Revision Question 7 — 06/05/14

Let X be the set $\{a, b, c, d\}$.

a) Which of the following define a topology on X? For those which do not, give a reason.

   i) $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$

   ii) $\{\emptyset, \{d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, b, c, d\}\}$

   iii) $\{\emptyset, \{c\}, \{a, d\}, \{b, c\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c, d\}\}$.

   [3 marks]

Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}.$$

b) What is the boundary of $\{b, c\}$ in X with respect to $\mathcal{O}_X$? [5 marks]

View $(S^1, \mathcal{O}_{S^1})$ as $(I/\sim, \mathcal{O}_{I/\sim})$, where $\sim$ is the equivalence relation on I generated by $0 \sim 1$.

c) Let

$$S^1 \xrightarrow{f} X$$

be the surjective map given by

$$[x] \mapsto \begin{cases} a & \text{if } x = 0, \\ b & \text{if } 0 < x < \frac{1}{2}, \\ c & \text{if } x = \frac{1}{2}, \\ d & \text{if } \frac{1}{2} < x < 1. \end{cases}$$

Prove that f is continuous, where X is equipped with the topology $\mathcal{O}_X$. Justify every assertion that you make that a given set belongs to any of the topologies that you consider. [7 marks]

d) Prove in two different ways that $(S^1, \mathcal{O}_{S^1})$ is not homeomorphic to $(X, \mathcal{O}_X)$. You may quote without proof any results from the course. [6 marks]

Let $\mathcal{O}'_X$ be the topology on X given by

$$\{\emptyset, \{b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}\}.$$

e) Demonstrate that the set

$$\{(a,b),(a,c),(b,b),(b,c),(b,d),(d,b),(d,c)\}$$

belongs to the product topology with respect to $\mathcal{O}_X$ and $\mathcal{O}'_X$ on $X \times X$.

Let $\mathcal{O}''_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,d\}, \{a,b,d\}, \{a,c,d\}, \{a,b,c,d\}\}\,.$$

f) Is $(X, \mathcal{O}''_X)$ homeomorphic to $(X, \mathcal{O}_X)$? Justify your answer. [5 marks]