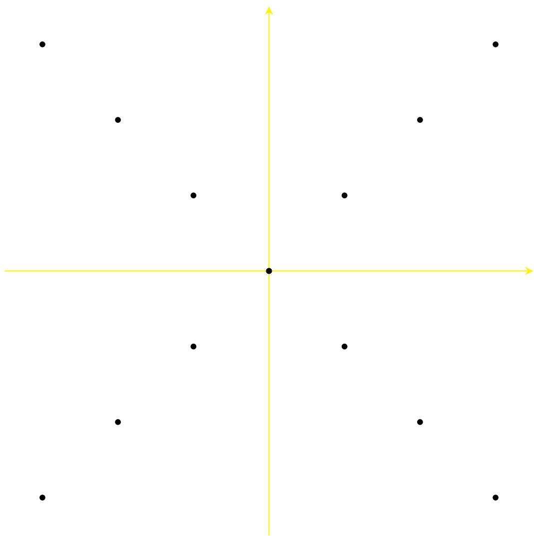
Revision Question 5 — 04/05/14

Let X be the subset of $\mathbb{Z}^2$ given by

$$\left\{(x, y) \in \mathbb{Z}^2 \mid x = y \text{ or } x = -y\right\}.$$



a) Given $z \in \mathbb{Z}$, let U_z^g be the subset of X given by

$$\{(x, y) \in X \mid x \geq z\},$$

and let U_z^l be the subset of X given by

$$\{(x, y) \in X \mid x \leq z\}.$$

Give a reason why the set

$$\{U_z^g \mid z \in \mathbb{Z}\} \cup \{U_z^l \mid z \in \mathbb{Z}\}$$

does not define a topology on X.

b) Given $n \in \mathbb{N}$, let U_n be the subset of X given by

$$X \cap \left\{(x, y) \in \mathbb{Z}^2 \mid -n \leq x \leq n \text{ and } -n \leq y \leq n\right\}.$$

Let $\mathcal{O}_X$ be the topology on X given by the set of subsets U of X such that, for every x which belongs to U, one of the following holds.

i) For some $n \leq 10$ which belongs to $\mathbb{N}$, we have both that x belongs to U_n and that U_n is a subset of U.

ii) We have both that x belongs to

$$\{(x,y) \in X \mid |x| > 10 \text{ or } |y| > 10\}$$

and that this set is a subset of U.

Demonstrate that $(X, \mathcal{O}_X)$ is not connected. [5 marks]

c) Let $\mathcal{O}'_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Is $(X, \mathcal{O}_X)$ homeomorphic to $(X, \mathcal{O}'_X)$? Justify your answer. [8 marks]

d) Define a topology $\mathcal{O}''_X$ on X such that $(X, \mathcal{O}''_X)$ has the following properties.

i) It is connected.

ii) It is not compact.

Give a proof that $(X, \mathcal{O}''_X)$ has property ii). You do not need to prove that $(X, \mathcal{O}''_X)$ has property i). [6 marks]