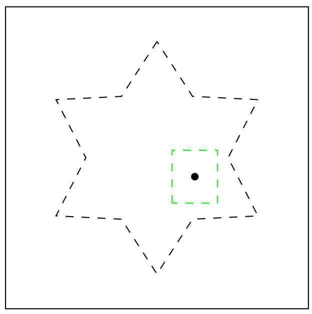
(1) We can find an open interval $U_X =]a, b[$ such that $0 < a < b < 1$ and $x \in U_X$.



(2) We can find an open interval $U_Y = \,]a', b'[$ such that $0 < a' < b' < 1$ and $y \in U_Y$.



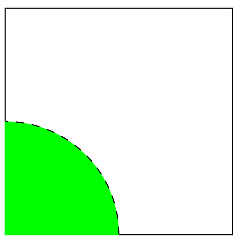
(3) We have that $U_X \times U_Y \subset U$.



As we observed in Example 2.3.3, both U_X and U_Y belong to $\mathcal{O}_I$. Thus (1) – (3) together demonstrate that U belongs to $\mathcal{O}_{I^2}$.

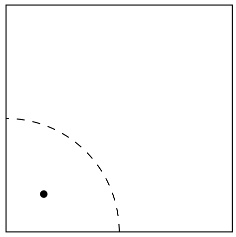
Example 4.1.9. Let U be the subset of I^2 given by

$$\left\{ (x, y) \in I^2 \mid |(x, y)| < \tfrac{1}{2} \right\}.$$

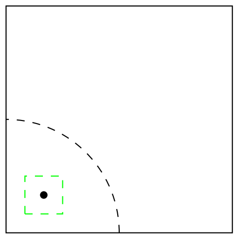


We have the following.

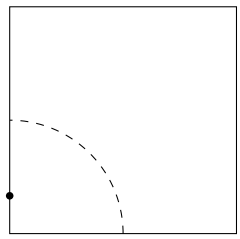
(1) Let (x, y) be a point of U which does not lie on the boundary of I^2.



As in Example 4.1.8, we can find an 'open rectangle' around (x, y) which is a subset of U.



(2) Let (x, y) be a point of U with $x = 0$.



Let ϵ be a real number such that

$$0 < \epsilon < \tfrac{1}{2} - y.$$

Let U_X denote the half open interval

$$\left[0, \tfrac{\epsilon\sqrt{2}}{2}\right[.$$

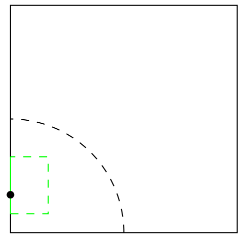
Let U_Y denote the open interval

$$\left]y - \tfrac{\epsilon\sqrt{2}}{2}, y + \tfrac{\epsilon\sqrt{2}}{2}\right[.$$

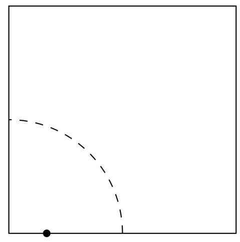
We have that $(0, y) \in U_X \times U_Y$. As we saw in Example 2.3.4, we have that U_X belongs to $\mathcal{O}_I$. As we saw in Example 2.3.3, we have that U_Y belongs to $\mathcal{O}_I$. Moreover, let (x', y') be a point of $U_X \times U_Y$. Arguing as in Example 3.2.3, we have that

$$\|(x', y')\| < \tfrac{1}{2}.$$

Thus $U_X \times U_Y \subset U$.



(3) Let (x, y) be a point of U with $y = 0$.



Let ϵ be a real number such that

$$0 < \epsilon < \tfrac{1}{2} - x.$$

Let U_X denote the open interval

$$\left] x - \tfrac{\epsilon\sqrt{2}}{2}, x + \tfrac{\epsilon\sqrt{2}}{2} \right[.$$

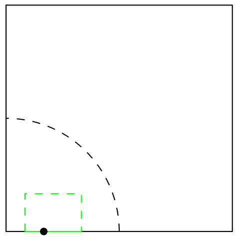
Let U_Y denote the half open interval

$$\left[0, \tfrac{\epsilon\sqrt{2}}{2} \right[.$$

We have that $(x, 0) \in U_X \times U_Y$. As we saw in Example 2.3.3, U_X belongs to $\mathcal{O}_I$. As we saw in Example 2.3.4, U_Y belongs to $\mathcal{O}_I$. Moreover, let (x', y') be a point of $U_X \times U_Y$. Arguing as in Example 3.2.3, we have that

$$\|(x', y')\| < \tfrac{1}{2}.$$

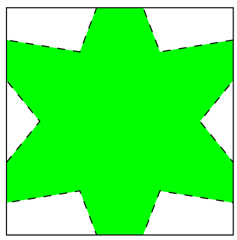
Thus $U_X \times U_Y \subset U$.



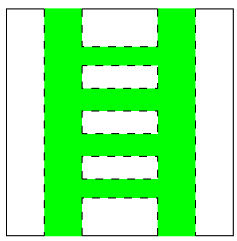
We conclude that U belongs to $\mathcal{O}_{I^2}$.

Remark 4.1.10. Many more subsets of I^2 with 'segments on the boundary' belong to $\mathcal{O}_{I^2}$.

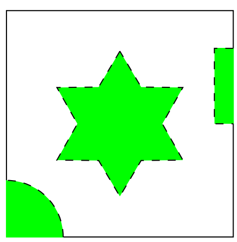
Example 4.1.11. A 'truncated star' belongs to $\mathcal{O}_{I^2}$.



Example 4.1.12. A 'half open ladder' belongs to $\mathcal{O}_{I^2}$.



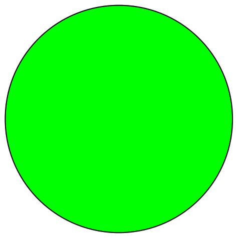
Example 4.1.13. The following subset of I^2 belongs to $\mathcal{O}_{I^2}$.



Remark 4.1.14. We now introduce a few more important examples of product and subspace topologies. Exploring them is the topic of Tasks E4.1.3 – E4.1.6.

Notation 4.1.15. Let D^2 denote the set

$$\left\{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\| \leq 1\right\}.$$

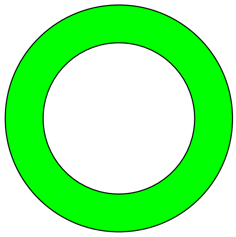


We denote by $\mathcal{O}_{D^2}$ the subspace topology on D^2 with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Terminology 4.1.16. We refer to $(D^2, \mathcal{O}_{D^2})$ as the *unit disc*.

Notation 4.1.17. Let k be a real number such that $0 < k < 1$. Let A_k denote the set

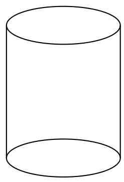
$$\left\{(x,y) \in \mathbb{R}^2 \mid k \leq \|(x,y)\| \leq 1\right\}.$$



We denote by $\mathcal{O}_{A_k}$ the subspace topology on A_k with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Terminology 4.1.18. We refer to $(A_k, \mathcal{O}_{A_k})$ as an *annulus*.

Notation 4.1.19. We denote by $\mathcal{O}_{S^1 \times I}$ the product topology on $S^1 \times I$ with respect to $(S^1, \mathcal{O}_{S^1})$ and $(I, \mathcal{O}_I)$.



This cylinder is hollow!

Terminology 4.1.20. We refer to $(S^1 \times I, \mathcal{O}_{S^1 \times I})$ as the *cylinder*.

4.2 Definition of a continuous map

Notation 4.2.1. Let X and Y be sets. Let

$$X \xrightarrow{f} Y$$

be a map. Let U be a subset of Y. We denote by $f^{-1}(U)$ the set

$$\{x \in X \mid f(x) \in U\}.$$

Terminology 4.2.2. We refer to $f^{-1}(U)$ as the *inverse image of* U *under* f.

Definition 4.2.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is *continuous* if, for every $U \in \mathcal{O}_Y$, the subset $f^{-1}(U)$ of X belongs to $\mathcal{O}_X$.

Remark 4.2.4. A map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous with respect to the standard topology on both copies of $\mathbb{R}$ if and only if it is continuous in the $\epsilon - \delta$ sense that you have met in earlier courses. To prove this is the topic of Task E4.2.9.

4.3 Examples of continuous maps between finite topological spaces

Example 4.3.1. Let X be a set with two elements $\{a, b\}$. Let $\mathcal{O}_X$ denote the topology on X given by

$$\{\emptyset, \{b\}, X\}.$$

In other words, $(X, \mathcal{O}_X)$ is the Sierpiński interval. Let Y denote the set with three elements $\{a', b', c'\}$. Let $\mathcal{O}_Y$ denote the topology on Y given by

$$\big\{\emptyset, \{a'\}, \{c'\}, \{a', c'\}, \{b', c'\}, Y\big\}.$$

Let

$$X \xrightarrow{f} Y$$

denote the map given by $a \mapsto b'$ and $b \mapsto c'$. We have the following.

(1) $f^{-1}(\emptyset) = \emptyset$.

(2) $f^{-1}(\{a'\}) = \emptyset$.

(3) $f^{-1}(\{c'\}) = \{b\}$.

(4) $f^{-1}(\{a', c'\}) = \{b\}$.

(5) $f^{-1}(\{b', c'\}) = X$.

(6) $f^{-1}(Y) = X$.

We see that $f^{-1}(U) \in \mathcal{O}_X$ for every $U \in \mathcal{O}_Y$. Thus f is continuous.

Example 4.3.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be as in Example 4.3.1. Let

$$Y \xrightarrow{g} X$$

denote the map given by $a' \mapsto a$, $b' \mapsto b$, and $c' \mapsto a$. We have that

$$g^{-1}(\{b\}) = \{b'\}.$$

Thus g is not continuous, since $\{b\}$ belongs to $\mathcal{O}_X$, but $\{b'\}$ does not belong to $\mathcal{O}_Y$.

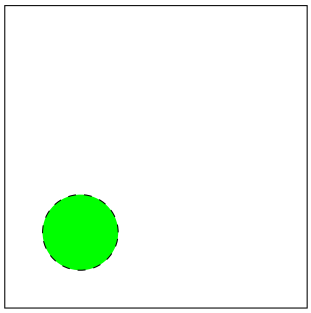
E4 Exercises for Lecture 4

E4.1 Exam questions

Task E4.1.1. Are the following subsets of I^2 open, closed, both, or neither, with respect to $\mathcal{O}_{I^2}$?

(1) The disc given by

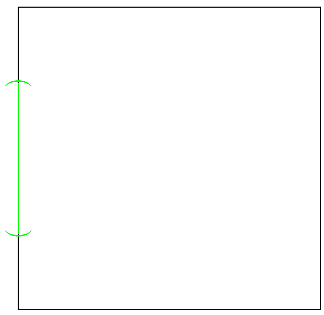
$$\left\{(x,y) \in \mathbb{R}^2 \mid \left\|(x - \tfrac{1}{4}, y - \tfrac{1}{4})\right\| < \tfrac{1}{8}\right\}.$$



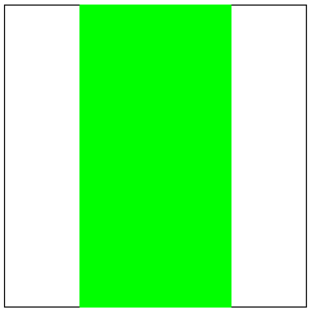
Can you justify your answer rigorously?

(2) The set

$$\left\{(0,y) \in I^2 \mid \tfrac{1}{4} < y < \tfrac{3}{4}\right\}.$$



(3) $\left[\frac{1}{4}, \frac{3}{4}\right] \times I$.

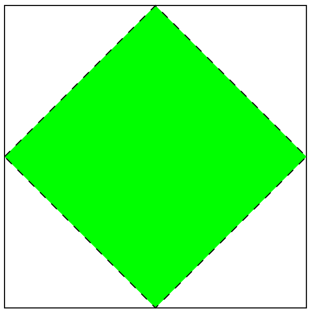


(4) The union of the set

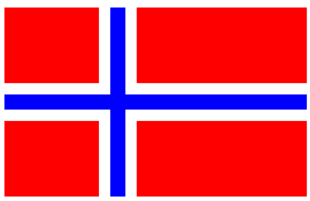
$$\left\{(x, y) \in I^2 \mid 0 < x \leq \tfrac{1}{2} \text{ and } |y| < 2x\right\}$$

and the set

$$\left\{(x, y) \in I^2 \mid \tfrac{1}{2} \leq x < 1 \text{ and } |y| < 2 - 2x\right\}.$$



Task E4.1.2. Let X denote the subset of $\mathbb{R}^2$ consisting of the red and blue parts of the flag below.



Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. For each of the following, draw an example of a subset U of X which has the required property, and which belongs to $\mathcal{O}_X$. Use dashes to indicate which parts of the boundary U in your picture are not to be thought of as belonging to U.

(1) U intersects none of the rectangles except the upper right red rectangle; and U does not intersect the boundary of this rectangle.

(2) U intersects all four red rectangles and both of the blue rectangles; but U does not intersect the boundary of X.

(3) U intersects both of the blue rectangles; but U does not intersect any of the red rectangles.

(4) U intersects only the horizontal blue rectangle, the upper left red rectangle, and the lower left red rectangle; U contains a segment of the border of both the upper left red rectangle and the lower left red rectangle; but U does not contain the entirety of either of the upper left red rectangle or the lower left red rectangle.

(5) U intersects only the vertical blue rectangle and the two upper red rectangles; U contains a segment on all four sides of both of the two upper red rectangles; but U does not contain the entirety of either of the upper red rectangles.

Task E4.1.3. For each of the following, give an example of a subset U of the unit disc D^2 which has the required property.

(1) U belongs to $\mathcal{O}_{D^2}$ but, when viewed as a subset of $\mathbb{R}^2$, does not belong to $\mathcal{O}_{\mathbb{R}^2}$.

(2) U belongs to $\mathcal{O}_{D^2}$ and, when viewed as a subset of $\mathbb{R}^2$, also belongs to $\mathcal{O}_{\mathbb{R}^2}$.

(3) U does not belong to $\mathcal{O}_{D^2}$ and, when viewed as a subset of $\mathbb{R}^2$, also does not belong to $\mathcal{O}_{\mathbb{R}^2}$.

(4) U is closed with respect to $\mathcal{O}_{D^2}$ but, when viewed as a subset of $\mathbb{R}^2$, is not closed with respect to $\mathcal{O}_{\mathbb{R}^2}$.

(5) U belongs to $\mathcal{O}_{D^2}$ and, when viewed as a subset of $\mathbb{R}^2$, is closed with respect to $\mathcal{O}_{\mathbb{R}^2}$.

Task E4.1.4. Draw the subset U of the annulus $A_{\frac{1}{2}}$ given by

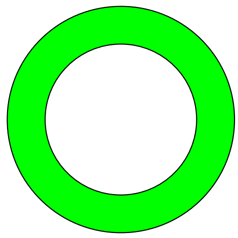
$$\left\{(x, y) \in A_{\frac{1}{2}} \mid \tfrac{1}{4} < x \leq 1\right\}.$$

Does U belong to $\mathcal{O}_{A_{\frac{1}{2}}}$? Does the set V given by

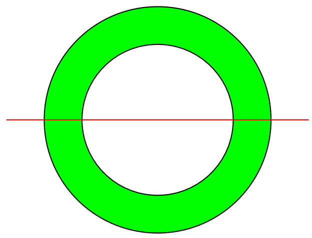
$$\left\{(x, y) \in A_{\frac{1}{2}} \mid \tfrac{1}{4} \leq x \leq 1\right\}$$

belong to $\mathcal{O}_{A_{\frac{1}{2}}}$?

Task E4.1.5. Let $(A_k, \mathcal{O}_{A_k})$ be an annulus.



The horizontal line depicted below is a segment of the x-axis in $\mathbb{R}^2$.



For each of the following, give and draw an example of a subset U of A_k which has the required property, and which belongs to $\mathcal{O}_{A_k}$.

(1) U contains a segment of the inner circle which is above the horizontal line, and does not contain a segment of the inner circle which is below the horizontal line; U contains a segment of the outer circle which is below the horizontal line, and does not contain a segment of the outer circle which is above the horizontal line.

(2) U contains a segment of the inner circle which is above the horizontal line, and its reflection in the horizontal line; U does not contain any segment of the outer circle.

(3) U contains the entire outer circle, but does not contain any point of the inner circle.

(4) U contains neither a segment of the inner circle, nor a segment of the outer circle.

Task E4.1.6. Draw the following subsets U of the cylinder $S^1 \times I$, and decide whether or not they belong to $\mathcal{O}_{S^1 \times I}$.

(1) $S^1 \times \{1\}$.

(2) $U \times \{0\}$, where U is the subset of S^1 given by

$$\left\{(x, y) \in S^1 \mid -\tfrac{1}{4} < y < \tfrac{1}{4}\right\}.$$

(3) $U \times \left]\tfrac{1}{4}, \tfrac{1}{2}\right[$, where U is the subset of S^1 given in (2).

(4) $\{(0,1)\} \times I$.

(5) $S^1 \times \left]\frac{3}{4}, 1\right]$.

(6) $\left(U_0 \times \left]\frac{1}{4}, \frac{1}{2}\right[\right) \cup \left(U_1 \times \left]\frac{1}{2}, \frac{3}{4}\right[\right)$, where U_0 is the subset of S^1 given by

$$\left\{(x,y) \in S^1 \mid \tfrac{1}{4} \leq x < \tfrac{1}{2}\right\},$$

and U_1 is the subset of S^1 given by

$$\left\{(x,y) \in S^1 \mid -\tfrac{1}{2} < x < -\tfrac{1}{4}\right\}.$$

(7) $(U_0 \times I) \cup \left(U_1 \times \left]\frac{1}{2}, \frac{3}{4}\right[\right)$, where U_0 is the subset of S^1 given by

$$\left\{(x,y) \in S^1 \mid \tfrac{1}{8} < x < \tfrac{1}{4}\right\}$$

and U_1 is the subset of S^1 given in (6).

Task E4.1.7. Let X be the set $\{a, b, c\}$. Let $\mathcal{O}_X$ denote the topology on X given by

$$\{\emptyset, \{b\}, \{a,b\}, \{b,c\}, X\}.$$

Let Y be the set $\{a', b', c', d', e'\}$. Let $\mathcal{O}_Y$ denote the topology on Y given by

$$\left\{\emptyset, \{a'\}, \{e'\}, \{a', e'\}, \{b', c'\}, \{a', b', c'\}, \{b', c', e'\}, \{a', b', c', e'\}, \{b', c', d', e'\}, Y\right\}.$$

Which of the following maps

$$X \xrightarrow{f} Y$$

are continuous?

(1) $a \mapsto d'$, $b \mapsto e'$, $c \mapsto d'$.

(2) $a \mapsto e'$, $b \mapsto e'$, $c \mapsto c'$.

(3) $a \mapsto c'$, $b \mapsto a'$, $c \mapsto d'$.

(4) $a \mapsto b'$, $b \mapsto c'$, $c \mapsto d'$.

Remark E4.1.8. It may save you some work to appeal to Task E4.2.5.

E4.2 For a deeper understanding

Definition E4.2.1. Let $(X, \mathcal{O})$ be a topological space. Let $\mathcal{B}$ be a set of subsets of X which belong to $\mathcal{O}$. Then $\mathcal{B}$ is a *basis* for $(X, \mathcal{O})$ if, for every subset of U of X which belongs to $\mathcal{O}$, there is a set $\{U_j\}_{j \in J}$ of (possibly infinitely many) subsets of X which belong to $\mathcal{B}$ such that $U = \bigcup_{j \in J} U_j$.

Task E4.2.2. Let $\mathcal{B}$ denote the set of open intervals. Prove that $\mathcal{B}$ is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Task E4.2.3. Let

$$\mathcal{B} = \{]x - \epsilon, x + \epsilon[\mid x, \epsilon \in \mathbb{R} \text{ and } \epsilon > 0\}.$$

Prove that $\mathcal{B}$ is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Remark E4.2.4. You may find it a little difficult at first to find the idea needed to accomplish Tasks E4.2.2 and E4.2.3. Don't worry if so, feel free to ask me about it. The idea will be used in different forms several times in the course.

Task E4.2.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\mathcal{B}$ be a basis for $(Y, \mathcal{O}_Y)$. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if $f^{-1}(U)$ belongs to $\mathcal{O}_X$ for every subset U of Y which belongs to $\mathcal{B}$.

Corollary E4.2.6. Let $(X, \mathcal{O}_X)$ be a topological space. A map

$$X \xrightarrow{f} \mathbb{R}$$

is continuous with respect to the standard topology $\mathcal{O}_{\mathbb{R}}$ on $\mathbb{R}$ if and only if $f^{-1}(]a, b[)$ belongs to $\mathcal{O}_X$, for every open interval $]a, b[$.

Proof. Follows immediately from Task E4.2.2 and Task E4.2.5. □

Definition E4.2.7. A map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

is *continuous in the ϵ-δ sense* if, for all $x, c, \epsilon \in \mathbb{R}$ with $\epsilon > 0$, there is a $\delta \in \mathbb{R}$ with $\delta > 0$ such that, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Remark E4.2.8. This is the notion of a continuous map that you have met in earlier courses.

Task E4.2.9. Prove that a map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous with respect to the standard topology $\mathcal{O}_{\mathbb{R}}$ on both copies of $\mathbb{R}$ if and only if it is continuous in the $\epsilon - \delta$ sense. You may find it helpful to appeal to Task E4.2.3 and to Task E4.2.5.

Definition E4.2.10. Let $(X, \mathcal{O})$ be a topological space. Let $\mathcal{S}$ be a set of subsets of X which belong to $\mathcal{O}$. Let $\mathcal{B}$ denote the set of subsets U of X such that

$$U = \bigcap_{j \in J} U_j,$$

for a set $\{U_j\}_{j \in J}$ of subsets of X which belong to $\mathcal{S}$, where J is finite. Then $\mathcal{S}$ is a *subbasis* for $(X, \mathcal{O})$ if $\mathcal{B}$ is a basis for $(X, \mathcal{O})$.

Task E4.2.11. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\mathcal{S}$ be a subbasis for $(Y, \mathcal{O}_Y)$. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if $f^{-1}(U)$ belongs to $\mathcal{O}_X$ for every subset U of Y which belongs to $\mathcal{S}$. You may wish to appeal to Task E4.2.5.

Task E4.2.12. Let $\mathcal{S}$ denote the union of the set

$$\{]-\infty, x[\mid x \in \mathbb{R}\}$$

and the set

$$\{]x, \infty[\mid x \in \mathbb{R}\}.$$

Prove that $\mathcal{S}$ is a subbasis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. You may wish to appeal to Task E4.2.2.

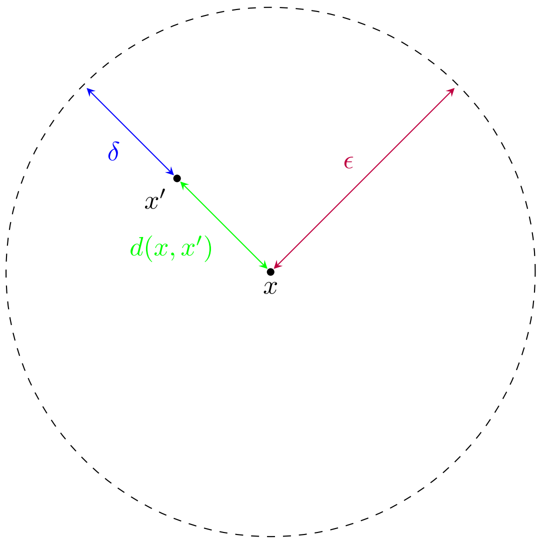
E4.3 Exploration — continuity for metric spaces

Definition E4.3.1. Let (X, d_X) and (Y, d_Y) be metric spaces. A map

$$X \xrightarrow{f} Y$$

is *continuous in the metric sense* if, for all $x \in X$, and all $\epsilon \in \mathbb{R}$ with $\epsilon > 0$, there is a $\delta \in \mathbb{R}$ with $\delta > 0$ such that $f(B_\delta(x))$ is a subset of $B_\epsilon(f(x))$.

Task E4.3.2. Let (X, d) be a metric space. Prove that for any x which belongs to X, any $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$, and any x' which belongs to $B_\epsilon(x)$, there is a $\delta \in \mathbb{R}$ such that $\delta > 0$, and such that $B_\delta(x')$ is a subset of $B_\epsilon(x)$. You may wish to let δ be $\epsilon - d(x, x')$.



You may then wish to observe that, for every x'' which belongs to $B_\delta(x)$, the following holds, by definition of d.

$$\begin{aligned}
d(x, x'') &\leq d(x, x') + d(x', x'') \\
&< d(x, x') + \delta \\
&= d(x, x') + \epsilon - d(x, x') \\
&= \epsilon.
\end{aligned}$$

Task E4.3.3. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\mathcal{O}_{d_X}$ be the topology on X corresponding to d_X of Task E3.4.9, and let $\mathcal{O}_{d_Y}$ be the topology on Y corresponding to d_Y. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if it is continuous in the metric sense. You may wish to proceed as follows.

(1) Suppose that f is continuous in the metric sense. Suppose that U belongs to $\mathcal{O}_{d_Y}$. Suppose that x belongs to $f^{-1}(U)$. By definition of $\mathcal{O}_{d_Y}$, observe that there is an $\epsilon \in \mathbb{R}$ with $\epsilon > 0$ such that $B_\epsilon(f(x))$ is a subset of U.

(2) Since f is continuous in the metric sense, there is a $\delta \in \mathbb{R}$ with $\delta > 0$ such that $f(B_\delta(x))$ is a subset of $B_\epsilon(f(x))$. Deduce that $f(B_\delta(x))$ is a subset of U, and thus that $B_\delta(x)$ is a subset of $f^{-1}(U)$.

(3) By definition of $\mathcal{O}_{d_X}$, we have that $B_\delta(x)$ belongs to $\mathcal{O}_{d_X}$.

(4) By Task E8.3.1, deduce from (2) and (3) that f is continuous.

(5) Suppose instead that f is continuous. Let $\epsilon \in \mathbb{R}$ be such that $\epsilon > 0$. Suppose that x belongs to X. By Task E4.3.2, for every y which belongs to $B_\epsilon(f(x))$, there is a $\zeta \in \mathbb{R}$ with $\zeta > 0$ such that $B_\zeta(y)$ is a subset of $B_\epsilon(f(x))$. By definition of $\mathcal{O}_{d_Y}$, we have that $B_\zeta(y)$ belongs to $\mathcal{O}_{d_Y}$. By Task E8.3.1, deduce that $B_\epsilon(f(x))$ belongs to $\mathcal{O}_{d_Y}$.

(6) Since f is continuous, deduce that $f^{-1}(B_\epsilon(f(x)))$ belongs to $\mathcal{O}_{d_X}$.

(7) By definition of $\mathcal{O}_{d_X}$, deduce that there is a $\delta \in \mathbb{R}$ with $\delta > 0$ such that $B_\delta(x)$ is a subset of $f^{-1}(B_\epsilon(f(x)))$.

(8) Deduce that $f(B_\delta(x))$ is a subset of $B_\epsilon(f(x))$. Conclude that f is continuous in the metric sense.

Definition E4.3.4. Let X be a set. A metric d on X is *symmetric* if, for all x_0 and x_1 which belong to X, we have that $d(x_0, x_1) = d(x_1, x_0)$.

Definition E4.3.5. A metric space (X, d) is *symmetric* if d is symmetric.

Definition E4.3.6. Let (X, d) be a metric space. Let A_0 and A_1 be subsets of X. The *distance from* A_0 *to* A_1 with respect to d is

$$\inf \{d(x_0, x_1) \mid x_0 \in A_0 \text{ and } x_1 \in A_1\}.$$

Notation E4.3.7. Let (X, d) be a metric space. Let A_0 and A_1 be subsets of X. We denote the distance from A_0 to A_1 with respect to d by $d(A_0, A_1)$. Suppose that x belongs to X, and that A is a subset of X. We shall denote $d(\{x\}, A)$ simply by $d(x, A)$.

Remark E4.3.1. Let (X, d) be a symmetric metric space. Let A be a subset of X. Suppose that a belongs to A. By (1) of Definition E3.4.2, we have that $d(a, A) = 0$.

Task E4.3.8. Let (X, d) be a symmetric metric space. Let A be a subset of X. Suppose that x belongs to X. Let X be equipped with the topology $\mathcal{O}_d$ corresponding to d of Task E3.4.9. Prove that the map

$$X \xrightarrow{d(-,A)} \mathbb{R}$$

given by $x \mapsto d(x, A)$ is continuous. You may wish to proceed as follows.

(1) By Task E3.4.12, we have that $\mathcal{O}_\mathbb{R} = \mathcal{O}_{d_\mathbb{R}}$. By Task E4.3.3, it therefore suffices to demonstrate that $d(-, A)$ is continuous in the metric sense.

(2) Suppose that a belongs to A. By definition of d, we have that $d(y, a) \leq d(y, x) + d(x, a)$. Since $d(y, A) \leq d(y, a)$, we deduce that $d(y, A) \leq d(y, x) + d(x, a)$.

(3) Deduce that $d(x, a) \geq d(y, A) - d(y, x)$. Since this inequality holds for all a which belong to A, deduce that $d(x, A) \geq d(y, A) - d(y, x)$. Deduce that $d(y, A) \leq d(x, A) + d(y, x)$.

(4) Carrying out exactly the same argument, but swapping x and y, observe that $d(y, A) \geq d(x, A) - d(x, y)$.

(5) Let $\epsilon \in \mathbb{R}$ be such that $\epsilon > 0$. Suppose that $d(x, y) < \epsilon$. Deduce from (3), (4), and the fact that d is symmetric, that

$$d(x, A) - \epsilon \leq d(y, A) \leq d(x, A) + \epsilon.$$

(6) Deduce from (5) that $d\left(B_\epsilon(x), A\right)$ is a subset of $B_\epsilon\left(d(x, A)\right)$. Conclude that $d(-, A)$ is continuous in the metric sense, as required.