MA3002 Generell Topologi — Vår 2014

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May 19, 2014

7 Monday 27th January

7.1 Homeomorphisms

Definition 7.1.1. Let X and Y be sets. A map

$$X \xrightarrow{f} Y$$

is *bijective* if there is a map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$.

Remark 7.1.2. Here id_X and id_Y denote the respective identity maps, in the terminology of E5.1.2.

Notation 7.1.3. Let X and Y be sets, and let

$$X \xrightarrow{f} Y$$

be a bijective map. We often denote the corresponding map

$$Y \xrightarrow{g} X$$

by f^{-1}.

Remark 7.1.4. Let X and Y be sets. A map

$$X \xrightarrow{f} Y$$

is bijective in the sense of Definition 7.1.1 if and only if f is both injective and surjective. To prove this is Task E7.2.1.

Definition 7.1.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is a *homeomorphism* if the following hold.

(1) We have that f is continuous,

(2) There is a continuous map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$.

Remark 7.1.6. An equivalent definition of a homeomorphism is the topic of Task E7.3.1.

Definition 7.1.7. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then $(X, \mathcal{O}_X)$ is *homeomorphic* to $(Y, \mathcal{O}_Y)$ if there exists a homeomorphism

$$X \longrightarrow Y.$$

Remark 7.1.8. By Task E7.3.2, we have that $(X, \mathcal{O}_X)$ is homeomorphic to $(Y, \mathcal{O}_Y)$ if and only if there exists a homeomorphism

$$Y \longrightarrow X.$$

7.2 Examples of homeomorphisms between finite topological spaces

Example 7.2.1. Let $X = \{a, b, c\}$ be a set with three elements. Let

$$X \xrightarrow{f} X$$

be the bijective map given by $a \mapsto b$, $b \mapsto c$, and $c \mapsto a$. Let $\mathcal{O}_0$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b, c\}, X\}.$$

Let $\mathcal{O}_1$ be the topology on X given by

$$\{\emptyset, \{a, c\}, \{b\}, X\}.$$

Let us regard the copy of X in the source of f as equipped with the topology $\mathcal{O}_0$, and regard the copy of X in the target of f as equipped with the topology $\mathcal{O}_1$. We have the following.

$$\begin{aligned} f^{-1}(\emptyset) &= \emptyset \\ f^{-1}(\{a, c\}) &= \{b, c\} \\ f^{-1}(\{b\}) &= \{a\} \\ f^{-1}(X) &= X. \end{aligned}$$

Thus f is continuous. Let

$$X \xrightarrow{g} X$$

be the inverse to f, given by $a \mapsto c$, $b \mapsto a$, and $c \mapsto b$. We have the following.

$$\begin{aligned} g^{-1}(\emptyset) &= \emptyset \\ g^{-1}(\{a\}) &= \{b\} \\ g^{-1}(\{b,c\}) &= \{a,c\} \\ g^{-1}(X) &= X. \end{aligned}$$

Thus g is continuous. We conclude that f is a homeomorphism. In other words, we have that $(X, \mathcal{O}_0)$ and $(X, \mathcal{O}_1)$ are homeomorphic.

Example 7.2.2. Let X be as in Example 7.2.1. Let $\mathcal{O}_2$ be the topology on X given by

$$\{\emptyset, \{a,b\}, \{c\}, X\}.$$

Let f be as in Example 7.2.1. Let us again regard the copy of X in the source of f as equipped with the topology $\mathcal{O}_0$, but let us now regard the copy of X in the target of f as equipped with the topology $\mathcal{O}_2$. Then f is not continuous, since $f^{-1}(\{c\}) = \{b\}$, and $\{b\}$ does not belong to $\mathcal{O}_0$. Thus f is not a homeomorphism.

Remark 7.2.3. Let

$$X \xrightarrow{h} X$$

be the bijective map given by $a \mapsto c$, $b \mapsto b$, and $c \mapsto a$. We have the following.

$$\begin{aligned} h^{-1}(\emptyset) &= \emptyset \\ h^{-1}(\{a,b\}) &= \{b,c\} \\ h^{-1}(\{c\}) &= \{a\} \\ h^{-1}(X) &= X. \end{aligned}$$

Thus h is continuous. Moreover we have that h is its own inverse. We conclude that h is a homeomorphism. In other words, we have that $(X, \mathcal{O}_0)$ and $(X, \mathcal{O}_2)$ are homeomorphic.

Example 7.2.2 and Remark 7.2.3 demonstrate that a pair of topological spaces can be homeomorphic, even though a particular map that we consider might not be a homeomorphism. It is very important to remember this!

Example 7.2.4. Let X be as in Example 7.2.1. Let $\mathcal{O}_3$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, X\}.$$

Let f be as in Example 7.2.1. Let us regard the copy of X in the source of f as equipped with the topology $\mathcal{O}_3$, and regard the copy of X in the target of f as equipped with the topology $\mathcal{O}_1$. Since $\mathcal{O}_0$ is a subset of $\mathcal{O}_3$, the calculation of Example 7.2.1 demonstrates that f is continuous. The inverse of f is the map g of Example 7.2.1. We have that $g^{-1}(\{b\}) = \{c\}$, and $\{c\}$ does not belong to $\mathcal{O}_1$. Thus g is not continuous. We conclude that f is not a homeomorphism.

Remark 7.2.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homeomorphic topological spaces. Then $\mathcal{O}_X$ and $\mathcal{O}_Y$ must have the same cardinality. To prove this is Task E7.3.3. Thus $(X, \mathcal{O}_3)$ is not homeomorphic to $(X, \mathcal{O}_1)$.

Example 7.2.4 and Remark 7.2.5 demonstrate that there can be a continuous bijective map from one topological space to another, and yet these topological spaces might not be homeomorphic. It is very important to remember this! This phenomenon does not occur in group theory or linear algebra, for instance.

7.3 Geometric examples of homeomorphisms

Remark 7.3.1. Two geometric examples of topological spaces are, intuitively, homeomorphic if we can bend, stretch, compress, twist, or otherwise 'manipulate in a continuous manner', one to obtain the other. We can sharpen or smooth edges. We cannot cut or tear.

Remark 7.3.2. It may help you to think of geometric examples of topological spaces as made of dough, or of clay that has not yet been fired!

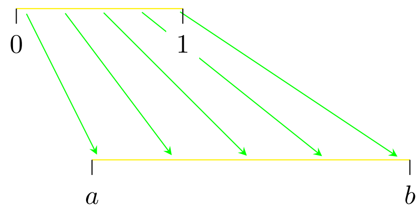
Example 7.3.3. Suppose that $a, b \in \mathbb{R}$, and that $a < b$. Let the open interval $]a, b[$ be equipped with the subspace topology $\mathcal{O}_{]a,b[}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let the open interval $]0, 1[$ also be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Then $\left(]a, b[\,, \mathcal{O}_{]a,b[}\right)$ is homeomorphic to $\left(]0, 1[\,, \mathcal{O}_{]0,1[}\right)$. Intuitively we can stretch or shrink, and translate, $]0, 1[$ to obtain $]a, b[$. To be rigorous, the map

$$]0,1[\xrightarrow{f}]a,b[$$

given by $t \mapsto a(1-t)+bt$ is a homeomorphism.



For the following hold.

(1) By Task E5.3.14, we have that f is continuous.

(2) Let

$$]a,b[\xrightarrow{g}]0,1[$$

be the map given by $t \mapsto \frac{t-a}{b-a}$. By Task E5.3.14, we have that g is continuous. Moreover we have the following, for every $t \in]0,1[$.

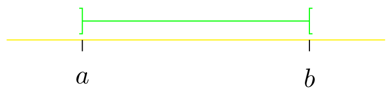
$$\begin{aligned} g(f(t)) &= g(a(1-t)+bt) \\ &= \frac{a(1-t)+bt-a}{b-a} \\ &= \frac{t(b-a)}{b-a} \\ &= t. \end{aligned}$$

Thus we have that $g \circ f = id_{]0,1[}$. We also have the following, for every $t \in]a,b[$.

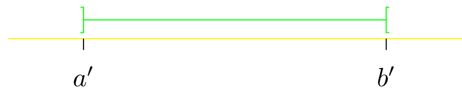
$$\begin{aligned} f(g(t)) &= f\left(\frac{t-a}{b-a}\right) \\ &= a\left(1-\frac{t-a}{b-a}\right)+b\left(\frac{t-a}{b-a}\right) \\ &= \frac{a(b-a)-a(t-a)+b(t-a)}{b-a} \\ &= \frac{t(b-a)}{b-a} \\ &= t. \end{aligned}$$

Thus we have that $f \circ g = id_{]a,b[}$.

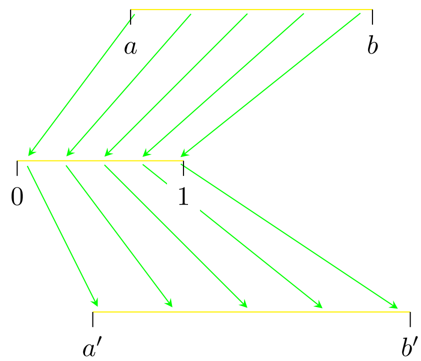
Example 7.3.4. Suppose that $a, b \in \mathbb{R}$, and that $a < b$. Suppose also that $a', b' \in \mathbb{R}$, and that $a' < b'$. Let $]a, b[$ be equipped with the subspace topology $\mathcal{O}_{]a,b[}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Let $]a', b'[$ be equipped with the subspace topology $\mathcal{O}_{]a',b'[}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



By Example 7.3.3 and Remark E7.1.11, we have that $\left(]a, b[\,, \mathcal{O}_{]a,b[}\right)$ is homeomorphic to $\left(]a', b'[\,, \mathcal{O}_{]a',b'[}\right)$. In other words, we use the homeomorphism of Example 7.3.3 to construct a homeomorphism from $(]a, b[\,, \mathcal{O}_{]a,b[})$ to $(]a', b'[\,, \mathcal{O}_{]a',b'[})$ in two steps.

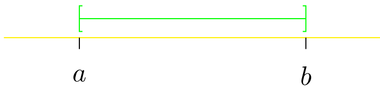


Remark 7.3.5. The technique of Example 7.3.4 and Example 7.3.4 is a good one to keep in mind when trying to prove that a pair of topological spaces are homeomorphic.

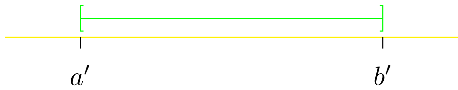
(1) Look for an intermediate special case, which in this case is where one of the topological spaces is $\left(]0, 1[\,, \mathcal{O}_{]0,1[}\right)$, for which we can explicitly write down a homeomorphism without too much difficulty.

(2) Apply a ‘machine’, which in this case is the fact that we can compose and invert homeomorphisms, to achieve our original goal.

Example 7.3.6. Suppose that $a, b \in \mathbb{R}$, and that $a < b$. Suppose also that $a', b' \in \mathbb{R}$, and that $a' < b'$. Let $[a, b]$ be equipped with the subspace topology $\mathcal{O}_{[a,b]}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

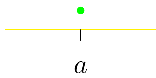


Let $[a', b']$ be equipped with the subspace topology $\mathcal{O}_{[a',b']}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

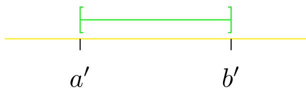


Then $\left([a, b], \mathcal{O}_{[a,b]}\right)$ is homeomorphic to $\left([a', b'], \mathcal{O}_{[a',b']}\right)$. Intuitively, we can stretch or shrink, and translate, $[a, b]$ to obtain $[a, b]$. To be rigorous, we can argue in exactly the same way as in Example 7.3.4 and Example 7.3.4, with the unit interval $(I, \mathcal{O}_I)$ as the intermediate special case.

Remark 7.3.7. The assumption that $a < b$ and $a' < b'$ is crucial in Example 7.3.6. Let $a \in \mathbb{R}$. Let $\{a\} = [a, a]$ be equipped with the subspace topology $\mathcal{O}_{\{a\}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Suppose that $a', b' \in \mathbb{R}$, and that $a' < b'$.



We have the following.

(1) A homeomorphism is in particular a bijection, as observed in Task E7.3.1.

(2) There is no bijective map

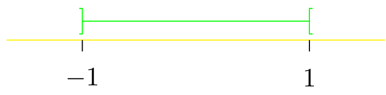
$$\{a\} \longrightarrow [a', b'] .$$

To check that you understand this is Task E7.2.2.

We conclude that $\left(\{a\}, \mathcal{O}_{\{a\}}\right)$ is not homeomorphic to $\left([a', b'], \mathcal{O}_{[a',b']}\right)$. Can you see where the argument of Example 7.3.4 breaks down? This is Task E7.2.3.

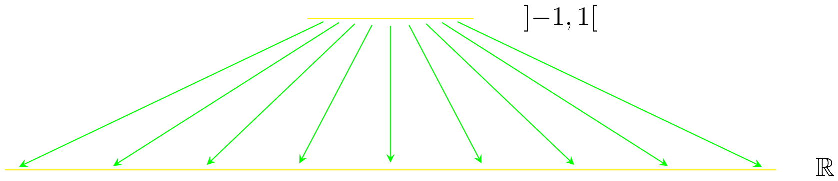
Remark 7.3.8. In a nutshell, we can shrink a closed interval to a closed interval which has as small a strictly positive length as we wish, but not to a point.

Example 7.3.9. Let the open interval $]-1,1[$ be equipped with the subspace topology $\mathcal{O}_{]-1,1[}$ with respect to $(\mathbb{R},\mathcal{O}_{\mathbb{R}})$.



Then $\left(]-1,1[\,,\mathcal{O}_{]-1,1[}\right)$ is homeomorphic to $(\mathbb{R},\mathcal{O}_{\mathbb{R}})$. Intuitively, think a cylindrical piece of dough. The dough can be worked in such a way that the cylinder becomes a longer and longer piece of spaghetti. We can think of open intervals in topology in a similar way!

With dough, our piece of spaghetti would eventually snap, but the mathematical dough of which an open interval is made can be stretched as much as we like, to the end of time! If we 'wait long enough', our mathematical piece of spaghetti will be longer than the distance between any pair of real numbers! A way to visualise this is depicted below.



To be rigorous, the map

$$]-1,1[\xrightarrow{\;f\;}\mathbb{R}$$

given by $t\mapsto\frac{t}{1-|t|}$ is a homeomorphism. For the following hold.

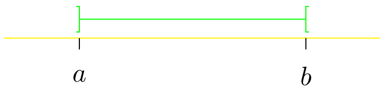
(1) We have that f is continuous. To prove this is the topic of Task E7.2.4.

(2) Let

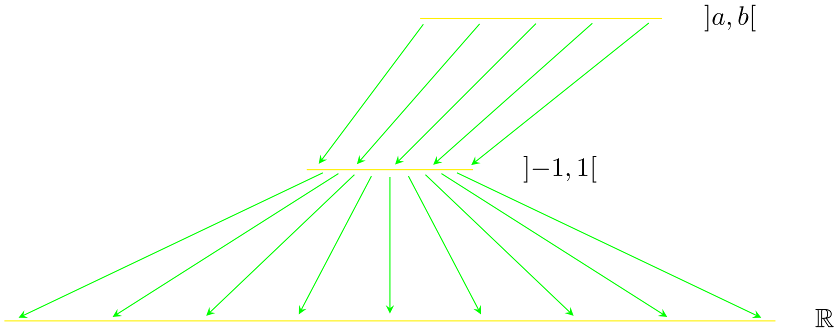
$$\mathbb{R}\xrightarrow{\;g\;}]-1,1[$$

be the map given by $t\mapsto\frac{t}{1+|t|}$. We have that g is continuous. To prove this is the topic of Task E7.2.5. Moreover we have that $g(f(t))=t$ for all $t\in\,]-1,1[$, so that $g\circ f=id_{]-1,1[}$. In addition we have that $f(g(t))=t$ for all $t\in\mathbb{R}$, so that $f\circ g=id_{\mathbb{R}}$. To prove the last two statements is the topic of Task E7.2.6.

Example 7.3.10. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Let $\mathcal{O}_{]a,b[}$ denote the subspace topology on $]a,b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

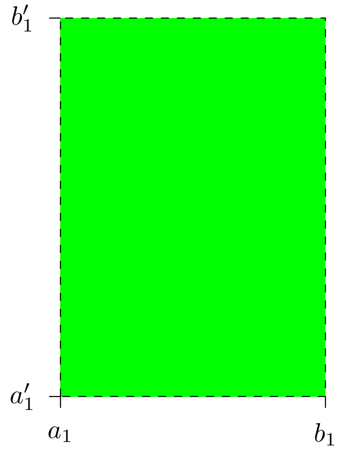


By Example 7.3.4, Example 7.3.9 and Remark E7.1.11, we have that $(]a,b[\,, \mathcal{O}_{]a,b[})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Following the technique described in Remark 7.3.5, we use the homeomorphisms of Example 7.3.4 and Example 7.3.9 to construct a homeomorphism from $(]a,b[\,, \mathcal{O}_{]a,b[})$ to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ in two steps.



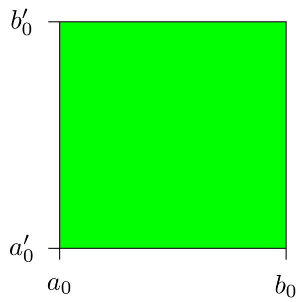
Example 7.3.11. Let $a_0, b_0, a_0', b_0' \in \mathbb{R}$ be such that $a_0 < b_0$ and $a_0' < b_0'$. Let X_0 be the 'open rectangle' given by $]a_0, b_0[\times]a_0', b_0'[$, equipped with the subspace topology $\mathcal{O}_{X_0}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let $a_1, b_1, a_1', b_1' \in \mathbb{R}$ be such that $a_1 < b_1$ and $a_1' < b_1'$. Let X_1 be the 'open rectangle' given by $]a_1, b_1[\times]a_1', b_1'[$, equipped with the subspace topology $\mathcal{O}_{X_1}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

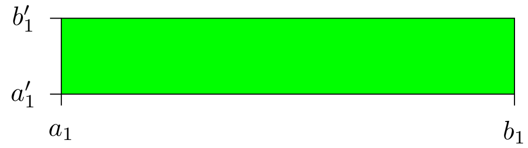


By Example 7.3.4, we have that $\left(]a_0, b_0[\,, \mathcal{O}_{]a_0,b_0[}\right)$ is homeomorphic to $\left(]a_1, b_1[\,, \mathcal{O}_{]a_1,b_1[}\right)$, and that $\left(]a'_0, b'_0[\,, \mathcal{O}_{]a'_0,b'_0[}\right)$ is homeomorphic to $\left(]a'_1, b'_1[\,, \mathcal{O}_{]a'_1,b'_1[}\right)$. By Task E7.1.14, we deduce that $(X_0, \mathcal{O}_{X_0})$ is homeomorphic to $(X_1, \mathcal{O}_{X_1})$.

Example 7.3.12. Let $a_0, b_0, a'_0, b'_0 \in \mathbb{R}$ be such that $a_0 < b_0$ and $a'_0 < b'_0$. Let X_0 be the 'closed rectangle' given by $[a_0, b_0] \times [a'_0, b'_0]$, equipped with the subspace topology $\mathcal{O}_{X_0}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

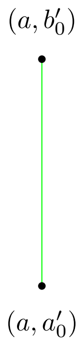


Let $a_1, b_1, a'_1, b'_1 \in \mathbb{R}$ be such that $a_1 < b_1$ and $a'_1 < b'_1$. Let X_1 be the 'closed rectangle' given by $[a_1, b_1] \times [a'_1, b'_1]$, equipped with the subspace topology $\mathcal{O}_{X_1}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

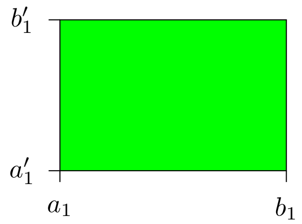


By Example 7.3.6, we have that $\left([a_0, b_0]\,, \mathcal{O}_{[a_0,b_0]}\right)$ is homeomorphic to $\left([a_1, b_1]\,, \mathcal{O}_{[a_1,b_1]}\right)$, and that $\left([a'_0, b'_0]\,, \mathcal{O}_{[a'_0,b'_0]}\right)$ is homeomorphic to $\left([a'_1, b'_1]\,, \mathcal{O}_{]a'_1,b'_1[}\right)$. By Task E7.1.14, we deduce that $(X_0, \mathcal{O}_{X_0})$ is homeomorphic to $(X_1, \mathcal{O}_{X_1})$.

Remark 7.3.13. As in Remark 7.3.7, it is crucial in Example 7.3.12 that the inequalities are strict. For instance, let $a, a'_0, b'_0 \in \mathbb{R}$ be such that $a'_0 < b'_0$. Let X_0 be the line $\{a\} \times [a'_0, b'_0]$, equipped with the subspace topology $\mathcal{O}_{X_0}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

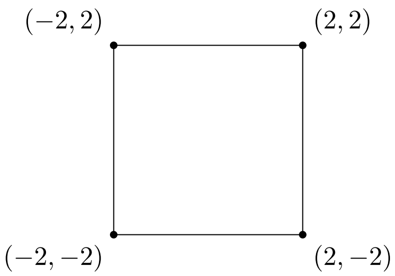


Let $a_1, b_1, a'_1, b'_1 \in \mathbb{R}$ be such that $a_1 < b_1$ and $a'_1 < b'_1$. Let X_1 be the 'closed rectangle' given by $[a_1, b_1] \times [a'_1, b'_1]$, equipped with the subspace topology $\mathcal{O}_{X_1}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Then $(X_0, \mathcal{O}_{X_0})$ is not homeomorphic to $(X_1, \mathcal{O}_{X_1})$. We cannot prove this yet, but we shall be able to soon, after we have studied *connectedness*.

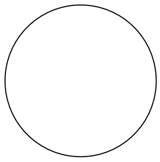
Example 7.3.14. Let X be the square depicted below, consisting of just the four lines, with no 'inside'.



In other words, X is given by

$$(\{-2, 2\} \times [-2, 2]) \cup ([-2, 2] \times \{-2, 2\}).$$

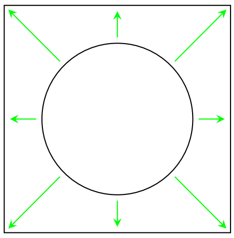
Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Then $(X, \mathcal{O}_X)$ is homeomorphic to the circle $(S^1, \mathcal{O}_{S^1})$.



A way to construct a homeomorphism

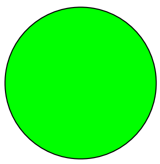
$$S^1 \xrightarrow{f} X$$

is to send each $x \in S^1$ to the unique $y \in X$ such that $y = kx$, where $k \in \mathbb{R}$ has the property that $k \geq 0$. To rigorously write down the details is the topic of Task E7.2.7.

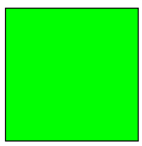


Remark 7.3.15. Think of a circular piece of string on a table. Even without stretching it, you could manipulate it so that it becomes a square!

Example 7.3.16. A similar argument to that of Example 7.3.14 demonstrates that the unit disc $(D^2, \mathcal{O}_{D^2})$



is homeomorphic to the unit square $(I^2, \mathcal{O}_{I^2})$.



To prove this is the topic of Task E7.2.9.

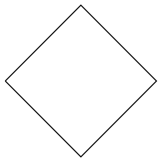
Example 7.3.17. Let Y denote the union of the set

$$\left\{(x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 0 \text{ and } |y| = 1 + x\right\}$$

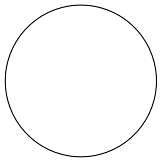
and the set

$$\left\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ and } |y| = 1 - x\right\}.$$

Let $\mathcal{O}_Y$ denote the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



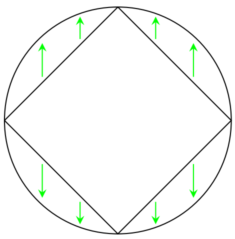
Then $(Y, \mathcal{O}_Y)$ is homeomorphic to the circle $(S^1, \mathcal{O}_{S^1})$.



A way to construct a homeomorphism

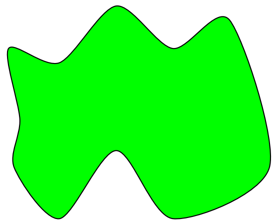
$$Y \xrightarrow{f} S^1$$

is to send each $(x, y_0) \in Y$ to the unique $(x, y_1) \in S^1$ such that $y_1 = k y_0$, where $k \in \mathbb{R}$ has the property that $k \geq 0$. To rigorously write down the details is the topic of Task **??**.



Remark 7.3.18. By Remark E7.1.11, we have that the topological space $(X, \mathcal{O}_X)$ of Example 7.3.14 is homeomorphic to the topological space $(Y, \mathcal{O}_Y)$ of Example 7.3.17, since both are homeomorphic to $(S^1, \mathcal{O}_{S^1})$. To prove this in a different way is the topic of Task 7.3.17.

Example 7.3.19. Let X be a 'blob' in $\mathbb{R}^2$, equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Then $(X, \mathcal{O}_X)$ is homeomorphic to the unit square $(I^2, \mathcal{O}_{I^2})$. If X were made of dough, it would be possible to knead it to obtain a square! To rigorously prove that $(X, \mathcal{O}_X)$ is homeomorphic to $(I^2, \mathcal{O}_{I^2})$ is the topic of Task **??**.

Remark 7.3.20. In Task **??**, we shall not explicitly describe a subset of $\mathbb{R}^2$ such as the 'blob' above. We shall work a little more abstractly, with subsets of $\mathbb{R}^2$ which can be 'cut into star shaped pieces'. Here 'star shaped' has a technical meaning, discussed before Task **??**.

E7 Exercises for Lecture 7

E7.1 Exam questions

Task E7.1.1. Let $X = \{a, b, c, d\}$ be a set with four elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}.$$

Let $Y = \{1, 2, 3, 4\}$ be a set with four elements. Let $\mathcal{O}_Z$ be the topology on Z given by

$$\{\emptyset, \{2\}, \{3\}, \{2, 3\}, \{1, 3, 4\}, Z\}.$$

Let

$$X \xrightarrow{f} Y$$

be the map given by $a \mapsto 3$, $b \mapsto 1$, $c \mapsto 2$, and $d \mapsto 4$. Is f a homeomorphism?

Task E7.1.2. Let $X = \{a, b, c\}$ be a set with three elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b, c\}, X\}.$$

Let $Y = \{a', b'\}$ be a set with two elements. Let $\mathcal{O}_Y$ be the topology on Y given by

$$\left\{\emptyset, \{a'\}, Y\right\}.$$

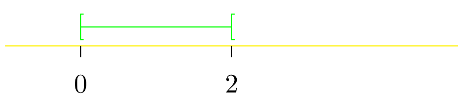
Let $Z = \{1, 2, \ldots, 6\}$ be a set with six elements. Let $\mathcal{O}_Z$ be the topology on Z given by

$$\{\emptyset, \{2\}, \{2, 5\}, \{1, 4\}, \{1, 3, 4, 6\}, \{1, 2, 4\}, \{1, 2, 4, 5\}, \{1, 2, 3, 4, 6\}, Z\}.$$

Let $X \times Y$ be equipped with the product topology $\mathcal{O}_{X \times Y}$. Find a homeomorphism

$$X \times Y \xrightarrow{f} Z.$$

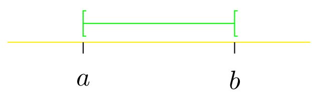
Task E7.1.3. Let $[0, 2[$ be equipped with the subspace topology $\mathcal{O}_{[0,2[}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Let $]3,4]$ be equipped with the subspace topology $\mathcal{O}_{]3,4]}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Prove that $\left([0,2[\,, \mathcal{O}_{[0,2[}\right)$ is homeomorphic to $\left(]3,4]\,, \mathcal{O}_{]3,4]}\right)$.

Task E7.1.4. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Suppose that $a', b' \in \mathbb{R}$, and that $a' < b'$. Let $\mathcal{O}_{[a,b[}$ be the subspace topology on $[a,b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Let $\mathcal{O}_{[a',b'[}$ be the subspace topology on $]a',b']$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Generalise your argument from Task E7.1.3 to prove that $\left([a,b[\,, \mathcal{O}_{[a,b[}\right)$ is homeomorphic to $\left(]a',b']\,, \mathcal{O}_{]a',b]}\right)$.

Task E7.1.5. Suppose that a belongs to $\mathbb{R}$. Let $\mathcal{O}_{]a,\infty[}$ denote the subspace topology on $]a,\infty[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Prove that $(]a,\infty[\,, \mathcal{O}_{]a,\infty[})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. You may wish to proceed as follows.

(1) Let

$$]0,1[\xrightarrow{f}]a,\infty[$$

be the map given by $x \mapsto a + \frac{x}{1-x}$. Demonstrate that f is a homeomorphism. You may wish to appeal to Task E5.3.15.

(2) By Task E7.3.2, deduce that there is a homeomorphism

$$]a, \infty[\longrightarrow]0,1[\,.$$

By Example 7.3.10, there is a homeomorphism

$$]0,1[\longrightarrow \mathbb{R}.$$

By Task E7.1.10, conclude that there is a homeomorphism

$$]a, \infty[\longrightarrow \mathbb{R}.$$

Task E7.1.6. Suppose that b belongs to $\mathbb{R}$. Let $\mathcal{O}_{]-\infty,b[}$ denote the subspace topology on $]-\infty, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Prove that $(]-\infty, b[\,, \mathcal{O}_{]-\infty,b[})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Task E7.1.7. Let $k \in \mathbb{R}$, and let $c \in \mathbb{R}$. Let $L_{k,c}$ be the set given by

$$\left\{(x, y) \in \mathbb{R}^2 \mid y = kx + c\right\}.$$

Let $\mathcal{O}_{L_{k,c}}$ denote the subspace topology on $L_{k,c}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Prove that $(L_{k,c}, \mathcal{O}_{L_{k,c}})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

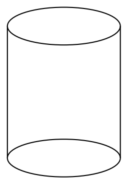
Task E7.1.8. Let $k \in \mathbb{R}$, and let $c \in \mathbb{R}$. Let $a, b \in \mathbb{R}$ be such that $a < b$. Let $L_{k,c}^{[a,b]}$ be the set given by

$$\left\{(x, y) \in \mathbb{R}^2 \mid y = kx + c \text{ and } a \leq x \leq b\right\}.$$

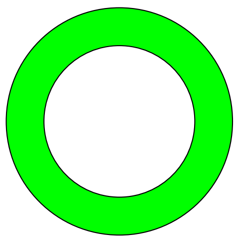
Let $\mathcal{O}_{L_{k,c}^{[a,b]}}$ denote the subspace topology on $L_{k,c}^{[a,b]}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Prove that $\left(L_{k,c}^{[a,b]}, \mathcal{O}_{L_{k,c}^{[a,b]}}\right)$ is homeomorphic to $(I, \mathcal{O}_I)$. You may quote without proof anything from the lectures, and any of the other tasks.

Task E7.1.9. Find an intuitive argument to demonstrate that the cylinder $(S^1 \times I, \mathcal{O}_{S^1 \times I})$



is homeomorphic to an annulus $(A_k, \mathcal{O}_{A_k})$, where $0 < k < 1$.



Can you find a way to give a rigorous proof, along the lines of your intuitive argument?

Task E7.1.10. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ be topological spaces. Let

$$X \xrightarrow{f_0} Y$$

and

$$Y \xrightarrow{f_1} Z$$

be homeomorphisms. Prove that

$$X \xrightarrow{f_1 \circ f_0} Z$$

is a homeomorphism.

Remark E7.1.11. Together with Task E7.3.2, it follows that if any two of $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ are homeomorphic, then each is homeomorphic to the other two.

Remark E7.1.12. In other words, the relation on the set of topological spaces given by $(X_0, \mathcal{O}_{X_0}) \sim (X_1, \mathcal{O}_{X_1})$ if $(X_0, \mathcal{O}_{X_0})$ is homeomorphic to $(X_1, \mathcal{O}_{X_1})$ is an equivalence relation.

Remark E7.1.13. If it worries you, we do have to be careful about how we make sense of something as large as the set of topological spaces. This is a foundational matter which can be addressed in many different ways, and which we can harmlessly ignore!

Task E7.1.14. Let $(X_0, \mathcal{O}_{X_0})$, $(X_1, \mathcal{O}_{X_1})$, $(Y_0, \mathcal{O}_{Y_0})$, and $(Y_1, \mathcal{O}_{Y_1})$ be topological spaces. Let

$$X_0 \xrightarrow{f_0} Y_0$$

and

$$X_1 \xrightarrow{f_1} Y_1$$

be homeomorphisms. Prove that the map

$$X_0 \times X_1 \xrightarrow{f_0 \times f_1} Y_0 \times Y_1$$

given by $(x_0, x_1) \mapsto (f_0(x_0), f_1(x_1))$ is a homeomorphism.

Definition E7.1.15. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is *open* if, for every subset U of X which belongs to $\mathcal{O}_X$, we have that $f(U)$ belongs to $\mathcal{O}_Y$.

Task E7.1.16. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Prove that f is open.

Definition E7.1.17. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is *closed* if, for every subset V of X which is closed with respect to $\mathcal{O}_X$, we have that $f(V)$ is closed with respect to $\mathcal{O}_Y$.

Task E7.1.18. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Prove

$$X \xrightarrow{f} Y$$

is closed.

Task E7.1.19. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that

$$X \xrightarrow{f} Y$$

is a homeomorphism. Let A be a subset of X. Let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Let $f(A)$ be equipped with the subspace topology $\mathcal{O}_{f(A)}$ with respect to $(Y, \mathcal{O}_Y)$. Prove that $(A, \mathcal{O}_A)$ is homeomorphic to $\big(f(A), \mathcal{O}_{f(A)}\big)$.

Task E7.1.20. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that

$$X \xrightarrow{f} Y$$

is a homeomorphism. Let A be a subset of X. Let $X \setminus A$ be equipped with the subspace topology $\mathcal{O}_{X \setminus A}$ with respect to $(X, \mathcal{O}_X)$. Let $Y \setminus f(A)$ be equipped with the subspace topology with respect to $(Y, \mathcal{O}_Y)$. Deduce from Task E7.1.19 that $\big(X \setminus A, \mathcal{O}_{X \setminus A}\big)$ is homeomorphic to $\big(Y \setminus f(A), \mathcal{O}_{Y \setminus f(A)}\big)$.

E7.2 In the lecture notes

Task E7.2.1. Let X and Y be sets. Prove that a map

$$X \xrightarrow{f} Y$$

is bijective in the sense of Definition 7.1.1 if and only if it is both injective and surjective.

Task E7.2.2. Let $X = \{x\}$ be a set with one element. Let Y be a set with at least two elements. Why can there not be a bijective map between X and Y? This was appealed to in Task 7.3.7.

Task E7.2.3. In the notation of Example 7.3.6, where does the analogue of the argument of Example 7.3.6 for closed intervals break down if we assume that $a = b$?

Task E7.2.4. Prove that the map

$$]-1,1[\xrightarrow{f} \mathbb{R}$$

given by $t \mapsto \frac{t}{1-|t|}$ is continuous. You may wish to proceed as follows.

(1) Prove that the map

$$]-1,1[\xrightarrow{g_1} \mathbb{R}$$

given by $t \mapsto 1-|t|$ is continuous. For this, you may wish to express g_1 as a composition of maps, allowing you to deduce continuity from Task E5.3.3and from Task E5.3.14.

(2) Taking g_0 to be the inclusion map

$$]-1,1[\longrightarrow \mathbb{R}$$

and g_1 to be the map of (1), deduce the continuity of f from Proposition 5.2.2, (1), and Task E5.3.10.

Task E7.2.5. Prove that the map

$$\mathbb{R} \xrightarrow{g}]-1,1[$$

given by $t \mapsto \frac{t}{1+|t|}$ is continuous. You may wish to proceed in a similar way as in Task E7.2.4.

Task E7.2.6. Let

$$]-1,1[\xrightarrow{f} \mathbb{R}$$

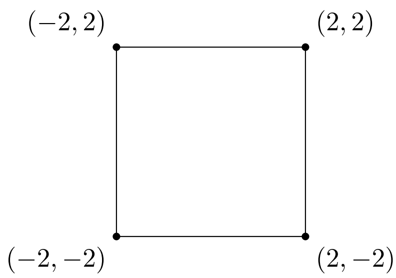
be the map of Task E7.2.4. Let

$$\mathbb{R} \xrightarrow{g}]-1,1[$$

be the map of Task E7.2.5. Prove that for all $t \in]-1,1[$ we have that $g(f(t)) = t$, and that for all $t \in \mathbb{R}$ we have that $f(g(t)) = t$.

Task E7.2.7. Let X be the square of Example 7.3.14, given by

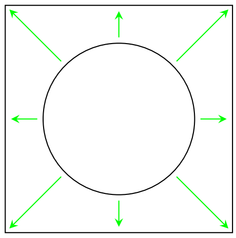
$$(\{-2,2\} \times [-2,2]) \cup ([-2,2] \times \{-2,2\}).$$



Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Construct a homeomorphism

$$S^1 \xrightarrow{f} X$$

in the manner indicated in Example 7.3.14.



You may wish to proceed as follows.

(1) Let A_{east} be the subset of S^1 given by

$$\left\{ (x, y) \in S^1 \mid x > 0 \text{ and } -\tfrac{1}{\sqrt{2}} \leq y \leq \tfrac{1}{\sqrt{2}}. \right\}.$$

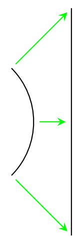
Let A_{east} be equipped with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$. Let B_{east} be the subset of X given by

$$\left\{ (2, y) \in \mathbb{R}^2 \mid -2 \leq y \leq 2 \right\}.$$

Let B_{east} be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Prove that the map

$$A_{\mathsf{east}} \xrightarrow{f_{\mathsf{east}}} B_{\mathsf{east}}$$

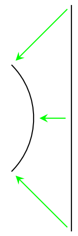
given by $(x, y) \mapsto \left(2, \frac{2y}{x}\right)$ is continuous. Quote any tasks which you appeal to.



(2) Prove that the map

$$B_{\mathsf{east}} \xrightarrow{g_{\mathsf{east}}} A_{\mathsf{east}}$$

given by $(x, y) \mapsto \frac{1}{\|(x,y)\|}(x, y)$ is continuous. In particular, quote any tasks which you appeal to.



(3) Verify that $g_{\mathsf{east}} \circ f_{\mathsf{east}} = id_{A_{\mathsf{east}}}$, and that $f_{\mathsf{east}} \circ g_{\mathsf{east}} = id_{B_{\mathsf{east}}}$. Conclude that f_{east} is a homeomorphism.

(4) Let A_{west} be the subset of S^1 given by

$$\left\{ (x, y) \in S^1 \mid x < 0 \text{ and } -\frac{1}{\sqrt{2}} \leq y \leq \frac{1}{\sqrt{2}} \right\}.$$

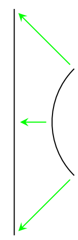
Let A_{west} be equipped with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$. Let B_{west} be the subset of X given by

$$\left\{(-2, y) \in \mathbb{R}^2 \mid -2 \leq y \leq 2\right\}.$$

Let B_{west} be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. As in (1) – (3), prove that the map

$$A_{\mathsf{west}} \xrightarrow{f_{\mathsf{west}}} B_{\mathsf{west}}$$

given by $(x, y) \mapsto \left(-2, \frac{2y}{x}\right)$ is a homeomorphism. Quote any tasks which you appeal to.



(5) Let A_{north} be the subset of S^1 given by

$$\left\{(x, y) \in S^1 \mid y > 0 \text{ and } -\tfrac{1}{\sqrt{2}} \leq x \leq \tfrac{1}{\sqrt{2}}\right\}.$$

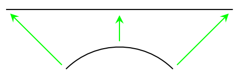
Let A_{north} be equipped with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$. Let B_{north} be the subset of X given by

$$\left\{(x, 2) \in \mathbb{R}^2 \mid -2 \leq x \leq 2\right\}.$$

Let B_{north} be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Along the lines of (1) – (3), prove that the map

$$A_{\mathsf{north}} \xrightarrow{f_{\mathsf{north}}} B_{\mathsf{north}}$$

given by $(x, y) \mapsto \left(\frac{2x}{y}, 2\right)$ is a homeomorphism. Quote any tasks which you appeal to.



(6) Let A_{south} be the subset of S^1 given by

$$\left\{(x, y) \in S^1 \mid y < 0 \text{ and } -\tfrac{1}{\sqrt{2}} \leq x \leq \tfrac{1}{\sqrt{2}}\right\}.$$

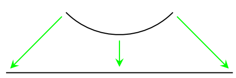
Let A_{north} be equipped with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$. Let B_{north} be the subset of X given by

$$\left\{(x, 2) \in \mathbb{R}^2 \mid -2 \leq x \leq 2\right\}.$$

Let B_{south} be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Along the lines of (1) – (3), prove that the map

$$A_{\mathsf{south}} \xrightarrow{f_{\mathsf{south}}} B_{\mathsf{south}}$$

given by $(x, y) \mapsto \left(\frac{2x}{y}, -2\right)$ is a homeomorphism. Quote any tasks which you appeal to.

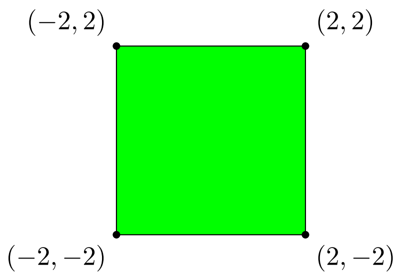


(7) Appeal to Task E7.3.6 three times to build a homeomorphism

$$S^1 \longrightarrow X$$

from the homeomorphisms f_{east}, f_{south}, f_{west}, and f_{north}.

Task E7.2.8. Let X be the subset $[-2,2] \times [-2,2]$ of $\mathbb{R}^2$, equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



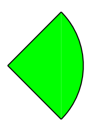
Construct a homeomorphism

$$D^2 \xrightarrow{f} X.$$

You may wish to proceed by adapting your argument from Task E7.2.7, in the following manner.

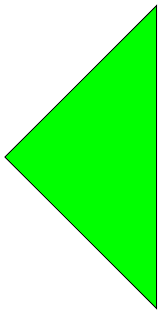
(1) Let A_{east} be the subset of D^2 given by

$$\left\{(x,y) \in D^2 \mid x > 0 \text{ and } -\tfrac{1}{\sqrt{2}} \leq y \leq \tfrac{1}{\sqrt{2}}\right\}.$$



Let A_{east} be equipped with the subspace topology with respect to $(D^2, \mathcal{O}_{D^2})$. Let B_{east} be the subset of X given by

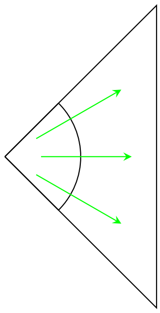
$$\left\{(x,y) \in X \mid x > 0 \text{ and } -\tfrac{1}{\sqrt{2}} \leq y \leq \tfrac{1}{\sqrt{2}}\right\}.$$



Let B_{east} be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Prove that the map

$$A_{\mathsf{east}} \xrightarrow{f_{\mathsf{east}}} B_{\mathsf{east}}$$

given by $(x, y) \mapsto \left(\frac{2}{\|(x,y)\|}, \frac{2y}{x} \right)$ is a homeomorphism. Quote any tasks which you appeal to.

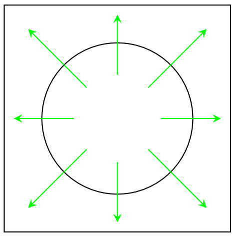


(2) Modify (4) – (6) of Task E7.2.7 in a similar way.

(3) Let $D^2 \backslash \{0\}$ be equipped with the subspace topology $\mathcal{O}_{D^2 \backslash \{0\}}$ with respect to $(D^2, \mathcal{O}_{D^2})$. Let $X \setminus \{0\}$ be equipped with the subspace topology $\mathcal{O}_{X \setminus \{0\}}$ with respect to $(X, \mathcal{O}_X)$. Appeal to Task E7.3.6 three times to build a homeomorphism

$$D^2 \setminus \{0\} \xrightarrow{f} X \setminus \{0\}$$

from the homeomorphisms f_{east}, f_{south}, f_{west}, and f_{north} of (1) and (2).



(4) By Task E7.3.9, deduce that the homeomorphism f of (3) gives rise to a homeomorphism

$$D^2 \longrightarrow X.$$

Task E7.2.9. Prove that $(D^2, \mathcal{O}_{D^2})$ is homeomorphic to $(I^2, \mathcal{O}_{I^2})$. You may wish to appeal to Task E7.2.8, Example 7.3.12, and Task E7.1.10.

E7.3 For a deeper understanding

Task E7.3.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that a map

$$X \xrightarrow{f} Y$$

is a homeomorphism if and only if f is bijective, continuous, and open.

Task E7.3.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. By definition of a homeomorphism, there is a continuous map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$. Prove that g is a homeomorphism.

Task E7.3.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homeomorphic topological spaces. Prove that there is a bijection between $\mathcal{O}_X$ and $\mathcal{O}_Y$.

Remark E7.3.4. In particular, if X and Y are finite sets such that $\mathcal{O}_X$ has a different cardinality to $\mathcal{O}_Y$, then $(X, \mathcal{O}_X)$ cannot be homeomorphic to $(Y, \mathcal{O}_Y)$.

Task E7.3.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\{A_j\}_{j \in J}$ be a set of subsets of X such that $X = \bigcup_{j \in J} A_j$. For every $j \in J$, let A_j be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Suppose that the following hold.

(1) For all j_0 and j_1 which belong to J, the restriction of f_{j_0} to $A_{j_0} \cap A_{j_1}$ is equal to the restriction of f_{j_1} to $A_{j_0} \cap A_{j_1}$.

(2) We have that A_j belongs to $\mathcal{O}_X$ for every j which belongs to J.

(3) For every j which belongs to J, we have a continuous map

$$A_j \xrightarrow{f_j} Y$$

such that the map

$$A_j \xrightarrow{f'_j} f(A_j)$$

given by $x \mapsto f_j(x)$ is a homeomorphism, where $f(A_j)$ is equipped with the subspace topology with respect to $(Y, \mathcal{O}_Y)$.

(4) Let

$$X \xrightarrow{f} Y$$

denote the map of Notation E5.3.22 corresponding to the maps $\{f_j\}_{j \in J}$. Suppose that f is bijective.

Prove that f is a homeomorphism. You may wish to proceed as follows.

(1) By (1) of Task E5.3.23, observe that f is continuous.

(2) Suppose that j belongs to J. Since f'_j is a homeomorphism, there is a continuous map

$$f(A_j) \xrightarrow{g'_j} A_j$$

such that $g'_j \circ f'_j = id_{A_j}$ and $f'_j \circ g'_j = id_{f(A_j)}$. Let

$$f(A_j) \xrightarrow{g_j} X$$

be the map given by $y \mapsto g'_j(y)$. By Task E5.1.10, observe that g_j is continuous.

(3) Suppose that j_0 and j_1 belong to J. We have that

$$f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1}) = f(A_{j_0}) \cap f(A_{j_1}).$$

Since f is bijective, we have that

$$f(A_{j_0}) \cap f(A_{j_1}) = f(A_{j_0} \cap A_{j_1}).$$

By definition of f, we have that

$$f(A_{j_0} \cap A_{j_1}) = f_{j_0}(A_{j_0} \cap A_{j_1})$$

and that

$$f(A_{j_0} \cap A_{j_1}) = f_{j_1}(A_{j_0} \cap A_{j_1}).$$

Thus we have that

$$f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1}) = f_{j_0}(A_{j_0} \cap A_{j_1})$$

and that

$$f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1}) = f_{j_1}(A_{j_0} \cap A_{j_1}).$$

Deduce that the restriction of g_{j_0} to $f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1})$ is equal to the restriction of g_{j_0} to $f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1})$. Let

$$Y \xrightarrow{\;g\;} X$$

be the map of Notation E5.3.22 corresponding to the maps $\{g_j\}_{j \in J}$.

(4) By Task E7.1.16, observe that $f_j(A_j)$ belongs to $\mathcal{O}_Y$.

(5) By (1) of Task E5.3.23, deduce from (2) and (4) that g is continuous.

(6) Observe that $g \circ f = id_X$, and that $f \circ g = id_Y$.

Task E7.3.6. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, $\{A_j\}_{j \in J}$, $\{f_j\}_{j \in J}$, and f be as in Task E7.3.5, except that instead of assuming that A_j belongs to $\mathcal{O}_X$ for every $j \in J$, suppose that $\{A_j\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_X$, and that A_j is closed with respect to $\mathcal{O}_X$ for every $j \in J$. Suppose that $\{f(A_j)\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_Y$. Prove that f is a homeomorphism. You may wish to proceed as follows.

(1) By (2) of Task E5.3.23, observe that f is continuous.

(2) Define

$$f(A_j) \xrightarrow{\;g_j\;} Y$$

as in (2) of Task E7.3.5. By Task E5.1.10, observe that g_j is continuous.

(3) As in (3) of Task E7.3.5, demonstrate that the restriction of g_{j_0} to $f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1})$ is equal to the restriction of g_{j_0} to $f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1})$. Let

$$Y \xrightarrow{g} X$$

be the map of Notation E5.3.22 corresponding to the maps $\{g_j\}_{j \in J}$.

(4) By Task E7.1.18, observe that $f_j(A_j)$ is closed in Y with respect to $\mathcal{O}_Y$.

(5) By (2) of Task E5.3.23, deduce from (2), (4), and our assumption that $\{f(A_j)\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_Y$, that g is continuous.

(6) As in (6) of Task E7.3.5, observe that $g \circ f = id_X$, and that $f \circ g = id_Y$.

Task E7.3.7. Let $\mathcal{O}_{[0,1[}$ be the subspace topology on $[0,1[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

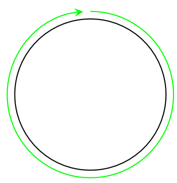
Let

$$[0,1[\xrightarrow{f} S^1$$

be the map given by $t \mapsto \phi(t)$, where

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is the map of Task E5.3.27.



Prove that f is a continuous bijection. Find a set $\{A_j\}_{j \in J}$ of subsets of $[0,1[$ with the following properties.

(1) We have that $\{A_j\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_{[0,1[}$.

(2) For every j which belongs to J, we have that A_j is closed in $[0,1[$ with respect to $\mathcal{O}_{[0,1[}$.

(3) Suppose that j belongs to J. Let A_j be equipped with the subspace topology with respect to $\big([0,1[\,, \mathcal{O}_{[0,1[}\big)$. Let $f(A_j)$ be equipped with the subspace topology with respect to $\big(S^1, \mathcal{O}_{S^1}\big)$. Then the map

$$A_j \xrightarrow{f_j} f(A_j)$$

given by $t \mapsto f(t)$ is a homeomorphism.

(4) We have that $\{f(A_j)\}_{j \in J}$ is not locally finite.

Remark E7.3.8. In Task E11.3.2, you are asked to prove that f is not homeomorphism.

Task E7.3.9. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that x belongs to X, and that $\{x\}$ is closed in X with respect to $\mathcal{O}_X$. Let $X \setminus \{x\}$ be equipped with the subspace topology $\mathcal{O}_{X \setminus \{x\}}$ with respect to $(X, \mathcal{O}_X)$. Let

$$X \xrightarrow{f} Y$$

be a bijective map. Let $Y \setminus f(x)$ be equipped with the subspace topology $\mathcal{O}_{Y \setminus \{f(x)\}}$ with respect to $(Y, \mathcal{O}_Y)$. Suppose that the map

$$X \setminus \{x\} \xrightarrow{g} Y \setminus \{f(x)\}$$

given by $x' \mapsto f(x')$ is a homeomorphism. Prove that f is a homeomorphism. You may wish to appeal to Task E5.3.29.

Task E7.3.10. As in Example 6.3.1, let $\sim$ be the equivalence relation on I generated by $0 \sim 1$.

Prove that $(I/\sim, \mathcal{O}_{I/\sim})$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$.

You may wish to proceed as follows.

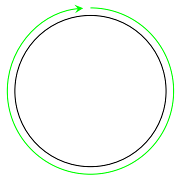
(1) Let

$$I \xrightarrow{\phi'} S^1$$

be the map given by $t \mapsto \phi(t)$, where

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is the map of Notation E5.3.26. By Task E5.3.27 and Task E5.2.3, observe that ϕ' is continuous.



(2) Observe that $\phi'(0) = \phi'(1)$. By Task E6.2.7, deduce that the map

$$I/\sim \xrightarrow{f} S^1$$

given by $[t] \mapsto \phi'(t)$ is continuous.

(3) Let A_0 be the set given by

$$\left\{(x, y) \in S^1 \mid x \geq 0\right\}.$$

Let A_0 be equipped with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$. Appealing to Task E2.3.1, Proposition 5.4.3, Task E5.3.14, and Proposition 5.3.1, observe that the map

$$A_0 \longrightarrow I$$

given by $(x, y) \mapsto -\frac{y}{4} + \frac{1}{4}$ is continuous.

(4) Let

$$I \xrightarrow{\pi} I/\sim$$

denote the quotient map. By Remark 6.1.9 and Proposition 5.3.1, deduce from (3) that the map

$$A_0 \xrightarrow{g_0} I/{\sim}$$

given by $(x, y) \mapsto [-\frac{y}{4} + \frac{1}{4}]$ is continuous.

(5) Let A_1 be the set given by

$$\left\{(x, y) \in S^1 \mid x \leq 0\right\}.$$

Let A_1 be equipped with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$. Appealing to Task E2.3.1, Proposition 5.4.3 and Task E5.3.14, observe that the map

$$A_1 \longrightarrow I$$

given by $(x, y) \mapsto \frac{y}{4} + \frac{3}{4}$ is continuous.

(6) By Remark 6.1.9 and Proposition 5.3.1, deduce from (5) that the map

$$A_1 \xrightarrow{g_1} I/{\sim}$$

given by $(x, y) \mapsto [\frac{y}{4} + \frac{3}{4}]$ is continuous.

(7) Let

$$S^1 \xrightarrow{g} I/{\sim}$$

denote the map given by

$$(x, y) \mapsto \begin{cases} g_0(x, y) & \text{if } (x, y) \text{ belongs to } A_0, \\ g_1(x, y) & \text{if } (x, y) \text{ belongs to } A_1. \end{cases}$$

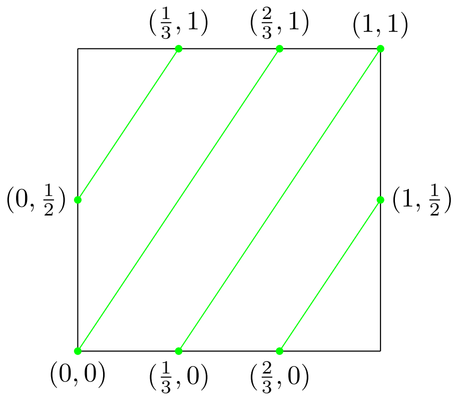
By (2) of Task E5.3.23, deduce from (4) and (6) that g is continuous.

(8) Observe that $g \circ f = id_{I/\sim}$, and that $f \circ g = id_{S^1}$.

(9) Conclude by (2), (7), and (8) that f is a homeomorphism.

E7.4 Exploration — torus knots

Task E7.4.1. Let K be the subset of T^2 of Task E6.3.1.



Let $\pi(K)$ be equipped with the subspace topology $\mathcal{O}_{\pi(K)}$ with respect to $(T^2, \mathcal{O}_{T^2})$. Prove that $\big(\pi(K), \mathcal{O}_{\pi(K)}\big)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$.