MA3002 Generell Topologi — Vår 2014

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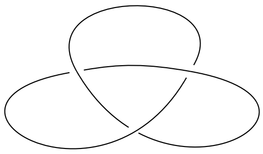
8 Tuesday 28th January

8.1 Further geometric examples of homeomorphisms

Example 8.1.1. Let K be a subset of $\mathbb{R}^3$ such as the following.

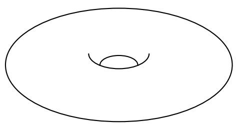


Let $\mathcal{O}_K$ denote the subspace topology on K with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$. Then $(K, \mathcal{O}_K)$ is an example of a *knot*. We have that $(K, \mathcal{O}_K)$ is homeomorphic to $(S^1, \mathcal{O}_S^1)$.

Remark 8.1.2. The crucial point is that both K and a circle can be obtained from a piece of string by glueing together the ends together. We may bend, twist, and stretch the string as much as we wish before we glue the ends together.

Remark 8.1.3. We shall explore knot theory later in the course.

Example 8.1.4. We have that $(T^2, \mathcal{O}_{T^2})$ is homeomorphic to $(S^1 \times S^1, \mathcal{O}_{S^1 \times S^1})$.

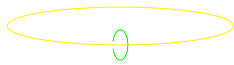


To prove this is the topic of Task E8.2.1.

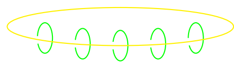
Remark 8.1.5. We can think of the left copy of S^1 in $S^1 \times S^1$ as the circle depicted below.

Suppose that x belongs to S^1.

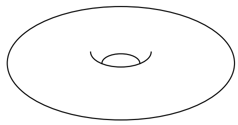
We can think of $\{x\} \times S^1$ as a circle around x.



In this way, we can think $S^1 \times S^1 = \bigcup_{x \in S^1} \{x\} \times S^1$ as a 'circle of circles'.



A 'circle of circles' is intuitively exactly a torus.



8.2 Neighbourhoods

Definition 8.2.1. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. A *neighbourhood* of x in X with respect to $\mathcal{O}_X$ is a subset U of X such that x belongs to U, and such that U belongs to $\mathcal{O}_X$.

In other references, you may see a neighbourhood U of x defined simply to be a subset of X to which x belongs, without the requirement that U belongs to $\mathcal{O}_X$.

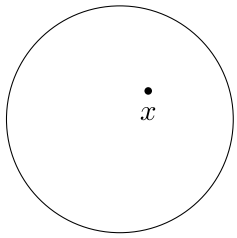
Example 8.2.2. Let $X = \{a, b, c, d\}$ be a set with four elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X\}.$$

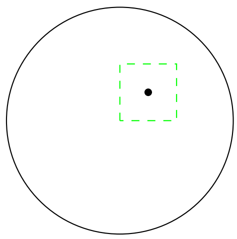
Here is a list of the neighbourhoods in X with respect to $\mathcal{O}_X$ of the elements of X.

Element	Neighbourhoods
a	$\{a\}$, $\{a,b\}$, $\{a,d\}$, $\{a,b,d\}$, $\{a,c,d\}$, X
b	$\{b\}$, $\{a,b\}$, $\{b,d\}$, $\{a,b,d\}$, $\{b,c,d\}$, X
c	$\{c,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, X
d	$\{d\}$, $\{a,d\}$, $\{b,d\}$, $\{c,d\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, X

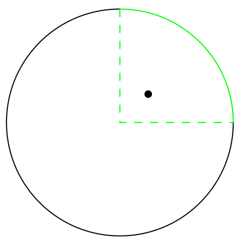
Example 8.2.3. Suppose that x belongs to D^2. For instance, we can take x to be $\left(\frac{1}{4}, \frac{1}{4}\right)$.



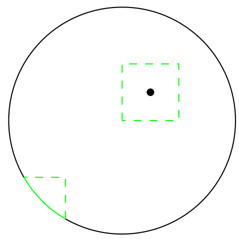
A typical example of a neighbourhood of x in D^2 with respect to $\mathcal{O}_{D^2}$ is a subset U of D^2 which is an 'open rectangle', and to which x belongs. When x is $\left(\frac{1}{4}, \frac{1}{4}\right)$, we can, for instance, take U to be $\left]0, \frac{1}{2}\right[\times \left]0, \frac{1}{2}\right[$.



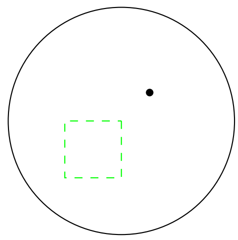
We could also take the intersection U with D^2 of any open rectangle in $\mathbb{R}^2$ to which x belongs. By definition of $\mathcal{O}_{D^2}$, we have that U belongs to $\mathcal{O}_{D^2}$. For instance, when x is $\left(\frac{1}{4}, \frac{1}{4}\right)$, we can take U to be the intersection with D^2 of $]0, 1[\times]0, 1[$.



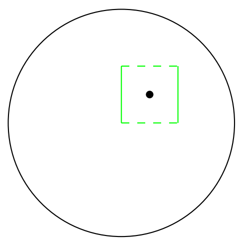
A disjoint union $U_0 \cup U_1$ of a pair of subsets of D^2 which both belong to $\mathcal{O}_{D^2}$, with the property that x belongs to either U_0 or U_1, is also a neighbourhood of x in D^2 with respect to $\mathcal{O}_{D^2}$. For $U_0 \cup U_1$ belongs to $\mathcal{O}_{D^2}$, and x belongs to $U_0 \cup U_1$. When x is $\left(\frac{1}{4}, \frac{1}{4}\right)$, we can for instance take U_0 to be $\left]0, \frac{1}{2}\right[\times \left]0, \frac{1}{2}\right[$, and take U_1 to be the intersection with D^2 of $\left]-1, -\frac{1}{2}\right[\times \left]-1, -\frac{1}{2}\right[$.



A subset of D^2 to which x does not belong is not a neighbourhood of x in D^2 with respect to $\mathcal{O}_{D^2}$, even if it belongs to $\mathcal{O}_{D^2}$. When x is $\left(\frac{1}{4}, \frac{1}{4}\right)$, the subset $\left]-\frac{1}{2}, 0\right[\times \left]-\frac{1}{2}, 0\right[$ is not a neighbourhood of x, for instance.



A subset of D^2 to which x belongs, but which does not belong to $\mathcal{O}_{D^2}$, is not a neighbourhood of x in D^2 with respect to $\mathcal{O}_{D^2}$. When x is $\left(\frac{1}{4}, \frac{1}{4}\right)$, the subset $\left]0, \frac{1}{2}\right[\times \left[0, \frac{1}{2}\right]$ is not a neighbourhood of x, for instance.



8.3 Limit points

Definition 8.3.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Suppose that x belongs to X. Then x is a *limit point* of A in X with respect to $\mathcal{O}_X$ if, for every neighbourhood U of x in X with respect to $\mathcal{O}_X$, there is an $a \in U$ such that a belongs to A.

Remark 8.3.2. In other words, x is a limit point of A in X with respect to $\mathcal{O}_X$ if and only if for every neighbourhood U of x in X with respect to $\mathcal{O}_X$, we have that $A \cap U \neq \emptyset$.

Remark 8.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Suppose that a belongs to A. Then a is a limit point of A in X with respect to $\mathcal{O}_X$, since every neighbourhood of a in X with respect to $\mathcal{O}_X$ contains a.

8.4 Examples of limit points

Example 8.4.1. Let $X = \{a, b\}$ be a set with two elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{b\}, X\}.$$

Let $A = \{b\}$. By Remark 8.3.3, we have that b is a limit point of A in X with respect to $\mathcal{O}_X$. Moreover, a is a limit point of A in X with respect to $\mathcal{O}_X$. For the only neighbourhood of a in X with respect to $\mathcal{O}_X$ is X, and we have that b belongs to X.

Example 8.4.2. Let $X = \{a, b, c, d, e\}$ be a set with five elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, e\}, \{c, d\}, \{a, b, e\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}.$$

Let $A = \{d\}$. By Remark 8.3.3, we have that d is a limit point of A in X with respect to $\mathcal{O}_X$. To decide whether the other elements of X are limit points, we look at their neighbourhoods.

Element	Neighbourhoods
a	$\{a\}$, $\{a, b\}$, $\{a, b, e\}$, $\{a, c, d\}$, $\{a, b, c, d\}$, X
b	$\{b\}$, $\{a, b\}$, $\{b, e\}$, $\{a, b, e\}$, $\{b, c, d\}$, $\{a, b, c, d\}$, $\{b, c, d, e\}$, X
c	$\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{a, b, c, d\}$, $\{b, c, d, e\}$, X
e	$\{b, e\}$, $\{a, b, e\}$, $\{b, c, d, e\}$, X

For each element, we check whether d belongs to all of its neighbourhoods.

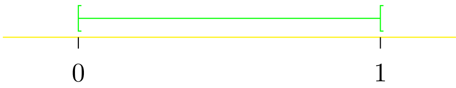
Element	Limit Point	Neighbourhoods to which d does not belong
a	✗	$\{a\}$, $\{a, b\}$, $\{a, b, e\}$
b	✗	$\{b\}$, $\{a, b\}$, $\{b, e\}$, $\{a, b, e\}$
c	✓	
e	✗	$\{b, e\}$, $\{a, b, e\}$

To establish that a, b, and e are not limit points, it suffices to observe that any *one* of the neighbourhoods listed in the table above does not contain d.

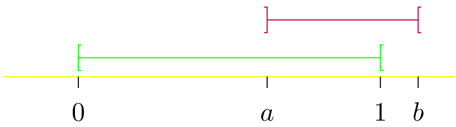
Example 8.4.3. Let $(X, \mathcal{O}_X)$ be as in Example 8.4.2. Let $A = \{b, d\}$. For each of the elements a, c, and e, we check whether every neighbourhood contains either b or d. The neighbourhoods are listed in a table in Example 8.4.2.

Element	Limit Point	Neighbourhoods U such that $A \cap U = \emptyset$
a	✗	$\{a\}$
c	✓	
e	✓	

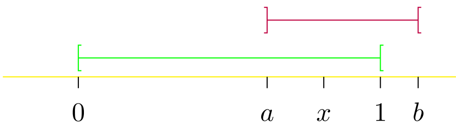
Example 8.4.4. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $A = [0, 1[$.



Let U be a neighbourhood of 1 in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. By definition of $\mathcal{O}_{\mathbb{R}}$, there is an open interval $]a, b[$ such that $a < 1 < b$ and which is a subset of U.

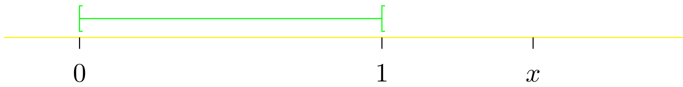


There is an $x \in \mathbb{R}$ such that $a < x < 1$, and $0 < x$. In particular, x belongs to $[0, 1[$.

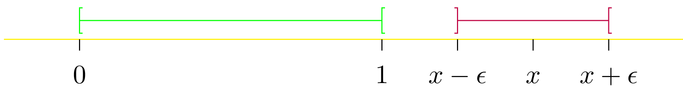


Since $]a, 1[$ is a subset of $]a, b[$, and since $]a, b[$ is a subset of U, we also have that x belongs to U. This proves that if U is a neighbourhood of 1 in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, then $[0, 1[\cap U$ is not empty. Thus 1 is a limit point of $[0, 1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

Suppose now that $x \in \mathbb{R}$ has the property that $x > 1$.

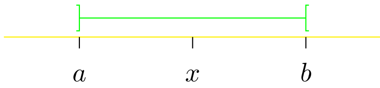


Let $\epsilon \in \mathbb{R}$ be such that $0 < \epsilon \leq x - 1$. Then $]x - \epsilon, x + \epsilon[$ is a neighbourhood of x in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, but $[0, 1[\cap]x - \epsilon, x + \epsilon[$ is empty.

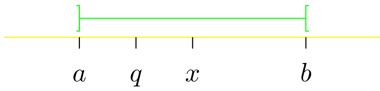


Thus x is not a limit point of $[0, 1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. In a similar way, one can demonstrate that if $x \in \mathbb{R}$ has the property that $x < 0$, then x is not a limit point of $[0, 1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. This is the topic of Task E8.2.2.

Example 8.4.5. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $A = \mathbb{Q}$, the set of rational numbers. Suppose that x belongs to $\mathbb{R}$. Let U be a neighbourhood of x in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. By definition of $\mathcal{O}_{\mathbb{R}}$, there is an open interval $]a, b[$ such that $a < x < b$ which is a subset of U.



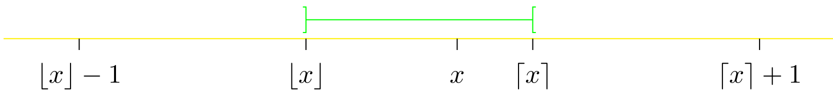
There is a $q \in \mathbb{Q}$ such that $a < q < x$. This is a consequence of the completeness of $\mathbb{R}$.



Since $]a, b[\cap \mathbb{Q}$ is a subset of $U \cap \mathbb{Q}$, we deduce that q belongs to U. We have proven that, for every neighbourhood U of x in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, $U \cap \mathbb{Q}$ is not empty. Thus x is a limit point of $\mathbb{Q}$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

Notation 8.4.6. Suppose that x belongs to $\mathbb{R}$. We denote by $\lfloor x \rfloor$ the largest integer z such that $z \leq x$. We denote by $\lceil x \rceil$ the smallest integer z such that $z \geq x$.

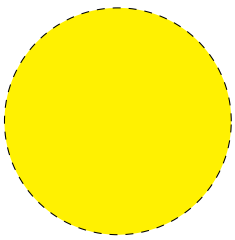
Example 8.4.7. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $A = \mathbb{Z}$, the set of integers. Suppose that x belongs to $\mathbb{R}$, and that x is not an integer. Then $]\lfloor x \rfloor, \lceil x \rceil[$ is a neighbourhood of x in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.



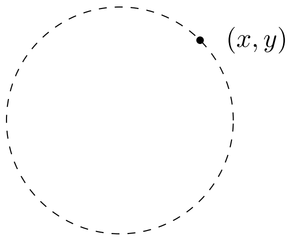
Moreover $\mathbb{Z} \cap]\lfloor x \rfloor, \lceil x \rceil[$ is empty. Thus x is not a limit point of $\mathbb{Z}$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

Example 8.4.8. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let A be the subset of $\mathbb{R}^2$ given by

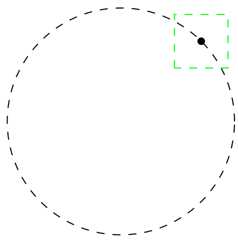
$$\left\{ (x, y) \in \mathbb{R}^2 \mid \|(x, y)\| < 1 \right\}.$$



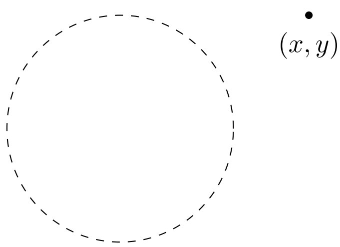
Suppose that $(x, y) \in \mathbb{R}^2$ belongs to S^1.



Then (x, y) is a limit point of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. Every neighbourhood of (x, y) in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ contains an 'open rectangle' U to which (x, y) belongs. We have that $A \cap U$ is not empty.



To fill in the details of this argument is the topic of Task E8.2.3. Suppose now that $(x, y) \in \mathbb{R}^2$ does not belong to D^2.



Then (x, y) is not a limit point of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. For let $\epsilon \in \mathbb{R}$ be such that

$$0 < \epsilon < \|(x, y)\| - 1.$$

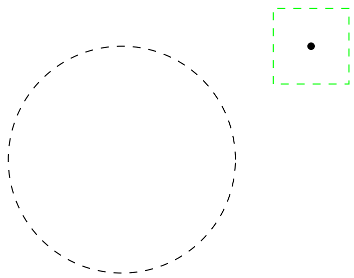
Let U_x be the open interval given by

$$\left] x - \tfrac{\epsilon\sqrt{2}}{\epsilon}, x + \tfrac{\epsilon\sqrt{2}}{\epsilon} \right[.$$

Let U_y be the open interval given by

$$\left] y - \tfrac{\epsilon\sqrt{2}}{\epsilon}, y + \tfrac{\epsilon\sqrt{2}}{\epsilon} \right[.$$

Then $U_x \times U_y$ is a neighbourhood of (x, y) in $\mathbb{R}^2$ whose intersection with A is empty.



To check this is the topic of Task E8.2.4.

8.5 Closure

Definition 8.5.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. The *closure* of A in X with respect to $\mathcal{O}_X$ is the set of limit points of A in X.

Notation 8.5.2. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. We shall denote the closure of A in X with respect to $\mathcal{O}_X$ by $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$.

Remark 8.5.3. The notation $\overline{A}$ is also frequently used to denote closure.

Remark 8.5.4. By Remark 8.3.3, we have that A is a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$.

Definition 8.5.5. Let $(X, \mathcal{O}_X)$ be a topological space. A subset A of X is *dense* in X with respect to $\mathcal{O}_X$ if the closure of A in X with respect to $\mathcal{O}_X$ is X.

8.6 Examples of closure

Example 8.6.1. Let $(X, \mathcal{O}_X)$ and A be as in Example 8.4.1. We found in Example 8.4.1 that the limit points of A in X with respect to $\mathcal{O}_X$ are a and b. Hence $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is X. Thus A is dense in X with respect to $\mathcal{O}_X$.

Example 8.6.2. Let $(X, \mathcal{O}_X)$ and A be as in Example 8.4.2. We found in Example 8.4.2 that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is $\{c, d\}$. Thus A is not dense in X with respect to $\mathcal{O}_X$.

Example 8.6.3. Let $(X, \mathcal{O}_X)$ and A be as in Example 8.4.3. We found in Example 8.4.3 that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is $\{b, c, d, e\}$. Thus A is not dense in X with respect to $\mathcal{O}_X$.

Example 8.6.4. We found in Example 8.4.4 that 1 is the only limit point of $[0, 1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$ which does not belong to $[0, 1[$. Thus $\mathsf{cl}_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})}([0, 1[)$ is $[0, 1]$. In particular, $[0, 1[$ is not dense in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

Example 8.6.5. We found in Example 8.4.5 that every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. In other words, $\mathsf{cl}_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})}(\mathbb{Q})$ is $\mathbb{R}$. Thus $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

Example 8.6.6. We found in Example 8.4.7 that if $x \in \mathbb{R}$ is not an integer, then x is not a limit point of $\mathbb{Z}$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. In other words, $\mathsf{cl}_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})}(\mathbb{Z})$ is $\mathbb{Z}$. In particular, $\mathbb{Z}$ is not dense in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

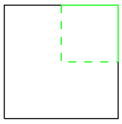
Example 8.6.7. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let A be as in Example 8.4.8. We found in Example 8.4.8 that if $(x, y) \in \mathbb{R}^2$ does not belong to A, then (x, y) is a limit point of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ if and only if (x, y) belongs to S^1. We conclude that $\mathsf{cl}_{(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})}(A)$ is D^2. In particular, A is not dense in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.

E8 Exercises for Lecture 8

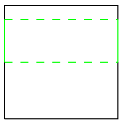
E8.1 Exam questions

Task E8.1.1. For which of the following subsets A of I^2 is $\pi(A)$ a neighbourhood of $\left[\left(\frac{3}{4}, \frac{3}{4}\right)\right]$ in K^2 with respect to $\mathcal{O}_{K^2}$? Take the equivalence relation on K^2 to be that of Example 6.4.11.

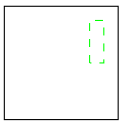
(1) $\left]\frac{1}{2}, 1\right] \times \left]\frac{1}{2}, 1\right]$



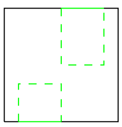
(2) $[0,1] \times \left]\frac{1}{2}, \frac{7}{8}\right[$



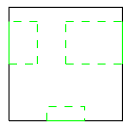
(3) $\left]\frac{3}{4}, \frac{7}{8}\right[\times \left]\frac{1}{2}, \frac{7}{8}\right[$



(4) $\left(\left]\frac{1}{2}, \frac{7}{8}\right[\times \left]\frac{1}{2}, 1\right]\right) \cup \left(\left]\frac{1}{8}, \frac{1}{2}\right[\times \left[0, \frac{1}{3}\right[\right)$



(5) $\left(\left]\frac{1}{2},1\right] \times \left]\frac{1}{2},\frac{7}{8}\right[\right) \cup \left(\left[0,\frac{1}{4}\right[\times \left]\frac{1}{2},\frac{7}{8}\right[\right) \cup \left(\left]\frac{1}{3},\frac{2}{3}\right[\times \left[0,\frac{1}{8}\right[\right)$

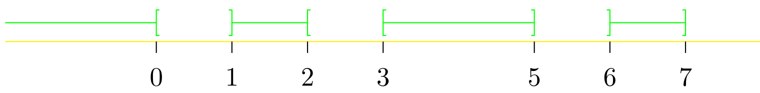


Task E8.1.2. Let $X = \{a, b, c, d\}$ be a set with four elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}\,.$$

What is the closure of $\{b\}$ in X with respect to $\mathcal{O}_X$? Find a subset A of X with two elements, neither of which is b, with the property that A is dense in X with respect to $\mathcal{O}_X$.

Task E8.1.3. Let $A = \left]-\infty, 0\right[\cup \left]1, 2\right[\cup [3, 5] \cup \left]6, 7\right]$.



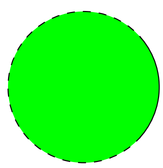
What is the closure of A in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$?

Task E8.1.4. Let A be the union of the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid -1 < x < \tfrac{3}{4} \text{ and } \|(x, y)\| < 1\right\}$$

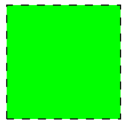
and the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid \tfrac{3}{4} \leq x < 1 \text{ and } \|(x, y)\| \leq 1\right\}.$$

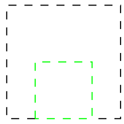


What is the closure of A in D^2 with respect to $\mathcal{O}_{D^2}$?

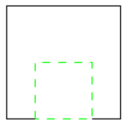
Task E8.1.5. Let $X = \left]0, 1\right[\times \left]0, 1\right[$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



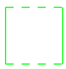
Let $A = \left]\frac{1}{4}, \frac{3}{4}\right[\times \left]0, \frac{1}{2}\right[$.



What is the closure of A in $(X, \mathcal{O}_X)$? What is the closure of A in $(I^2, \mathcal{O}_{I^2})$?



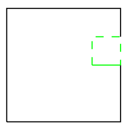
Find a subset Y of $\mathbb{R}^2$ such that the closure of A in Y with respect to $\mathcal{O}_Y$ is $\left[\frac{1}{4}, \frac{3}{4}\right] \times \left]0, \frac{1}{2}\right[$, where $\mathcal{O}_Y$ is the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Task E8.1.6. Let $A = \left]\frac{3}{4}, 1\right[\times \left[\frac{1}{2}, \frac{3}{4}\right[$. Let

$$I^2 \xrightarrow{\pi} T^2$$

denote the quotient map.



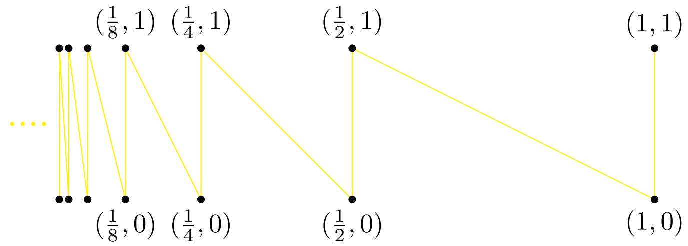
What is the closure of $\pi(A)$ in $(T^2, \mathcal{O}_{T^2})$?

Task E8.1.7. Let A be the subset of $\mathbb{R}^2$ given by the union of the sets

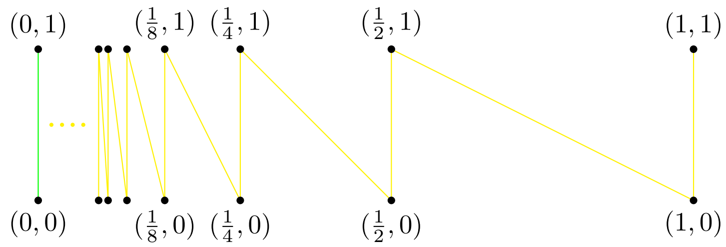
$$\bigcup_{n \in \mathbb{N}} \left\{ \left(\frac{1}{2^{n-1}}, y \right) \mid y \in [0, 1] \right\}$$

and

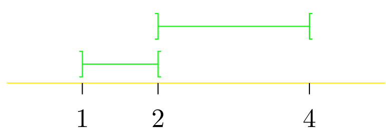
$$\bigcup_{n \in \mathbb{N}} \left\{ (x, -2^n x + 2) \mid x \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \right\}.$$



Prove that the closure of X in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ is the union of X and the line $\{0\} \times [0, 1]$.



Task E8.1.8. Let $X =]1, 2[\cup]2, 4[$. What is the closure of X in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$?



E8.2 In the lectures

Task E8.2.1. Prove that $(T^2, \mathcal{O}_{T^2})$ is homeomorphic to $(S^1 \times S^1, \mathcal{O}_{S^1 \times S^1})$, as discussed in Example 8.1.4. You may wish to proceed as follows.

(1) As in Example 6.3.1, work with S^1 throughout this task as the quotient of I by the equivalence relation generated by $0 \sim 1$. In particular, think of $\mathcal{O}_{S^1}$ as the quotient topology $\mathcal{O}_{I/\sim}$.

(2) Let

$$I \xrightarrow{\pi_{S^1}} S^1$$

denote the quotient map. Appealing to Remark 6.1.9 and Task **??**, observe that the map

$$I \times I \xrightarrow{\pi_{S^1} \times \pi_{S^1}} S^1 \times S^1$$

is continuous.

(3) Appealing to Task E6.2.7, deduce from (2) that the map

$$T^2 \xrightarrow{f} S^1 \times S^1$$

given by $[(s,t)] \mapsto ([s],[t])$ is continuous.

(4) Let $t \in I$. Appealing to Task E5.3.14, Task E5.1.5, and Task E5.3.17, observe that the map

$$I \xrightarrow{f_t^0} I^2$$

given by $s \mapsto (t,s)$ is continuous.

(5) Let

$$I^2 \xrightarrow{\pi_{T^2}} T^2$$

denote the quotient map. Appealing to Task 5.3.1, deduce from (1) and Remark 6.1.9 that the map

$$I \xrightarrow{\pi_{T^2} \circ f_t^0} T^2$$

given by $s \mapsto [(s,t)]$ is continuous.

(6) Observe that $\pi_{T^2}\left(f_t^0(0)\right) = \pi_{T^2}\left(f_t^0(1)\right)$. By Task E6.2.7, deduce that the map

$$S^1 \xrightarrow{g_t^0} T^2$$

given by $[s] \mapsto [(t,s)]$ is continuous.

(7) As in (4) – (6), use the map

$$I \xrightarrow{f_t^1} I^2$$

given by $s \mapsto (s,t)$ to prove that the map

$$S^1 \xrightarrow{g_t^1} T^2$$

given by $[t] \mapsto [(s,t)]$ is continuous.

(8) Let

$$S^1 \times S^1 \xrightarrow{g} T^2$$

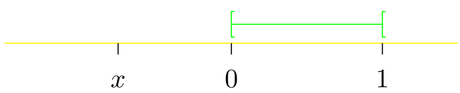
denote the map given by $([s],[t]) \mapsto [(s,t)]$. Observe that $g \circ f = id_{T^2}$, and that $f \circ g = id_{S^1 \times S^1}$.

(9) Let U be a subset of T^2 which belongs to $\mathcal{O}_{T^2}$. Suppose that $([x],[y])$ belongs to $g^{-1}(U)$. Let U_x denote the subset $\left(g_y^1\right)^{-1}(U)$ of S^1. By (6), we have that U_x belongs to $\mathcal{O}_{S^1}$. Let U_y denote the subset $\left(g_x^0\right)^{-1}(U)$ of S^1. By (5), we have that U_y belongs to $\mathcal{O}_{S^1}$. Observe that $([x],[y])$ belongs to $U_x \times U_y$, and that $U_x \times U_y$ is a subset of $g^{-1}(U)$.

(10) By definition of $\mathcal{O}_{S^1 \times S^1}$, deduce from (8) that $g^{-1}(U)$ belongs to $\mathcal{O}_{S^1 \times S^1}$. Conclude that g is continuous.

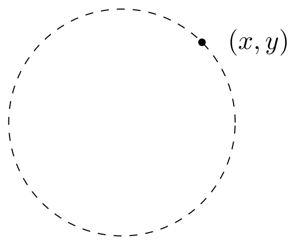
(11) Observe that (2), (8), and (10) together establish that f is a homeomorphism.

Task E8.2.2. Let $x \in \mathbb{R}$ be such that $x < 0$.



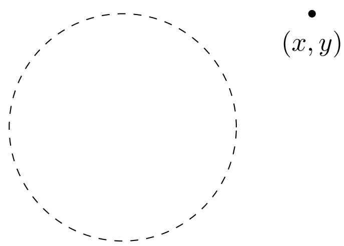
Prove that x is not a limit point of $[0,1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$).

Task E8.2.3. Let $(X, \mathcal{O}_X)$ and A be as in Example 8.4.8. Suppose that $(x,y) \in \mathbb{R}^2$ belongs to S^1,



Prove that (x, y) is a limit point of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.

Task E8.2.4. Let $(X, \mathcal{O}_X)$ and A be as in Example 8.4.8. Suppose that $(x, y) \in \mathbb{R}^2$ does not belong to D^2.



Prove that (x, y) is not a limit point of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$, following the argument outlined in Example 8.4.8. You may find it helpful to look back at Example 3.2.3.

E8.3 For a deeper understanding

Task E8.3.1. Let $(X, \mathcal{O}_X)$. Let U be a subset of X. Prove that U belongs to $\mathcal{O}_X$ if and only if, for every x which belongs to X, there is a neighbourhood U_x of x in $(X, \mathcal{O}_X)$ such that U_x is a subset of U.

Remark E8.3.2. Task E8.3.1 gives a 'local characterisation' of subsets of X which belong to $\mathcal{O}_X$.

Task E8.3.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a map. Prove that f is continuous if and only for every $x \in X$, and every neighbourhood $U_{f(x)}$ of $f(x)$ in Y with respect to $\mathcal{O}_Y$, there is a neighbourhood U_x of x in X with respect to $\mathcal{O}_X$ such that $f(U_x)$ is a subset of $U_{f(x)}$. You may wish to proceed as follows.

(1) Suppose that f satisfies this condition. Let U be a subset of Y which belongs to $\mathcal{O}_Y$. Suppose that x belongs $f^{-1}(U)$. Observe that U is a neighbourhood of $f(x)$ in Y with respect to $\mathcal{O}_Y$.

(2) By assumption, there is thus a neighbourhood U_x of x in X with respect to $\mathcal{O}_X$ such that $f(U_x)$ is a subset of U. Deduce that U_x is a subset of $f^{-1}(U)$.

(3) By Task E8.3.1, deduce that $f^{-1}(U)$ belongs to $\mathcal{O}_X$. Conclude that f is continuous.

(4) Conversely, suppose that f is continuous. Suppose that x belongs to X, and that $U_{f(x)}$ is a neighbourhood of $f(x)$ in Y with respect to $\mathcal{O}_Y$. We have that $f\left(f^{-1}(U_{f(x)})\right)$ is a subset of $U_{f(x)}$. Since f is continuous, observe that $f^{-1}(U_{f(x)})$ is moreover a neighbourhood of x in X with respect to $\mathcal{O}_X$.

Remark E8.3.4. Task E8.3.3 gives a 'local characterisation' of continuous maps.

Definition E8.3.5. Let $(X, \mathcal{O}_X)$ be a topological space. A set $\{A_j\}_{j \in J}$ of (possibly infinitely many) subsets of X is *locally finite* with respect to $\mathcal{O}_X$ if, for every $x \in X$, there is a neighbourhood U of x in $(X, \mathcal{O}_X)$ with the property that the set of $j \in J$ such that $U \cap A_j$ is non-empty is finite.

Remark E8.3.6. If J is finite, then $\{A_j\}_{j \in J}$ is locally finite.

Task E8.3.7. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{V_j\}_{j \in J}$ be a set of subsets of X which is locally finite with respect to $\mathcal{O}_X$. Suppose that V_j is closed with respect to $\mathcal{O}_X$, for every $j \in J$. Let K be a (possibly infinite) subset of J. Prove that $\bigcup_{j \in K} V_j$ is closed with respect to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) Let $x \in X \setminus \left(\bigcup_{j \in K} V_j\right)$. Observe that since $\{V_j\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_X$, there is a neighbourhood U_x of x in $(X, \mathcal{O}_X)$ with the property that the set L of $j \in J$ such that $U_x \cap V_j$ is non-empty is finite.

(2) Let $U = U_x \cap \left(\bigcap_{j \in L} X \setminus V_j\right)$. Prove that U belongs to $\mathcal{O}_X$.

(3) Observe that $x \in U$.

(4) Prove that $U \cap \left(\bigcup_{j \in K} V_j\right)$ is empty, and thus that U is a subset of $X \setminus V$.

(5) By Task E8.3.1, deduce that $X \setminus \left(\bigcup_{j \in K} V_j\right)$ belongs to $\mathcal{O}_X$.

Task E8.3.8. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{V_j\}_{j \in J}$ be a locally finite set of subsets of X, with the property that $X = \bigcup_{j \in J} V_j$. For every $j \in J$, let $\mathcal{O}_{V_j}$ denote the subspace topology on V_j with respect to $(X, \mathcal{O}_X)$. Suppose that V_j is closed with respect to $\mathcal{O}_X$ for every $j \in J$. Let V be a subset of X such that $V \cap V_j$ is closed with respect to $\mathcal{O}_{V_j}$ for every $j \in J$. Prove that V is closed with respect to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) Appealing to Task E2.3.3 (3), observe that $V \cap V_j$ is closed with respect to $\mathcal{O}_X$.

(2) Prove that since $\{V_j\}_{j \in J}$ is locally finite, so is $\{V \cap V_j\}_{j \in J}$.

(3) By Task E8.3.7, deduce that $\bigcup_{j \in J} V \cap V_j$ is closed with respect to $\mathcal{O}_X$.

(4) Observe that $V = \bigcup_{j \in J} V \cap V_j$.

Task E8.3.9. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{V_j\}_{j \in J}$ be a locally finite set of subsets of X, with the property that $X = \bigcup_{j \in J} V_j$. For every $j \in J$, let $\mathcal{O}_{V_j}$ denote the subspace topology on V_j with respect to $(X, \mathcal{O}_X)$. Suppose that V_j is closed with respect to $\mathcal{O}_X$ for every $j \in J$. Let U be a subset of X such that $U \cap V_j$ belongs to $\mathcal{O}_{V_j}$ for every $j \in J$. Prove that U belongs to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) Since $U \cap V_j$ belongs to $\mathcal{O}_{V_j}$, observe that $V_j \setminus (U \cap V_j)$ is closed with respect to $\mathcal{O}_{V_j}$, for every $j \in J$.

(2) Observe that $V_j \setminus (U \cap V_j) = V_j \cap (X \setminus U)$.

(3) By Task E8.3.8, deduce that $X \setminus U$ is closed with respect to $\mathcal{O}_X$.

Task E8.3.10. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\mathcal{O}'_X$ be a topology on X such that $\mathcal{O}'_X$ is a subset of $\mathcal{O}_X$. Let A be a subset of X. Suppose that x is a limit point of A in X with respect to $\mathcal{O}_X$. Prove that x is a limit point of A in X with respect to $\mathcal{O}'_X$.

Task E8.3.11. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let A be a subset of X, and let B be a subset of Y. Prove that $\mathsf{cl}_{(X \times Y, \mathcal{O}_{X \times Y})}(A \times B)$ is

$$\mathsf{cl}_{(X, \mathcal{O}_X)}(A) \times \mathsf{cl}_{(Y, \mathcal{O}_Y)}(B).$$

Task E8.3.12. Let $(X, \mathcal{O}_X)$ be a topological space. Let A and B be subsets of X such that A is a subset of B. Prove that $\mathsf{cl}_{(X, \mathcal{O}_X)}(A)$ is a subset of $\mathsf{cl}_{(X, \mathcal{O}_X)}(B)$.

Task E8.3.13. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Let $\mathcal{O}_A$ denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Let B be a subset of A which belongs to $\mathcal{O}_X$. Prove that $\mathsf{cl}_{(A, \mathcal{O}_A)}(B)$ is $A \cap \mathsf{cl}_{(X, \mathcal{O}_X)}(B)$. You may wish to proceed as follows.

(1) Suppose that x belongs to $\mathsf{cl}_{(A, \mathcal{O}_A)}(B)$. In particular, we have that x belongs to A. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_X$. By definition of $\mathcal{O}_A$, observe that $A \cap U$ is a neighbourhood of x in A with respect to $\mathcal{O}_A$.

(2) Since x belongs to $\mathsf{cl}_{(A, \mathcal{O}_A)}(B)$, observe that $B \cap (A \cap U)$ is not empty.

(3) Since $B \cap (A \cap U)$ is $(B \cap A) \cap U$, and since B is a subset of A, deduce that $B \cap U$ is not empty.

(4) Deduce that x belongs to $\mathsf{cl}_{(X, \mathcal{O}_X)}(B)$. Conclude that $\mathsf{cl}_{(A, \mathcal{O}_A)}(B)$ is a subset of $A \cap \mathsf{cl}_{(X, \mathcal{O}_X)}(B)$.

(5) Conversely, suppose that x belongs to $A \cap \mathsf{cl}_{(X, \mathcal{O}_X)}(B)$. Suppose that U is a neighbourhood of x in A with respect to $\mathcal{O}_A$. By definition of $\mathcal{O}_A$, observe that there is a subset U' of X which belongs to $\mathcal{O}_X$ with the property that $U = A \cap U'$.

(6) Since x belongs to $\mathsf{cl}_{(X,\mathcal{O}_X)}(B)$, observe that $B \cap U'$ is not empty.

(7) Since B is a subset of A, we have that $B = B \cap A$. Deduce that $(B \cap A) \cap U' = B \cap (A \cap U') = B \cap U$ is not empty.

(8) Deduce that x belongs to $\mathsf{cl}_{(A,\mathcal{O}_A)}(B)$. Conclude that $A \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$ is a subset of $\mathsf{cl}_{(A,\mathcal{O}_A)}(B)$.

(9) By (4) and (8), deduce that $\mathsf{cl}_{(A,\mathcal{O}_A)}(B)$ is $A \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$.