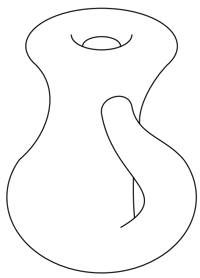
Revision Question 4 — 03/05/14

a) Prove that the Klein bottle is a surface: give an argument for why all the necessary conditions are satisfied. You may quote without proof any results from the course. [15 marks]

b) Let $(K^2, \mathcal{O}_{K^2})$ be the Klein bottle.



Let

$$I^2 \xrightarrow{\pi} K^2$$

be the quotient map. Find a subset C of I^2 with the following two properties.

i) We have that $\big(\pi(C), \mathcal{O}_{\pi(C)}\big)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$, where $\mathcal{O}_{\pi(C)}$ is the subspace topology on $\pi(C)$ with respect to $(K^2, \mathcal{O}_{K^2})$.

ii) We have that $(K^2 \setminus \pi(C), \mathcal{O}_{K^2 \setminus \pi(C)})$ is connected, where $\mathcal{O}_{K^2 \setminus \pi(C)}$ is the subspace topology on $K^2 \setminus \pi(C)$ with respect to $(K^2, \mathcal{O}_{K^2})$.

You do not need to prove anything. [3 marks]

c) Apply surgery to $(K^2, \mathcal{O}_{K^2})$ with respect to your curve C of part b). Outline an argument to demonstrate that we obtain a surface which, depending on your choice of C, is homeomorphic to either the real projective plane or $(S^2, \mathcal{O}_{S^2})$. You do not need to give a detailed proof. [8 marks]

d) Equip $(K^2, \mathcal{O}_{K^2})$ with the structure of a Δ-complex. [3 marks]

e) Use your Δ-complex structure of part d) to calculate the Euler characteristic of $(K^2, \mathcal{O}_{K^2})$. You may not use any other method. [2 marks]

f) Up to homeomorphism, how many surfaces have the same Euler characteristic as $(K^2, \mathcal{O}_{K^2})$? You may quote without proof any results from the course. [4 marks]