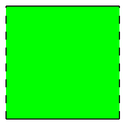
Revision Question 6 — 05/05/14

Let X be the subset of $\mathbb{R}^2$ given by $]0,1[\times [0,1]$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



a) Find an open covering of X with respect to $\mathcal{O}_X$ which does not admit a finite subcovering. [3 marks]

b) Find a subset Y of $\mathbb{R}^2$ such that the following hold, where $\mathcal{O}_Y$ is the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

   i) X is a subset of Y.

   ii) $(Y, \mathcal{O}_Y)$ is not locally compact.

   Prove that (ii) holds. [8 marks]

c) Define an equivalence relation $\sim$ on I^2 such that $(I^2/{\sim}, \mathcal{O}_{I^2/\sim})$ is a one point compactification of $(X, \mathcal{O}_X)$. You do not need to give a proof. You may wish to draw a picture. [6 marks]

d) Let

$$I^2 \xrightarrow{\pi} I^2/{\sim}$$

be the quotient map, where $\sim$ is your equivalence relation of part c). Find an example of a subset U of I^2 which belongs to $\mathcal{O}_{I^2}$, but for which $\pi(U)$ does not belong to $\mathcal{O}_{I^2/\sim}$. [4 marks]

e) Let $\sim$ be the equivalence relation on $\mathbb{R}$ given by $x_0 \sim x_1$ if $x_1 - x_0$ belongs to $\mathbb{Z}$. Let

$$\mathbb{R} \xrightarrow{\pi} \mathbb{R}/{\sim}$$

be the quotient map. Prove that if U belongs to $\mathcal{O}_{\mathbb{R}}$, then $\pi(U)$ belongs to $\mathcal{O}_{\mathbb{R}/\sim}$. [6 marks]

f) Let $\sim$ be the equivalence relation on $\mathbb{R}$ given by $x_0 \sim x_1$ if $x_1 - x_0$ belongs to $\mathbb{Q}$. Prove that for any topological space $(X', \mathcal{O}_{X'})$, we have that every map

$$X' \longrightarrow \mathbb{R}/\sim$$

is continuous. You may assume without proof any facts about $\mathbb{R}$. [8 marks]