MA3002 Generell Topologi — Vår 2014

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5 Monday 20th January

5.1 Geometric examples of continuous maps

Remark 5.1.1. Most of our continuous maps between geometric examples of topological spaces will be constructed from polynomial maps

$$\mathbb{R} \longrightarrow \mathbb{R}$$

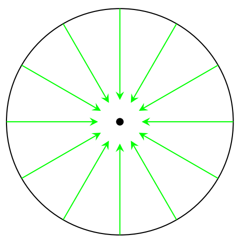
in 'canonical' ways: by restrictions, products, and quotients. Don't worry about this for now. We shall take it for granted, leaving details for the exercises, and instead focus on developing a geometric feeling for continuity.

Example 5.1.2. Let

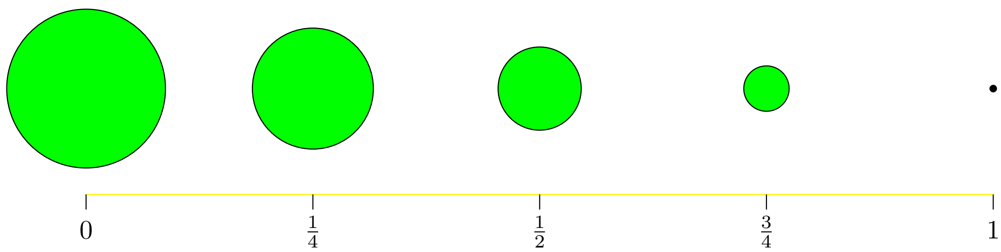
$$D^2 \times I \xrightarrow{f} D^2$$

be given by $(x, y, t) \mapsto ((1-t)x, (1-t)y)$. Then f is continuous. To prove this is Task E5.2.6.

Remark 5.1.3. We may think of f as 'shrinking D^2 onto its centre', as t moves from 0 to 1.



We can picture the image of $D^2 \times \{t\}$ under f as follows, as t moves from 0 to 1.



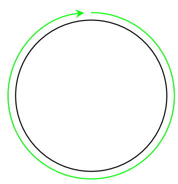
Example 5.1.4. Let $k \in \mathbb{R}$. There is a continuous map

$$I \xrightarrow{f} S^1$$

which can be thought of as travelling k times around a circle, starting at $(0, 1)$. To construct f rigorously is the topic of Task E5.2.7.

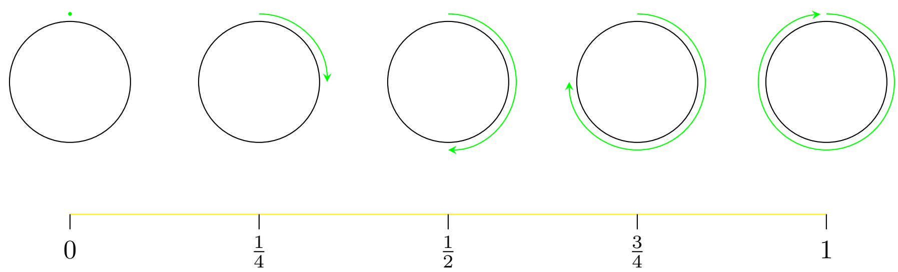
Remark 5.1.5. Let us picture f for a few values of k.

(1) Let $k = 1$. Then we travel exactly once around S^1.

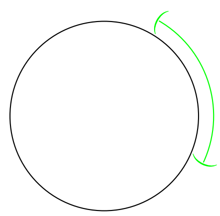


Don't be misled by the picture. The path really travels around the circle, not slightly outside it.

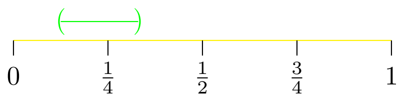
We may picture $f([0, t])$ as t moves from 0 to 1 as follows.



Recall from Examples 4.1.4 that a typical open subset U of S^1 is an 'open arc'.

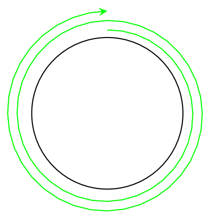


We have that $f^{-1}(U)$ is an open interval as follows.



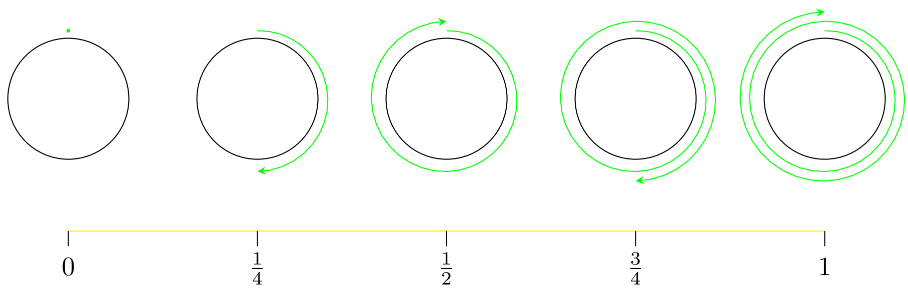
In particular, $f^{-1}(U)$ belongs to $\mathcal{O}_I$. Thus, even though we have not yet rigorously constructed f, we can intuitively believe that it is continuous.

(2) Let $k = 2$. Then we travel exactly twice around S^1.

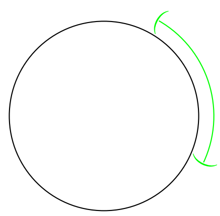


Again, don't be misled by the picture. The path really travels twice around the circle, thus passing through every point on the circle twice, not in a spiral outside the circle.

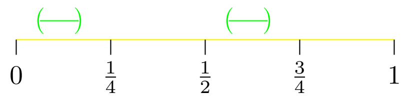
We may picture $f([0, t])$ as t moves from 0 to 1 as follows.



Let U denote the subset of S^1 given by the 'open arc' depicted below.

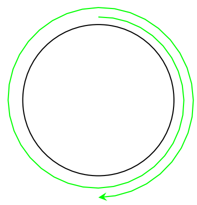


Then $f^{-1}(U)$ is a disjoint union of open intervals as follows.

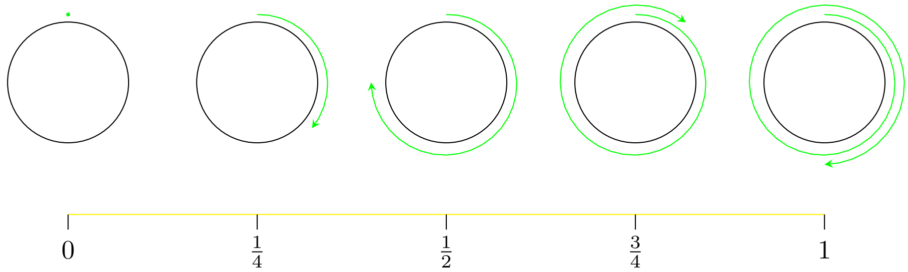


In particular, $f^{-1}(U)$ belongs to $\mathcal{O}_I$. Thus, again, even though we have not rigorously constructed f, we can believe intuitively that it is continuous.

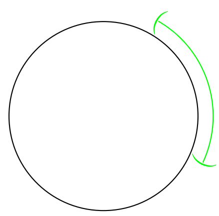
(3 Let $k = \frac{3}{2}$. Then we travel exactly one and a half times around S^1.



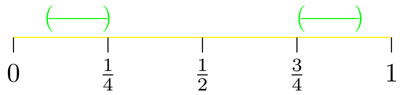
We may picture $f\left([0, t]\right)$ as t moves from 0 to 1 as follows.



Let U denote the subset of S^1 given by the 'open arc' depicted below.



Then $f^{-1}(U)$ is a disjoint union of open intervals as follows.



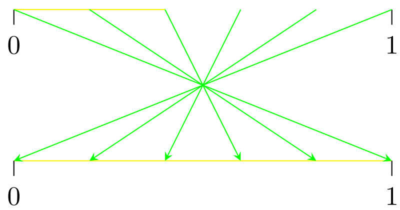
In particular, $f^{-1}(U)$ belongs to $\mathcal{O}_I$. Thus, once more, even though we have not rigorously constructed f, we can believe intuitively that it is continuous.

Example 5.1.6. Let

$$I \xrightarrow{f} I$$

be given by $t \mapsto 1 - t$. Then f is continuous, by Task E5.3.14.

Remark 5.1.7. We may picture f as follows.



Let U denote the subset of I given by the following open interval.

Then $f^{-1}(U)$ is the following open interval.

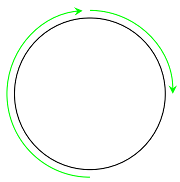
In particular, $f^{-1}(U)$ belongs to $\mathcal{O}_I$. Thus, even though this is not quite a proof yet, we can intuitively believe that f is continuous.

Example 5.1.8. There is a map

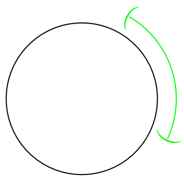
$$I \xrightarrow{f} S^1$$

travels around the circle at half speed from $(0,1)$ to $(1,0)$ for $0 \leq t \leq \frac{1}{2}$, and at normal speed from $(0,-1)$ to $(0,1)$ for $\frac{1}{2} < t \leq 1$. It is not continuous. To construct f rigorously, and to prove that it is not continuous, is the topic of Task E5.2.8.

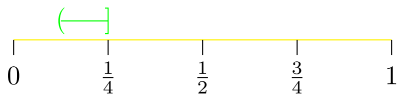
Remark 5.1.9. We may picture f as follows.



Let U denote the subset of S^1 given by the 'open arc' depicted below.



Then $f^{-1}(U)$ is a half open interval as follows.



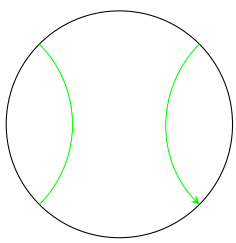
In particular, $f^{-1}(U)$ does not belong to $\mathcal{O}_I$. Thus we can see intuitively that f is not continuous.

Example 5.1.10. There is a map

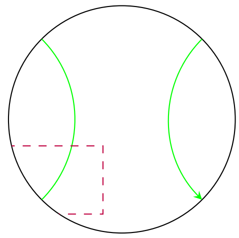
$$I \xrightarrow{f} D^2$$

which begins at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, travels around an arc of radius $\frac{1}{4}$ centred at $\left(-\frac{3}{4}, 0\right)$ to $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, jumps to $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and then travels around an arc of radius $\frac{1}{4}$ centred at $\left(\frac{3}{4}, 0\right)$ to $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. It is not continuous. To construct f rigorously, and to prove that it is not continuous, is the topic of Task E5.2.9.

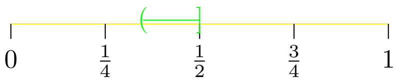
Remark 5.1.11. We may picture f as follows.



Let U denote the subset of D^2 given by the 'open rectangle' depicted below.



Then $f^{-1}(U)$ is a half open interval as follows.



In particular, $f^{-1}(U)$ does not belong to $\mathcal{O}_I$. Thus we can see intuitively that f is not continuous.

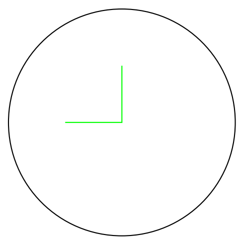
Remark 5.1.12. Intuitively, continuous maps cannot 'jump'!

Example 5.1.13. Let

$$\mathbb{R} \xrightarrow{f} D^2$$

be the map given by

$$x \mapsto \begin{cases} (-\frac{1}{2}, 0) & \text{for } x \leq -\frac{1}{2}, \\ (x, 0) & \text{for } -\frac{1}{2} \leq x \leq 0, \\ (0, x) & \text{for } 0 \leq x \leq \frac{1}{2}, \\ (0, \frac{1}{2}) & \text{for } x \geq \frac{1}{2}. \end{cases}$$



Then f is continuous. To prove this is the topic of Task E5.2.10.

Remark 5.1.14. In particular, continuous maps can have 'sharp edges'. In *differential topology*, maps are required to moreover be *smooth*: sharp edges are disallowed! The courses MA3402 Analyse på Mangfoldigheter and TMA4190 Mangfoldigheter both lead towards differential topology.

5.2 Inclusion maps are continuous

Terminology 5.2.1. Let X be a set, and let A be a subset of X. We refer to the map

$$A \xrightarrow{i} X$$

given by $x \mapsto x$ as an *inclusion map*.

Proposition 5.2.2. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X, and let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. The inclusion map

$$A \xrightarrow{i} X$$

is continuous.

Proof. Let U be a subset of X which belongs to $\mathcal{O}_X$. Then $i^{-1}(U) = A \cap U$. By definition of $\mathcal{O}_A$, we have that $A \cap U$ belongs to $\mathcal{O}_A$. We conclude that $i^{-1}(U)$ belongs to $\mathcal{O}_A$. $\square$

Notation 5.2.3. Let X, Y, and Z be sets. Let

$$X \xrightarrow{f} Y$$

and

$$Y \xrightarrow{g} Z$$

be maps. We denote by

$$X \xrightarrow{g \circ f} Z$$

the *composition* of f and g, given by $x \mapsto g\left(f(x)\right)$.

5.3 Compositions of continuous maps are continuous

Proposition 5.3.1. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

and

$$Y \xrightarrow{g} Z$$

be continuous maps. The map

$$X \xrightarrow{g \circ f} Z$$

is continuous.

Proof. Let U be a subset of Z which belongs to $\mathcal{O}_Z$. Then

$$\begin{aligned}(g \circ f)^{-1}(U) &= \{x \in X \mid g(f(x)) \in U\} \\ &= \left\{x \in X \mid f(x) \in g^{-1}(U)\right\} \\ &= f^{-1}\left(g^{-1}(U)\right).\end{aligned}$$

Since g is continuous, we have that $g^{-1}(U)$ belongs to $\mathcal{O}_Y$. We deduce, since f is continuous, that $f^{-1}\left(g^{-1}(U)\right)$ belongs to $\mathcal{O}_X$. Thus $(g \circ f)^{-1}(U)$ belongs to $\mathcal{O}_X$. □

5.4 Projection maps are continuous

Notation 5.4.1. Let X and Y be sets. We denote by

$$X \times Y \xrightarrow{p_1} X$$

the map given by $(x, y) \mapsto x$. We denote by

$$X \times Y \xrightarrow{p_2} Y$$

the map given by $(x, y) \mapsto y$.

Terminology 5.4.2. We refer to p_1 and p_2 as *projection maps*.

Proposition 5.4.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $X \times Y$ be equipped with the product topology $\mathcal{O}_{X \times Y}$. Then

$$X \times Y \xrightarrow{p_1} X$$

and

$$X \times Y \xrightarrow{p_2} Y$$

are continuous.

Proof. Suppose that U_X is a subset of X which belongs to $\mathcal{O}_X$. Then

$$p_1^{-1}(U_X) = U_X \times Y.$$

We have that $U_X \times Y$ belongs to $\mathcal{O}_{X \times Y}$. Thus p_1 is continuous.

Suppose now that U_Y is a subset of Y which belongs to $\mathcal{O}_Y$. Then

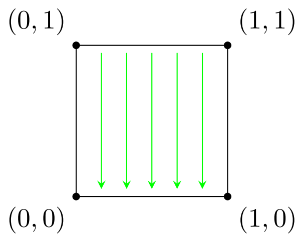
$$p_2^{-1}(U_Y) = X \times U_Y.$$

We have that $X \times U_Y$ belongs to $\mathcal{O}_{X \times Y}$. Thus p_2 is continuous. □

Remark 5.4.4. We can think of

$$I \times I \xrightarrow{p_1} I$$

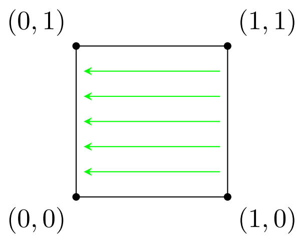
as the map $(x, y) \mapsto (x, 0)$. We can picture this as follows.



We can think of

$$I \times I \xrightarrow{p_2} I.$$

as the map $(x, y) \mapsto (0, y)$. We can picture this as follows.



E5 Exercises for Lecture 5

E5.1 Exam questions

Remark E5.1.1. You may find it helpful to carry out Tasks E5.2.3 – E5.2.5 before attempting the tasks in this section.

Terminology E5.1.2. Let X be a set. We refer to the map

$$X \longrightarrow X$$

given by $x \mapsto x$ as the *identity map* from X to itself.

Task E5.1.3. Let $(X, \mathcal{O}_X)$ be a topological space. Prove that the identity map

$$X \xrightarrow{id} X$$

is continuous.

Terminology E5.1.4. Let X and Y be sets. A map

$$X \xrightarrow{f} Y$$

is *constant* if $f(x) = f(x')$ for all $x, x' \in X$.

Task E5.1.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a constant map. Prove that f is continuous. You may wish to proceed as follows.

(1) Observe that if f is constant, then there is a $y \in Y$ such that $f(x) = y$ for all $x \in X$.

(2) Let U be a subset of Y which belongs to $\mathcal{O}_Y$. Determine $f^{-1}(U)$ in the cases that $y \in U$, and in the case that $y \notin U$.

Terminology E5.1.6. Let X and Y be sets, and let A be a subset of X. Let

$$X \xrightarrow{f} Y$$

be a map. The *restriction* of f to A is the map

$$A \longrightarrow Y$$

given by $x \mapsto f(x)$.

Remark E5.1.7. In other words, the restriction of f to A is the map

$$A \xrightarrow{f \circ i} Y,$$

where

$$A \xrightarrow{i} X$$

is the inclusion map.

Task E5.1.8. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a continuous map. Let A be a subset of X, and let A be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Prove that the restriction of f to A defines a continuous map

$$A \longrightarrow Y.$$

Task E5.1.9. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let A be a subset of Y, and let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(Y, \mathcal{O}_Y)$. Prove that if

$$X \xrightarrow{f} Y$$

is a continuous map such that $f(X) \subset A$, then the map

$$X \longrightarrow A$$

given by $x \mapsto f(x)$ is continuous.

Task E5.1.10. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let A be a subset of Y, and let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(Y, \mathcal{O}_Y)$. Prove that if

$$X \xrightarrow{f} A$$

is a continuous map, then the map

$$X \longrightarrow Y$$

given by $x \mapsto f(x)$ is continuous.

Terminology E5.1.11. Let X be a set. We refer to the map

$$X \xrightarrow{\Delta} X \times X$$

given by $x \mapsto (x, x)$ as the *diagonal map*.

Task E5.1.12. Let $(X, \mathcal{O}_X)$ be a topological space. Let $X \times X$ be equipped with the product topology $\mathcal{O}_{X \times X}$ with respect to two copies of $(X, \mathcal{O}_X)$. Prove that

$$X \xrightarrow{\Delta} X \times X$$

is continuous. You may wish to proceed as follows.

(1) Let U_0 and U_1 be subsets of X which belong to $\mathcal{O}_X$. Prove that $\Delta^{-1}(U_0 \times U_1)$ belongs to $\mathcal{O}_X$.

(2) Let U be a subset of $X \times X$ which belongs to $\mathcal{O}_{X \times X}$. Prove that $\Delta^{-1}(U)$ belongs to $\mathcal{O}_X$, by appealing to (1) and to Task E8.3.1.

Task E5.1.13. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if $f^{-1}(V)$ is closed with respect $\mathcal{O}_X$, for every subset V of Y which is closed with respect to $\mathcal{O}_Y$.

Task E5.1.14. Let X be a set. Let $\mathcal{O}_X$ be the discrete topology on X. Let $(Y, \mathcal{O}_Y)$ be a topological space. Prove that any map

$$X \xrightarrow{f} Y$$

is continuous.

Task E5.1.15. Let $(X, \mathcal{O}_X)$ be a topological space. Let Y be a set. Let $\mathcal{O}_Y$ be the indiscrete topology on X. Prove that any map

$$X \xrightarrow{f} Y$$

is continuous.

E5.2 In the lecture notes

Task E5.2.1. Let X, Y, and Z be sets. Let

$$X \xrightarrow{f} Y$$

and

$$Y \xrightarrow{g} Z$$

be maps. Prove that

$$\{x \in X \mid g\left(f(x)\right) \in U\} = \left\{x \in X \mid f(x) \in g^{-1}(U)\right\}.$$

This was appealed to in the proof of Proposition 5.3.1.

Task E5.2.2. Let $(X, \mathcal{O}_X$ and $(Y, \mathcal{O}_Y)$ be topological spaces.

(1) Let U_X be a subset of X which belongs to $\mathcal{O}_X$. Check that you understand why $U_X \times Y$ belongs to $\mathcal{O}_{X \times Y}$.

(2) Let U_Y be a subset of Y which belongs to $\mathcal{O}_Y$. Check that you understand why $X \times U_Y$ belongs to $\mathcal{O}_{X \times Y}$.

These observations were appealed to in the proof of Proposition 5.4.3.

Task E5.2.3. Do the same as in Task E2.2.2 for the proof of Proposition 5.2.2.

Task E5.2.4. Do the same as in Task E2.2.2 for the proof of Proposition 5.3.1.

Task E5.2.5. Do the same as in Task E2.2.2 for the proof of Proposition 5.4.3.

Task E5.2.6. Prove that the map

$$D^2 \times I \xrightarrow{f} D^2$$

of Example 5.1.2 is continuous. You may wish to proceed as follows.

(1) Express the map

$$\mathbb{R}^3 \xrightarrow{f_0} \mathbb{R}$$

given by $(x, y, t) \mapsto (1-t)x$ as a composition of four maps.

(I) The map

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

given by $(x, y, t) \mapsto (x, t)$.

(II) The twist map

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

given by $(x, y) \mapsto (y, x)$.

(III) The map

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

given by $(x, y) \mapsto (1 - x, y)$.

(IV) The map

$$\mathbb{R}^2 \xrightarrow{\times} \mathbb{R}$$

given by $(x, y) \mapsto xy$.

Appealing to Proposition 5.4.3, Task E5.3.17, Task E5.3.19, Task E5.3.14, Task E5.3.11, and Proposition 5.3.1, deduce that f_0 is continuous.

(2) In a similar way, prove that the map

$$\mathbb{R}^3 \xrightarrow{f_1} \mathbb{R}$$

given by $(x, y, t) \mapsto (1 - t)y$ is continuous.

(3) View $D^2 \times I$ as equipped with the subspace topology with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$. Appealing to Task E5.1.8, deduce from (1) that the map

$$D^2 \times I \xrightarrow{f_0} \mathbb{R}$$

given by $(x, y, t) \mapsto (1 - t)x$ is continuous, and deduce from (2) that the map

$$D^2 \times I \xrightarrow{f_1} \mathbb{R}$$

given by $(x, y, t) \mapsto (1 - t)y$ is continuous,

(4) Appealing to Task E5.3.17, deduce from (3) that the map

$$D^2 \times I \longrightarrow \mathbb{R}^2$$

given by $(x, y, t) \mapsto ((1-t)x, (1-t)y)$ is continuous.

(5) Appealing to Task E5.1.9, conclude from (4) that f is continuous.

Task E5.2.7. Let $k \in \mathbb{R}$. Construct a continuous map

$$I \longrightarrow S^1$$

which travels around the circle k times, as in Example 5.1.4. You may wish to proceed as follows.

(1) By Task E5.3.14, observe that the map

$$I \longrightarrow [0, k]$$

given by $t \mapsto kt$ is continuous.

(2) By Task E5.3.27 and Task E5.1.8, observe that the map

$$[0, k] \longrightarrow S^1$$

given by $t \mapsto \phi(t)$ is continuous, where

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is the map of Task E5.3.27.

(3) Appeal to Proposition 5.3.1.

Task E5.2.8. Use the map

$$\mathbb{R} \xrightarrow{\phi} S^1$$

of Task E5.3.27 to construct the map

$$I \xrightarrow{f} S^1$$

of Example 5.1.8. Prove that f is not continuous.

Task E5.2.9. Use the map

$$\mathbb{R} \xrightarrow{\phi} S^1$$

of Task E5.3.27 to construct the map

$$I \xrightarrow{f} D^2$$

of Example 5.1.10. Prove that f is not continuous.

Task E5.2.10. Let

$$\mathbb{R} \xrightarrow{f} D^2$$

be the map of Example 5.1.13. Prove that f is continuous. You may wish to proceed as follows.

(1) By Task E5.1.5, observe that the map

$$\left[-\infty, -\tfrac{1}{2}\right[\longrightarrow D^2$$

given by $x \mapsto (-\frac{1}{2}, 0)$ is continuous.

(2) By Task E5.1.3, Task E5.1.5, and Task E5.3.17, observe that the map

$$\left[-\tfrac{1}{2}, 0\right] \longrightarrow D^2$$

given by $x \mapsto (x, 0)$ is continuous.

(3) By Task E5.1.3, Task E5.1.5, and Task E5.3.17, observe that the map

$$\left[0, \tfrac{1}{2}\right] \longrightarrow D^2$$

given by $x \mapsto (0, x)$ is continuous.

(4) By Task E5.1.5, observe that the map

$$\left]\tfrac{1}{2}, \infty\right] \longrightarrow D^2$$

given by $x \mapsto (0, \frac{1}{2})$ is continuous.

(5) Appeal to (2) of Task E5.3.23.

E5.3 For a deeper understanding

Assumption E5.3.1. Throughout this section, let $\mathbb{R}$ be equipped with the standard topology $\mathcal{O}_{\mathbb{R}}$.

Remark E5.3.2. The proofs needed for Tasks E5.3.5 – E5.3.7 and Task E5.3.9 all follow the pattern of the proof of the following proposition, which is given to help you along.

Proposition E5.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Then the map

$$X \xrightarrow{|f|} \mathbb{R}$$

given by $x \mapsto |f(x)|$ is continuous.

Proof. By Corollary E4.2.6, to prove that f is continuous, it suffices to prove that $|f|^{-1}(]a,b[)$ belongs to $\mathcal{O}_X$, for every open interval $]a,b[$. We have that

$$\begin{aligned} |f|^{-1}(]a,b[) &= \left\{x' \in X \mid \left|f(x')\right| \in]a,b[\right\} \\ &= \left\{x' \in X \mid f(x') \in]a,b[\right\} \cup \left\{x' \in X \mid -f(x') \in]a,b[\right\} \\ &= \left\{x' \in X \mid f(x') \in]a,b[\right\} \cup \left\{x' \in X \mid f(x') \in]-b,-a[\right\} \\ &= f^{-1}(]a,b[) \cup f^{-1}(]-b,-a[) . \end{aligned}$$

Both $]a,b[$ and $]-b,-a[$ belong to $\mathcal{O}_{\mathbb{R}}$. Since f is continuous, we deduce that both $f^{-1}(]a,b[)$ and $f^{-1}(]-b,-a[)$ belong to $\mathcal{O}_X$. Since $\mathcal{O}_X$ is a topology on X, this implies that

$$f^{-1}(]a,b[) \cup f^{-1}(]-b,-a[)$$

belongs to $\mathcal{O}_X$. Hence $|f|^{-1}(]a,b[)$ belongs to $\mathcal{O}_X$. □

Remark E5.3.4. In a nutshell, the proof of Proposition E5.3.3 proceeds by expressing $\left|f^{-1}\right|(]a,b[)$ as a union of inverse images under f of subsets of $\mathbb{R}$ which belong to $\mathcal{O}_{\mathbb{R}}$. It is this idea that is also at the heart of the proofs needed for Tasks E5.3.5 – E5.3.7 and Task E5.3.9.

Task E5.3.5. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Prove that, for any $k \in \mathbb{R}$, the map

$$X \xrightarrow{kf} \mathbb{R}$$

given by $x \mapsto k \cdot f(x)$ is continuous. You may wish to proceed by considering separately the cases $k = 0$, $k > 0$, and $k < 0$.

Task E5.3.6. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$X \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{R}$$

be continuous maps. Prove that the map

$$X \xrightarrow{f+g} \mathbb{R}$$

given by $x \mapsto f(x) + g(x)$ is continuous. You may wish to proceed as follows.

(1) Observe that, by Task E4.2.12 and Task E4.2.11, to prove that $f+g$ is continuous, it suffices to prove that for any $y \in \mathbb{R}$, the sets

$$(f+g)^{-1}\left(]-\infty, y[\right)$$

and

$$(f+g)^{-1}\left(]y, \infty[\right)$$

belong to $\mathcal{O}_X$.

(2) Prove that $\{x \in X \mid f(x) + g(x) < y\}$ is equal to

$$\bigcup_{y' \in \mathbb{R}} \left(\left\{ x \in X \mid f(x) < y - y' \right\} \cap \left\{ x \in X \mid g(x) < y' \right\} \right),$$

and that $\{x \in X \mid f(x) + g(x) > y\}$ is equal to

$$\bigcup_{y' \in \mathbb{R}} \left(\left\{ x \in X \mid f(x) > y - y' \right\} \cap \left\{ x \in X \mid g(x) > y' \right\} \right).$$

Task E5.3.7. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map, with the property that $f(x) \geq 0$ for all $x \in X$. Prove that the map

$$X \xrightarrow{f^2} \mathbb{R}$$

given by $x \mapsto f(x) \cdot f(x)$ is continuous.

Task E5.3.8. Let

$$X \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{R}$$

be continuous maps. Prove that the map

$$X \xrightarrow{fg} \mathbb{R}$$

given by $x \mapsto f(x) \cdot g(x)$ is continuous. You may wish to proceed as follows.

(1) Observe that fg is

$$\tfrac{1}{4}\left(|f+g|^2 - |f-g|^2\right).$$

(2) Appeal to Proposition E5.3.3 and Tasks E5.3.5 – E5.3.7.

Task E5.3.9. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Suppose that $f(x) \neq 0$ for all $x \in X$. Prove that the map

$$X \xrightarrow{\frac{1}{f}} \mathbb{R}$$

given by $x \mapsto \frac{1}{f(x)}$ is continuous. You may wish to proceed as follows.

(1) Observe that, by Task E4.2.12 and Task E4.2.11, to prove that $\frac{1}{f}$ is continuous, it suffices to prove that for any $y \in \mathbb{R}$, the sets

$$\left(\frac{1}{f}\right)^{-1}(]-\infty, y[)$$

and

$$\left(\frac{1}{f}\right)^{-1}(]y, \infty[)$$

belong to $\mathcal{O}_X$.

(2) Prove that, for all $y \in \mathbb{R}$, the set

$$\left\{x \in X \mid \frac{1}{f(x)} < y\right\}$$

is equal to the union of

$$\{x \in X \mid f(x) > 0\} \cap \{x \in X \mid (yf)(x) > 1\}$$

and

$$\{x \in X \mid f(x) < 0\} \cap \{x \in X \mid (yf)(x) < 1\}.$$

(3) Prove that, for all $y \in \mathbb{R}$, the set

$$\left\{x \in X \mid \frac{1}{f(x)} > y\right\}$$

is equal to the union of

$$\{x \in X \mid f(x) > 0\} \cap \{x \in X \mid (yf)(x) < 1\}$$

and

$$\{x \in X \mid f(x) < 0\} \cap \{x \in X \mid (yf)(x) > 1\}.$$

(4) Appeal to Task E5.3.5.

Task E5.3.10. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$X \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{R}$$

be continuous maps. Suppose that $g(x) \neq 0$ for all $x \in X$. Prove that the map

$$X \xrightarrow{\frac{f}{g}} \mathbb{R}$$

given by $x \mapsto \frac{f(x)}{g(x)}$ is continuous. You may wish to appeal to two of the previous tasks.

Task E5.3.11. Let $\mathbb{R}^2$ be equipped with the topology $\mathcal{O}_{\mathbb{R}^2}$. Prove that the map

$$\mathbb{R}^2 \xrightarrow{\times} \mathbb{R}$$

given by $(x, y) \mapsto xy$ is continuous. You may wish to appeal to Proposition 5.4.3, and to Task E5.3.8.

Task E5.3.12. Let $\mathbb{R}^2$ be equipped with the topology $\mathcal{O}_{\mathbb{R}^2}$. Prove that the map

$$\mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R}$$

given by $(x, y) \mapsto x + y$ is continuous. You may wish to appeal to Proposition 5.4.3, and to Task E5.3.6.

Terminology E5.3.13. Let X and Y be subsets of $\mathbb{R}$. Let

$$X \xrightarrow{f} Y$$

be a map given by

$$x \mapsto k_n x^n + k_{n-1} x^{n-1} + \ldots + k_1 x_1 + k_0,$$

where $k_i \in \mathbb{R}$ for all $0 \leq i \leq n$. We refer to f as a *polynomial map.*

Task E5.3.14. Let X be a subset of $\mathbb{R}$, equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let Y also be a subset of $\mathbb{R}$, equipped with the subspace topology $\mathcal{O}_Y$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Prove that every polynomial map

$$X \xrightarrow{f} Y$$

is continuous. You may wish to proceed as follows.

(1) Demonstrate that a polynomial map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous. For this, you may wish to proceed by induction, appealing to Task E5.1.3, Task E5.3.5, Task E5.3.8, and Task E5.3.6.

(2) Appeal to Task E5.1.8 and to Task E5.1.9.

Corollary E5.3.15. Let X be a subset of $\mathbb{R}$, equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let Y also be a subset of $\mathbb{R}$, equipped with the subspace topology $\mathcal{O}_Y$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let

$$X \xrightarrow{f} Y$$

be a map given by $x \mapsto \frac{g_0(x)}{g_1(x)}$, where $g_1(x) \neq 0$ for all $x \in X$. Suppose that g_0 and g_1 are polynomial maps. Then f is continuous.

Proof. We can view f as a map

$$X \xrightarrow{f'} \mathbb{R}.$$

It follows immediately from Task E5.3.14 and Task E5.3.10 that f' is continuous. By Task E5.1.9, we conclude that f is continuous. $\square$

Task E5.3.16. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map such that $f(x) \geq 0$ for all $x \in X$. Prove that, for any $n \in \mathbb{N}$, the map

$$X \xrightarrow{\sqrt[n]{f}} \mathbb{R}$$

given by $x \mapsto \sqrt[n]{f(x)}$, where $\sqrt[n]{f(x)}$ denotes the positive n^{th} root of $f(x)$, is continuous.

Task E5.3.17. Let $(X_0, \mathcal{O}_{X_0})$, $(X_1, \mathcal{O}_{X_1})$, $(Y_0, \mathcal{O}_{Y_0})$, and $(Y_1, \mathcal{O}_{Y_1})$ be topological spaces. Let

$$X_0 \xrightarrow{f_0} Y_0$$

and

$$X_1 \xrightarrow{f_1} Y_1$$

be continuous maps. Prove that the map

$$X_0 \times X_1 \xrightarrow{f_0 \times f_1} Y_0 \times Y_1$$

given by $(x_0, x_1) \mapsto (f_0(x_0), f_1(x_1))$ is continuous.

Terminology E5.3.18. Let X and Y be sets. We refer to the map

$$X \times Y \xrightarrow{\tau} Y \times X$$

given by $(x, y) \mapsto (y, x)$ as the *twist map*.

Task E5.3.19. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that the twist map

$$X \times Y \xrightarrow{\tau} Y \times X$$

is continuous. You may wish to appeal to Task E5.3.17.

Task E5.3.20. Let $(X, \mathcal{O}_X)$, $(Y_0, \mathcal{O}_{Y_0})$, and $(Y_1, \mathcal{O}_{Y_1})$ be topological spaces. Let

$$X \xrightarrow{f_0} Y_0$$

and

$$X \xrightarrow{f_1} Y_1$$

be continuous maps. Prove that the map

$$X \xrightarrow{f_0 \times f_1} Y_0 \times Y_1$$

given by $x \mapsto (f_0(x), f_1(x))$ is continuous. You may wish to appeal to Task E5.1.12 and Task E5.3.17.

Task E5.3.21. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a continuous map. Let A be a subset of Y, and let A be equipped with the subspace topology with respect to $(Y, \mathcal{O}_Y)$. Let

$$A \xrightarrow{i} Y$$

be the inclusion map. Prove that a map

$$X \xrightarrow{f} A$$

is continuous if and only if the map

$$X \xrightarrow{i \circ f} Y$$

is continuous.

Notation E5.3.22. Let X and Y be sets. Let $\{A_j\}_{j \in J}$ be a set of subsets of X such that $X = \bigcup_{j \in J} A_j$. Suppose that, for every $j \in J$, we have a map

$$A_j \xrightarrow{f_j} Y.$$

Moreover, suppose that, for all $j_0, j_1 \in J$, the restriction of f_{j_0} to $A_{j_0} \cap A_{j_1}$ is equal to the restriction of f_{j_1} to $A_{j_0} \cap A_{j_1}$. We then obtain a map

$$X \longrightarrow Y$$

given by $x \mapsto f_j(x)$ if $x \in A_j$.

Task E5.3.23. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\{A_j\}_{j \in J}$ be a set of subsets of X such that $X = \bigcup_{j \in J} A_j$. For every $j \in J$, let A_j be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Suppose that, for every $j \in J$, we have a continuous map

$$A_j \xrightarrow{f_j} Y.$$

Moreover, suppose that, for all $j_0, j_1 \in J$, the restriction of f_{j_0} to A_{j_0} is equal to the restriction of f_{j_1} to A_{j_1}. Let

$$X \xrightarrow{f} Y$$

denote the map of Notation E5.3.22 corresponding to the maps $\{f_j\}_{j \in J}$.

(1) Suppose that A_j belongs to $\mathcal{O}_X$ for every $j \in J$. Prove that f is continuous. You may wish to appeal to Task E2.3.4.

(2) Suppose that $\{A_j\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_X$, and that A_j is closed with respect to $\mathcal{O}_X$, for every $j \in J$. Prove that f is continuous. You may wish to appeal to Task E8.3.9.

(3) Suppose that J finite. Give an example to demonstrate that if we do not assume that A_j is closed with respect to $\mathcal{O}_X$ for every $j \in J$, then f is not necessarily continuous.

(4) Suppose that A_j is closed with respect to $\mathcal{O}_X$ for every $j \in J$. Given an example to demonstrate that, if we do not assume that $\{A_j\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_X$, then f is not necessarily continuous.

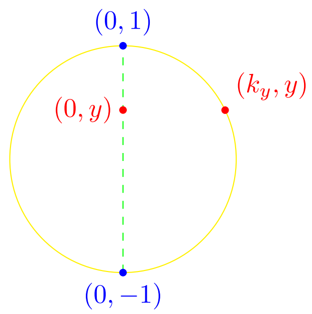
Remark E5.3.24. Taking into account Remark E8.3.6, we have that if J is finite, and A_j is closed with respect to $\mathcal{O}_X$, for every $j \in J$, then f is continuous.

Remark E5.3.25. The result of (1) and (2) of Task E5.3.23 is sometimes known as the *glueing lemma* or *pasting lemma*. Continuous maps constructed by means of (1) and (2) of Task E5.3.23 are sometimes said to be defined *piecewise*.

Notation E5.3.26. Given $y \in [-1, 1]$, let

$$k_y = \sqrt{1 - y^2}.$$

Here we take the positive square root. We have that $\|(k_y, y)\| = 1$.

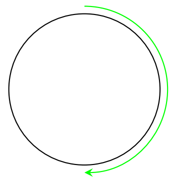


Let

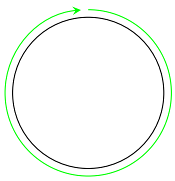
$$\mathbb{R} \xrightarrow{\phi} S^1$$

be the map defined as follows.

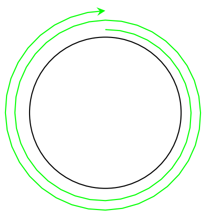
(1) Suppose that $x \in \left[0, \frac{1}{2}\right]$. Let $y = 1 - 4x$. We define $\phi(x)$ to be (k_y, y). We can picture ϕ on $\left[0, \frac{1}{2}\right]$ as follows.



(2) Suppose that $x \in \left[\frac{1}{2}, 1\right]$. Let $y = 4x - 3$. We define $\phi(x)$ to be $(-k_y, y)$. We can picture ϕ on $[0, 1]$ as follows.



(3) Suppose that $x \in [n, n+1]$, for some $n \in \mathbb{Z}$. We define $\phi(x)$ to be $\phi(x - n)$. We can picture ϕ on $[0, 2]$, for instance, as follows.



Task E5.3.27. Prove that the map

$$\mathbb{R} \xrightarrow{\phi} S^1$$

of Notation E5.3.26 is continuous. You may wish to proceed as follows.

(1) Let $\left[0, \frac{1}{2}\right]$ be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Observe that by Task E5.3.14 and Task E5.3.16, the map

$$\left[0, \tfrac{1}{2}\right] \xrightarrow{f} \mathbb{R}$$

given by $y \mapsto k_y$ is continuous.

(2) By Task E5.3.17, deduce from (1) that the map

$$\left[0, \tfrac{1}{2}\right] \xrightarrow{f \times id} \mathbb{R}^2$$

given by $y \mapsto (k_y, y)$ is continuous. By Task E5.1.9, deduce that the map

$$\left[0, \tfrac{1}{2}\right] \longrightarrow S^1$$

given by $x \mapsto \phi(x)$ is continuous.

(3) Let $\left[\frac{1}{2}, 1\right]$ be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. As in (1) and (2), demonstrate that the map

$$\left[\tfrac{1}{2}, 1\right] \longrightarrow S^1$$

given by $x \mapsto \phi(x)$ is continuous.

(4) Let the unit interval I be equipped with the topology $\mathcal{O}_I$. By (2) of Task E5.3.23, deduce from (2) and (3) that the map

$$I \longrightarrow S^1$$

given by $x \mapsto \phi(x)$ is continuous.

(5) Let $n \in \mathbb{Z}$. Let $[n, n+1]$ be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Task E5.3.14, observe that the map

$$[n, n+1] \xrightarrow{g} I$$

given by $x \mapsto x - n$ is continuous.

(6) Let $n \in \mathbb{Z}$. By Proposition 5.3.1, deduce from (4) and (5) that the map

$$[n, n+1] \longrightarrow S^1$$

given by $x \mapsto \phi(x - n)$ is continuous.

(7) By (2) of Task E5.3.23, deduce from (6) that

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is continuous.

Remark E5.3.28. The map ϕ allows us to construct paths around a circle without using, for instance, trigonometric maps. Sine and cosine do define continuous maps, but their construction, and the proof that they are continuous, is much more involved. We shall explore this in a later task.

Task E5.3.29. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that x belongs to X. Let $X \setminus f^{-1}(\{f(x)\})$ be equipped with the subspace topology $\mathcal{O}_{X \setminus f^{-1}(\{f(x)\})}$ with respect to $(X, \mathcal{O}_X)$. Let

$$X \xrightarrow{f} Y$$

be a map. Suppose that $f^{-1}(\{f(x)\})$ is closed in X with respect to $\mathcal{O}_X$. Let $Y \setminus \{f(x)\}$ be equipped with the subspace topology $\mathcal{O}_{Y \setminus \{f(x)\}}$ with respect to $(Y, \mathcal{O}_Y)$. Suppose that the map

$$X \setminus f^{-1}(\{f(x)\}) \xrightarrow{g} Y \setminus \{f(x)\}$$

given by $x' \mapsto f(x')$ is continuous. Prove that f is continuous. You may wish to proceed as follows.

(1) Let V be a subset of Y which is closed with respect to $\mathcal{O}_Y$. Suppose that $f(x)$ does not belong to V. Then V is a subset of $Y \setminus \{f(x)\}$. Thus $f^{-1}(V) = g^{-1}(V)$. Since g is continuous, deduce by Task **??** that $f^{-1}(V)$ is closed in X with respect to $\mathcal{O}_X$.

(2) Suppose that $f(x)$ belongs to V. Then

$$\begin{aligned} X \setminus f^{-1}(V) &= f^{-1}(Y \setminus V) \\ &= g^{-1}(Y \setminus V). \end{aligned}$$

Since V is closed in Y with respect to $\mathcal{O}_Y$, we have that $Y \setminus V$ belongs to $\mathcal{O}_Y$. Hence $Y \setminus V$ belongs to $\mathcal{O}_{Y \setminus \{f(x)\}}$. Since g is continuous, we thus have that $g^{-1}(Y \setminus V)$ belongs to $\mathcal{O}_{X \setminus f^{-1}(\{f(x)\})}$. Deduce that $X \setminus f^{-1}(V)$ belongs to $\mathcal{O}_{X \setminus f^{-1}(\{f(x)\})}$.

(3) Since $f^{-1}(\{f(x)\})$ is closed in X with respect to $\mathcal{O}_X$, we have that $X \setminus f^{-1}(\{f(x)\})$ belongs to $\mathcal{O}_X$. By Task E2.3.3 (1) and (2), deduce that $X \setminus f^{-1}(V)$ belongs to $\mathcal{O}_X$. Thus we have that $f^{-1}(V)$ is closed in X with respect to $\mathcal{O}_X$.

(3) By Task **??**, conclude from (1) and (2) that f is continuous.