MA3002 Generell Topologi — Vår 2014 – Revision Questions

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Contents

Revision Questions

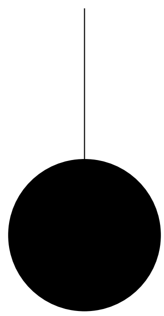
Revision Question 1 — 30/04/14

Let X be the subset of $\mathbb{R}^2$ given by the union of

$$\{(x, y) \mid \|(x-2, y)\| \leq 1\}$$

and

$$\{(2, y) \mid 1 \leq y \leq 3\}.$$



Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

a) Is $(X, \mathcal{O}_X)$ compact? Justify your answer. You may quote without proof any results from the syllabus. [5 marks]

b) Is $(X, \mathcal{O}_X)$ Hausdorff? Justify your answer. You may quote without proof any results from the syllabus. [5 marks]

c) Is $(X, \mathcal{O}_X)$ locally compact? You may quote without proof any results from the syllabus. [3 marks]

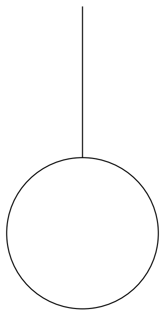
d) Suppose that (x, y) belongs to $\mathbb{R}^2$, and that $\|(x, y)\| \leq 1$. Prove that $D^2 \setminus \{(x, y)\}$, equipped with the subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$, is path connected. You may quote without proof any results from the syllabus. [6 marks]

Let Y be the subset of $\mathbb{R}^2$ given by the union of

$$\{(x, y) \mid \|(x-2, y)\| = 1\}$$

and

$$\{(2, y) \mid 1 \leq y \leq 3\}.$$

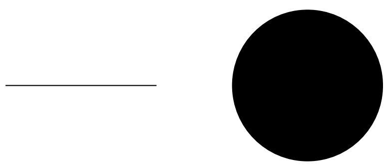


Let $\mathcal{O}_Y$ be the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

e) Is $(X, \mathcal{O}_X)$ homeomorphic to $(Y, \mathcal{O}_Y)$? Justify your answer. You may quote without proof any results from the syllabus, except that you may not appeal to the fact that homeomorphisms preserve Euler characteristic. [8 marks]

f) Equip $(X, \mathcal{O}_X)$ with the structure of a Δ-complex, and calculate its Euler characteristic. [6 marks]

Let Z be the union of $\{(x, 0) \mid -2 \leq x \leq 0\}$ and $\{(x, y) \mid \|(x-2, y)\| \leq 1\}$.

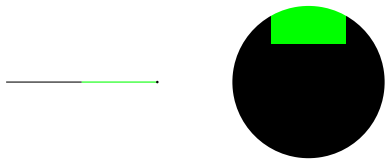


Let $\mathcal{O}_Z$ be the subspace topology on Z with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let $\sim$ be the equivalence relation on Z generated by $(0,0) \sim (2,1)$. Let $\mathcal{O}_{Z/\sim}$ be the quotient topology on $Z/\sim$ with respect to $(Z, \mathcal{O}_Z)$. Let A be the union of

$$\{(x, 0) \mid -1 < x < 0\}$$

and

$$\{(x, y) \mid \|(x-2, y)\| \leq 1\} \cap \left(\left]\tfrac{3}{2}, \tfrac{5}{2}\right[\times \left]\tfrac{1}{2}, \tfrac{3}{2}\right[\right).$$



Let

$$Z \xrightarrow{\pi} Z/{\sim}$$

be the quotient map.

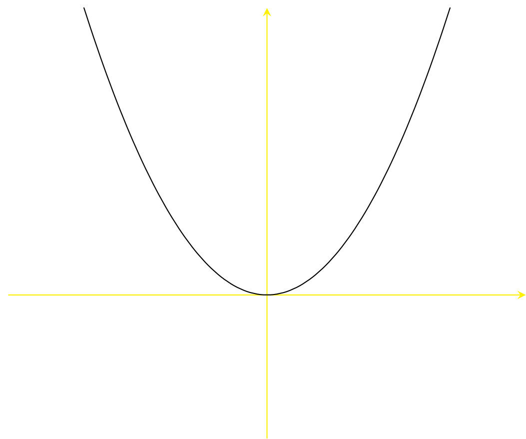
g) Does $\pi(A)$ belong to $\mathcal{O}_{Z/\sim}$? Justify your answer. [6 marks]

h) Prove that $(Z/{\sim}, \mathcal{O}_{Z/\sim})$ is homeomorphic to $(X, \mathcal{O}_X)$. You may quote without proof any results from the course. [8 marks]

Revision Question 2 — 01/05/14

Let X be the subset of $\mathbb{R}^2$ given by

$$\left\{(x, x^2) \mid x \in \mathbb{R}\right\}.$$



Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

a) Prove that $(X, \mathcal{O}_X)$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. You may quote without proof any facts from the course. [5 marks]

b) Give an example of a subset Y of $\mathbb{R}^2$ such that the following hold.

   i) Y is not closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.

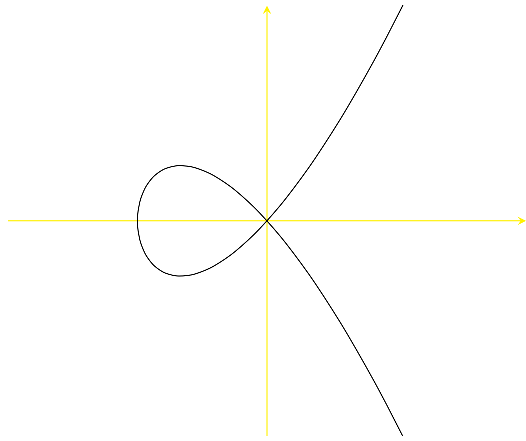
   ii) $(Y, \mathcal{O}_Y)$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, where $\mathcal{O}_Y$ is the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

   You do not need to prove that your set Y satisfies i) and ii). [5 marks]

c) What is the closure in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ of your set Y of part b)? You do not need to give a proof. [3 marks]

Let Z be the subset of $\mathbb{R}^2$ given by

$$\left\{(x^2 - 1, x^3 - x) \mid x \in \mathbb{R}\right\}.$$



Let $\mathcal{O}_Z$ be the subspace topology on Z with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

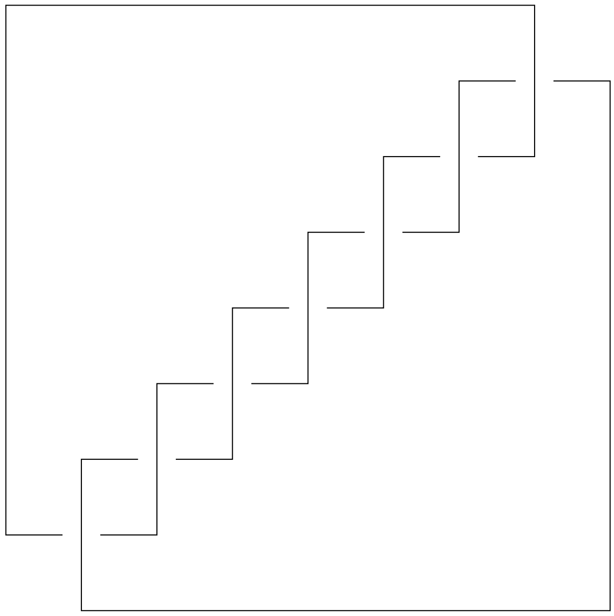
d) Is $(Z, \mathcal{O}_Z)$ homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$? Justify your answer. You may quote without proof any results from the syllabus. [6 marks]

e) Prove that $(Z, \mathcal{O}_Z)$ is connected. You may quote without proof any results from the syllabus. [5 marks]

f) Prove that $(-3, 0)$ is not a limit point of Z in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. [4 marks]

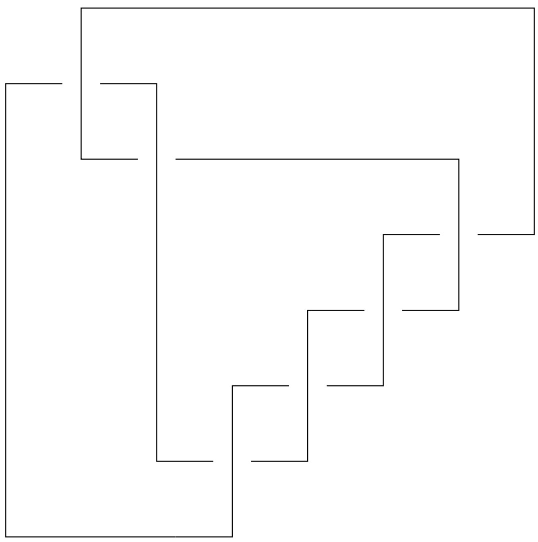
Revision Question 3 — 02/05/14

The knot 7_1, known as the *septafoil*, is pictured below.



a) There is an integer p such that 7_1 is m-colourable if and only if $p \mid m$. Find p. Justify your answer. [8 marks]

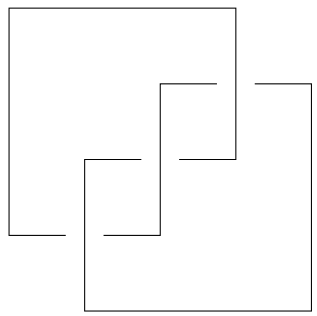
The knot 6_1, known as the *stevedore knot*, is pictured below.



b) Find a 15-colouring of 6_1. [6 marks]

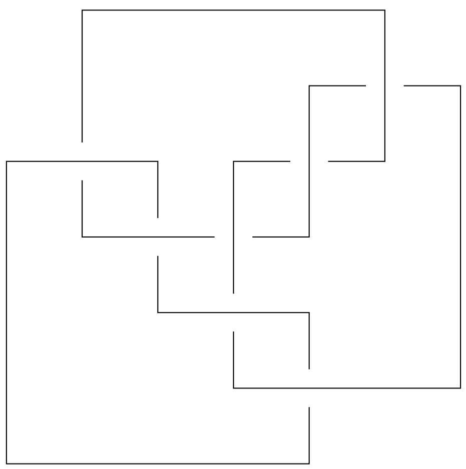
c) Is 6_1 isotopic to 7_1? Justify your answer. You may quote without proof any results from the course. [3 marks]

The knot 3_1, the trefoil, is pictured below.



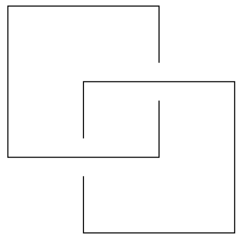
d) Prove that 6_1 is not isotopic to 3_1. You may quote without proof any results from the course. [12 marks]

The link 7^2_5 is pictured below.



e) What is the linking number of 7^2_5? [5 marks]

The link 2^2_1, the Hopf link, is pictured below.



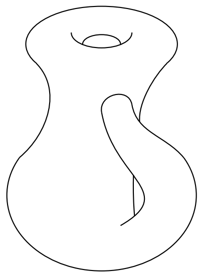
f) Prove that 7^2_5 is not isotopic to 2^2_1. Justify your answer. You may quote without proof any results from the course. [3 marks]

g) Can your answer to e) be used to prove that 7^2_5 is not isotopic to the unlink with two components? [3 marks]

Revision Question 4 — 03/05/14

a) Prove that the Klein bottle is a surface: give an argument for why all the necessary conditions are satisfied. You may quote without proof any results from the course. [15 marks]

b) Let $(K^2, \mathcal{O}_{K^2})$ be the Klein bottle.



Let

$$I^2 \xrightarrow{\pi} K^2$$

be the quotient map. Find a subset C of I^2 with the following two properties.

i) We have that $\left(\pi(C), \mathcal{O}_{\pi(C)}\right)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$, where $\mathcal{O}_{\pi(C)}$ is the subspace topology on $\pi(C)$ with respect to $(K^2, \mathcal{O}_{K^2})$.

ii) We have that $(K^2 \setminus \pi(C), \mathcal{O}_{K^2 \setminus \pi(C)})$ is connected, where $\mathcal{O}_{K^2 \setminus \pi(C)}$ is the subspace topology on $K^2 \setminus \pi(C)$ with respect to $(K^2, \mathcal{O}_{K^2})$.

You do not need to prove anything. [3 marks]

c) Apply surgery to $(K^2, \mathcal{O}_{K^2})$ with respect to your curve C of part b). Outline an argument to demonstrate that we obtain a surface which, depending on your choice of C, is homeomorphic to either the real projective plane or $(S^2, \mathcal{O}_{S^2})$. You do not need to give a detailed proof. [8 marks]

d) Equip $(K^2, \mathcal{O}_{K^2})$ with the structure of a Δ-complex. [3 marks]

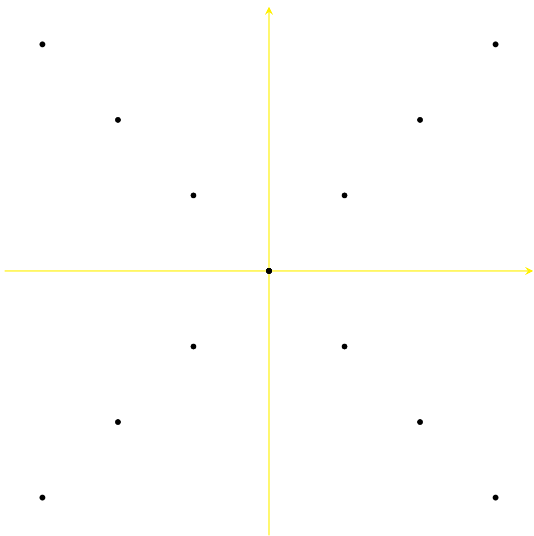
e) Use your Δ-complex structure of part d) to calculate the Euler characteristic of $(K^2, \mathcal{O}_{K^2})$. You may not use any other method. [2 marks]

f) Up to homeomorphism, how many surfaces have the same Euler characteristic as $(K^2, \mathcal{O}_{K^2})$? You may quote without proof any results from the course. [4 marks]

Revision Question 5 — 04/05/14

Let X be the subset of $\mathbb{Z}^2$ given by

$$\left\{(x, y) \in \mathbb{Z}^2 \mid x = y \text{ or } x = -y\right\}.$$



a) Given $z \in \mathbb{Z}$, let U_z^g be the subset of X given by

$$\{(x, y) \in X \mid x \geq z\},$$

and let U_z^l be the subset of X given by

$$\{(x, y) \in X \mid x \leq z\}.$$

Give a reason why the set

$$\{U_z^g \mid z \in \mathbb{Z}\} \cup \{U_z^l \mid z \in \mathbb{Z}\}$$

does not define a topology on X.

b) Given $n \in \mathbb{N}$, let U_n be the subset of X given by

$$X \cap \left\{(x, y) \in \mathbb{Z}^2 \mid -n \leq x \leq n \text{ and } -n \leq y \leq n\right\}.$$

Let $\mathcal{O}_X$ be the topology on X given by the set of subsets U of X such that, for every x which belongs to U, one of the following holds.

i) For some $n \leq 10$ which belongs to $\mathbb{N}$, we have both that x belongs to U_n and that U_n is a subset of U.

ii) We have both that x belongs to

$$\{(x, y) \in X \mid |x| > 10 \text{ or } |y| > 10\}$$

and that this set is a subset of U.

Demonstrate that $(X, \mathcal{O}_X)$ is not connected. [5 marks]

c) Let $\mathcal{O}'_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Is $(X, \mathcal{O}_X)$ homeomorphic to $(X, \mathcal{O}'_X)$? Justify your answer. [8 marks]

d) Define a topology $\mathcal{O}''_X$ on X such that $(X, \mathcal{O}''_X)$ has the following properties.

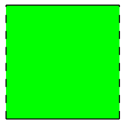
i) It is connected.

ii) It is not compact.

Give a proof that $(X, \mathcal{O}''_X)$ has property ii). You do not need to prove that $(X, \mathcal{O}''_X)$ has property i). [6 marks]

Revision Question 6 — 05/05/14

Let X be the subset of $\mathbb{R}^2$ given by $]0,1[\times [0,1]$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



a) Find an open covering of X with respect to $\mathcal{O}_X$ which does not admit a finite subcovering. [3 marks]

b) Find a subset Y of $\mathbb{R}^2$ such that the following hold, where $\mathcal{O}_Y$ is the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

   i) X is a subset of Y.

   ii) $(Y, \mathcal{O}_Y)$ is not locally compact.

   Prove that (ii) holds. [8 marks]

c) Define an equivalence relation $\sim$ on I^2 such that $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ is a one point compactification of $(X, \mathcal{O}_X)$. You do not need to give a proof. You may wish to draw a picture. [6 marks]

d) Let

$$I^2 \xrightarrow{\pi} I^2/\sim$$

   be the quotient map, where $\sim$ is your equivalence relation of part c). Find an example of a subset U of I^2 which belongs to $\mathcal{O}_{I^2}$, but for which $\pi(U)$ does not belong to $\mathcal{O}_{I^2/\sim}$. [4 marks]

e) Let $\sim$ be the equivalence relation on $\mathbb{R}$ given by $x_0 \sim x_1$ if $x_1 - x_0$ belongs to $\mathbb{Z}$. Let

$$\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\sim$$

   be the quotient map. Prove that if U belongs to $\mathcal{O}_{\mathbb{R}}$, then $\pi(U)$ belongs to $\mathcal{O}_{\mathbb{R}/\sim}$. [6 marks]

f) Let $\sim$ be the equivalence relation on $\mathbb{R}$ given by $x_0 \sim x_1$ if $x_1 - x_0$ belongs to $\mathbb{Q}$. Prove that for any topological space $(X', \mathcal{O}_{X'})$, we have that every map

$$X' \longrightarrow \mathbb{R}/{\sim}$$

is continuous. You may assume without proof any facts about $\mathbb{R}$. [8 marks]

Revision Question 7 — 06/05/14

Let X be the set $\{a, b, c, d\}$.

a) Which of the following define a topology on X? For those which do not, give a reason.

   i) $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$

   ii) $\{\emptyset, \{d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, b, c, d\}\}$

   iii) $\{\emptyset, \{c\}, \{a, d\}, \{b, c\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c, d\}\}$.

   [3 marks]

Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}.$$

b) What is the boundary of $\{b, c\}$ in X with respect to $\mathcal{O}_X$? [5 marks]

View $(S^1, \mathcal{O}_{S^1})$ as $(I/\sim, \mathcal{O}_{I/\sim})$, where $\sim$ is the equivalence relation on I generated by $0 \sim 1$.

c) Let

$$S^1 \xrightarrow{f} X$$

be the surjective map given by

$$[x] \mapsto \begin{cases} a & \text{if } x = 0, \\ b & \text{if } 0 < x < \frac{1}{2}, \\ c & \text{if } x = \frac{1}{2}, \\ d & \text{if } \frac{1}{2} < x < 1. \end{cases}$$

Prove that f is continuous, where X is equipped with the topology $\mathcal{O}_X$. Justify every assertion that you make that a given set belongs to any of the topologies that you consider. [7 marks]

d) Prove in two different ways that $(S^1, \mathcal{O}_{S^1})$ is not homeomorphic to $(X, \mathcal{O}_X)$. You may quote without proof any results from the course. [6 marks]

Let $\mathcal{O}'_X$ be the topology on X given by

$$\{\emptyset, \{b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}\}.$$

e) Demonstrate that the set

$$\{(a,b),(a,c),(b,b),(b,c),(b,d),(d,b),(d,c)\}$$

belongs to the product topology with respect to $\mathcal{O}_X$ and $\mathcal{O}'_X$ on $X \times X$.

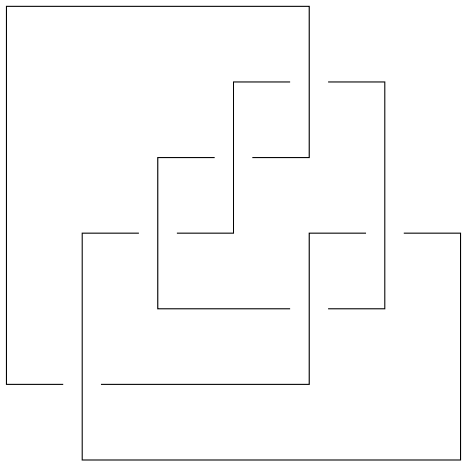
Let $\mathcal{O}''_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,d\}, \{a,b,d\}, \{a,c,d\}, \{a,b,c,d\}\}.$$

f) Is $(X, \mathcal{O}''_X)$ homeomorphic to $(X, \mathcal{O}_X)$? Justify your answer. [5 marks]

Revision Question 8 — 07/05/14

The knot 6_2 is pictured below.

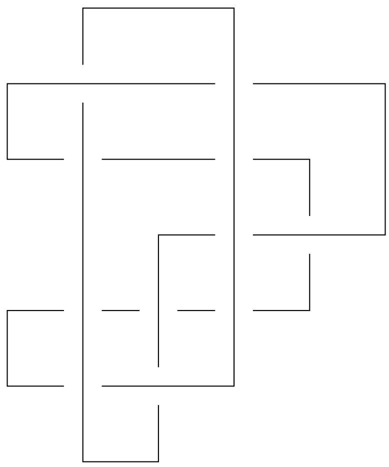


a) Using the skein relations or otherwise, calculate the Jones polynomial of 6_2. [10 marks]

b) Is 6_2 isotopic to its mirror image? Justify your answer. You may quote without proof any results from the course. [3 marks]

c) Explain how to construct the Seifert surface associated to 6_2. What is its genus? [5 marks]

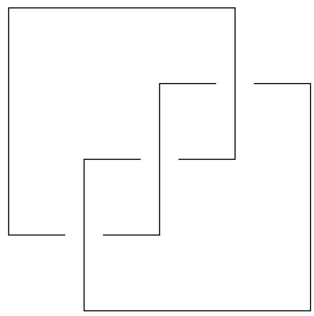
Let K be the knot depicted below.



d) Calculate the writhe $w(K)$ of K? [3 marks]

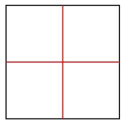
e) Find a knot which is isotopic to K, but whose writhe is $w(K) + 3$. [3 marks]

f) Using the Reidemeister moves, demonstrate that K is isotopic to 3_1, the trefoil, depicted below. For fewer marks, you may give a demonstration that does not use only the Reidemeister moves. [6 marks]



Revision Question 9 — 08/05/14

Let A be the subset of I^2 given by $\left(\frac{1}{2} \times I\right) \cup \left(I \times \frac{1}{2}\right)$.



a) Prove that A is closed in I^2 with respect to $\mathcal{O}_{I^2}$. [4 marks]

b) Give two different proofs that A is a connected subset of I^2 with respect to $\mathcal{O}_{I^2}$. You may quote without proof any results from the course. [6 marks]

c) Let $\mathcal{O}_{I^2 \setminus A}$ be the subspace topology on $I^2 \setminus A$ with respect to $(I^2, \mathcal{O}_{I^2})$. What is the connected component of $(\frac{1}{4}, \frac{3}{4})$ in $I^2 \setminus A$ with respect to $\mathcal{O}_{I^2 \setminus A}$? Justify your answer. [6 marks]

Let $\sim$ be the equivalence relation on I^2 given by $x \sim y$ if both x and y belong to $I^2 \setminus A$.

d) Prove that $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ is not Hausdorff. [5 marks]

e) Find a subset B of $I^2/\sim$ which is compact with respect to $\mathcal{O}_{I^2/\sim}$, but which is not closed in $I^2/\sim$ with respect to $\mathcal{O}_{I^2/\sim}$. Justify why your set B has these properties. You may quote without proof any results from the course. [6 marks]

f) Is there a compact subset of I^2 which is not closed? Justify your answer. You may quote without proof any results from the course. [4 marks]

Let $\approx$ be the equivalence relation on I^2 given by $x \approx y$ if both x and y belong to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} I^2$. Let

$$I^2 \xrightarrow{\pi} I^2/\approx$$

be the quotient map.

g) Let $\mathcal{O}_{\pi(A)}$ be the subspace topology on $\pi(A)$ with respect to $(I^2/\approx, \mathcal{O}_{I^2/\approx})$. Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(I^2, \mathcal{O}_{I^2})$. Is $\left(\pi(A), \mathcal{O}_{\pi(A)}\right)$ homeomorphic to $(A, \mathcal{O}_A)$? Justify your answer. [6 marks]

Revision Question 10 — 09/05/14

Let X be the set $[0,1] \times]0,1[\times]0,1[$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$.

a) Find an open covering of X which does not admit a finite subcovering. [3 marks]

b) Let A be the subset of X given by $\left]\frac{1}{2}, 1\right] \times \left]\frac{1}{4}, \frac{3}{4}\right[\times \left]\frac{1}{4}, \frac{3}{4}\right[$. Decide whether the following are true or false, and justify your answers.

   i) The set A belongs to $\mathcal{O}_X$.

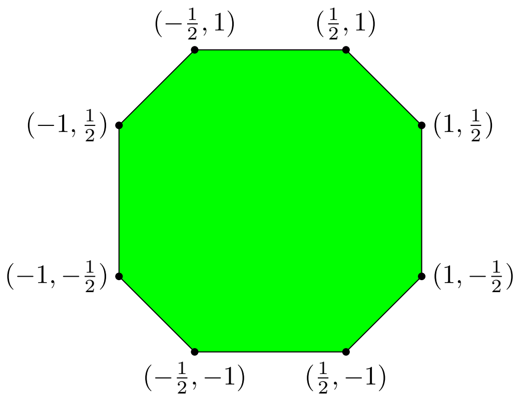
   ii) The set A is closed in X with respect to $\mathcal{O}_X$.

   [6 marks]

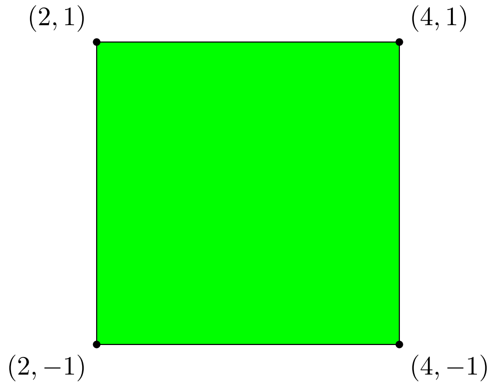
c) Let $\mathcal{O}_{[0,1]}$ be the subspace topology on $[0,1]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Give a counterexample to the following statement: for every subset W of X such that W belongs to $\mathcal{O}_X$, and such that $\left\{\frac{1}{2}\right\} \times]0,1[\times]0,1[$ is a subset of W, there is a neighbourhood U of $\frac{1}{2}$ in $[0,1]$ with respect to $\mathcal{O}_{[0,1]}$ such that $U \times]0,1[\times]0,1[$ is a subset of W. [4 marks]

d) Is it possible to find a counterexample if we replace X by $[0,1] \times [0,1] \times [0,1]$? Justify your answer. You may quote without proof any results from the course. [5 marks]

Let O be the octagon in $\mathbb{R}^2$ depicted below.



Let S be the square in $\mathbb{R}^2$ depicted below.

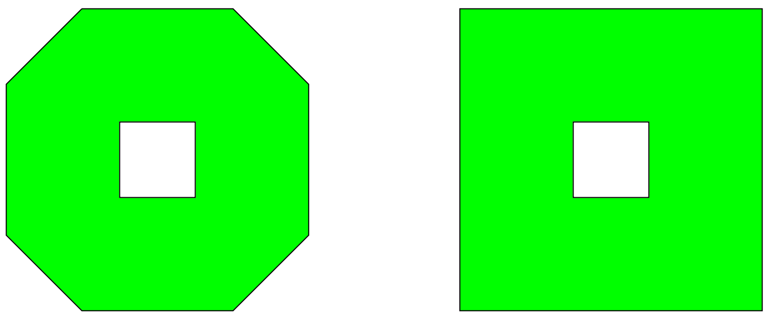


Let Y be the subset of $\mathbb{R}^2$ given by the union of

$$O \setminus \left(\left]-\tfrac{1}{4}, \tfrac{1}{4}\right[\times \left]-\tfrac{1}{4}, \tfrac{1}{4}\right[\right)$$

and

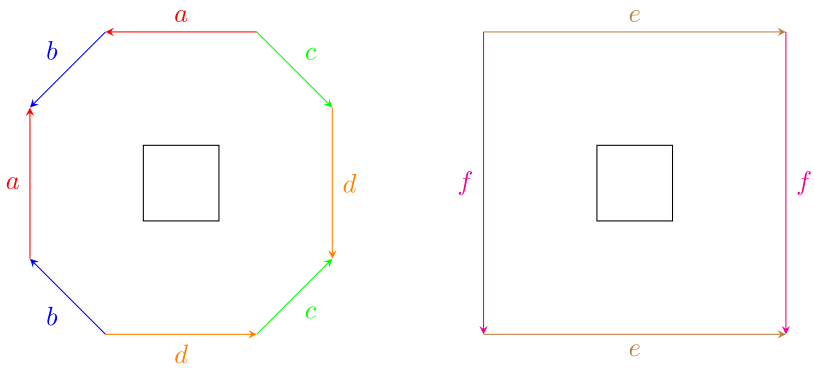
$$S \setminus \left(\left]\tfrac{11}{4}, \tfrac{13}{4}\right[\times \left]-\tfrac{1}{4}, \tfrac{1}{4}\right[\right).$$



Let $\mathcal{O}_Y$ be the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

e) Prove that $(Y, \mathcal{O}_Y)$ is not connected. [5 marks]

f) Let $\sim$ be the equivalence relation on Y which identifies, without a twist, the edges with the same letter in the figure below.



Define an equivalence relation $\approx$ on Y such that the following hold.

i) Suppose that x_0 and x_1 belong to $\mathbb{R}^2$, and that $x_0 \sim x_1$. Then $x_0 \approx x_1$.

ii) We have that $(Y/\!\approx, \mathcal{O}_{Y/\approx})$ is a surface.

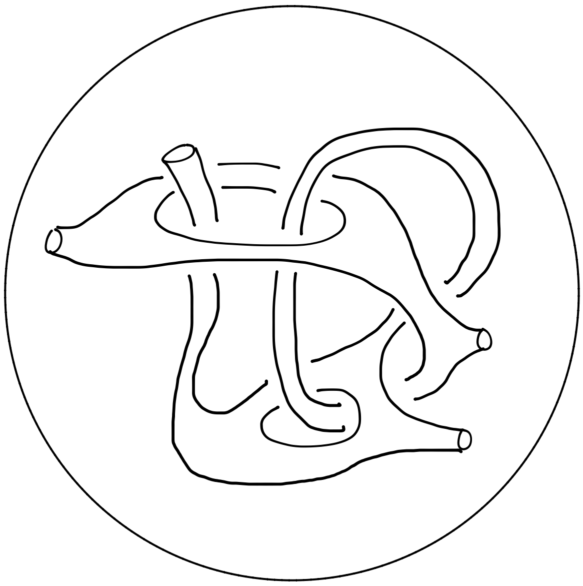
You do not need to prove that i) and ii) hold. [6 marks]

g) Equip $(Y/\!\approx, \mathcal{O}_{Y/\approx})$ with the structure of a Δ-complex. [4 marks]

h) Calculate the Euler characteristic of $(Y/\!\approx, \mathcal{O}_{Y/\approx})$. [3 marks]

i) Assume that $(Y/\!\approx, \mathcal{O}_{Y/\approx})$ is an n-handlebody for some n. Using your answer to part h) or otherwise, determine n. You may quote without proof any results from the course. [3 marks]

Let $(X, \mathcal{O}_X)$ be the surface given by the 'sphere with tunnels' depicted below.



j) Which of the surfaces in the classification is $(X, \mathcal{O}_X)$ homeomorphic to? Justify your answer by a surgery argument. [6 marks]

k) Hence or otherwise, calculate the Euler characteristic of $(X, \mathcal{O}_X)$.

Solutions and Discussion

The solutions here are not presented in as much detail as if they had appeared in the lecture notes. This is to help indicate the level of detail that I am looking for on the exam.

It is good to consider how you would in principle go about giving a proof with all the details, even if you do not actually write this proof down.

Just let me know if you would like me to elaborate upon or clarify anything.

Solutions 1

a) Yes, $(X, \mathcal{O}_X)$ is compact. By a result from the course, a subset of $\mathbb{R}^2$ is compact if and only if it is bounded and closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. The set X has both of these properties.

b) Yes, $(X, \mathcal{O}_X)$ is Hausdorff. This is a consequence of the following results from the course.

(1) $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff.

(2) Products of Hausdorff topological spaces are Hausdorff. By (1), we thus have that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff.

(3) Every subspace of a Hausdorff topological space is Hausdorff.

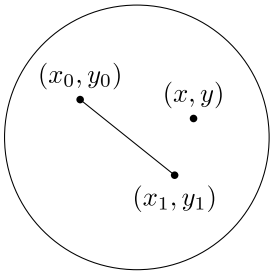
c) Yes, $(X, \mathcal{O}_X)$ is locally compact. By a result from the course, every compact Hausdorff topological space is locally compact. That $(X, \mathcal{O}_X)$ is locally compact thus follows from a) and b).

d) Suppose that (x_0, y_0) and (x_1, y_1) belong to $D^2 \setminus \{(x, y)\}$. Suppose that (x, y) does not lie on the straight line between these two points, or in other words in the image of the map

$$I \xrightarrow{f_{(x_0,y_0),(x_1,y_1)}} D^2$$

given by

$$t \mapsto (1-t)(x_0, y_0) + t(x_1, y_1).$$



This map is a product of polynomial maps, namely

$$t \mapsto (x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0)) .$$

Hence it is continuous. We also have that

$$f_{(x_0,y_0),(x_1,y_1)}(0) = (x_0, y_0),$$

and that

$$f_{(x_0,y_0),(x_1,y_1)}(1) = (x_1, y_1).$$

Thus the map

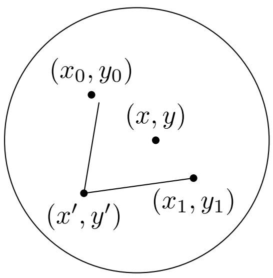
$$I \longrightarrow D^2 \setminus \{(x, y)\}$$

given by

$$t \mapsto f_{(x_0,y_0),(x_1,y_1)}(t)$$

defines a path from (x_0, y_0) to (x_1, y_1) in $(D^2 \setminus \{(x, y)\}, \mathcal{O}_{D^2\setminus\{(x,y)\}})$.

Suppose now that (x, y) does belong to the image of $f_{(x_0,y_0),(x_1,y_1)}$. Let (x', y') be a point of D^2 which does not belong to the image of $f_{(x_0,y_0),(x_1,y_1)}$. Then $f_{(x_0,y_0),(x',y')}$ is a path from (x_0, y_0) to (x', y') in $(D^2 \setminus \{(x, y)\}, \mathcal{O}_{D^2\setminus\{(x,y)\}})$, and $f_{(x',y'),(x_1,y_1)}$ is a path from (x', y') to (x_1, y_1) in $(D^2 \setminus \{(x, y)\}, \mathcal{O}_{D^2\setminus\{(x,y)\}})$.



By as result from the course, we can 'concatenate' these two paths to obtain a path from (x_0, y_0) to (x_1, y_1) in $(D^2 \setminus (x, y), \mathcal{O}_{D^2\setminus\{(x,y)\}})$.

e) No, $(X, \mathcal{O}_X)$ is not homeomorphic to $(Y, \mathcal{O}_Y)$. We have the following two facts from the course.

(1) If

$$X \xrightarrow{f} Y$$

is a homeomorphism, then the map

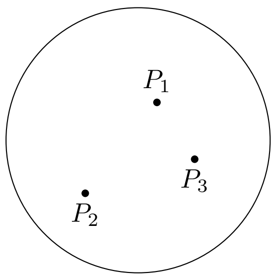
$$X \setminus A \longrightarrow Y \setminus f(A)$$

given by $x \mapsto f(x)$ is a homeomorphism for any subset A of X, where $X \setminus A$ has the subspace topology with respect to $(X, \mathcal{O}_X)$, and $Y \setminus f(A)$ has the subspace topology with respect to $(Y, \mathcal{O}_Y)$.

(2) Homeomorphisms preserve connectedness.

Suppose that P_0, P_1, and P_2 belong to

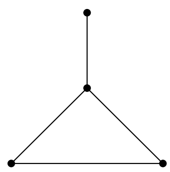
$$\{(2, y) \mid 1 < y < 3\}.$$



Then $X \setminus \{P_0, P_1, P_2\}$, equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$, is connected. However, by removing any three points from Y, and equipping this set with the subspace topology with respect to $(Y, \mathcal{O}_Y)$, we obtain a topological space which is not connected.

That $(X, \mathcal{O}_X)$ is not homeomorphic to $(Y, \mathcal{O}_Y)$ thus follows from (1) and (2).

f) A Δ-complex structure on $(X, \mathcal{O}_X)$ is depicted below. The triangle is to be regarded as a 2-simplex.



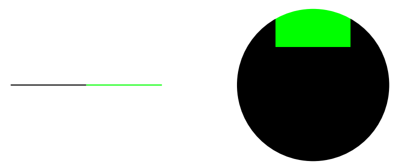
The Euler characteristic of $(X, \mathcal{O}_X)$ is $4 - 4 + 1 = 1$.

g) Yes, $\pi(A)$ belongs to $\mathcal{O}_{Z/\sim}$. We have that $\pi^{-1}(\pi(A))$ is the union of

$$\{(x, 0) \mid -1 < x \leq 0\}$$

and

$$\{(x, y) \mid \|(x-2, y)\| \leq 1\} \cap \left(\left]\tfrac{3}{2}, \tfrac{5}{2}\right[\times \left]\tfrac{1}{2}, \tfrac{3}{2}\right[\right).$$

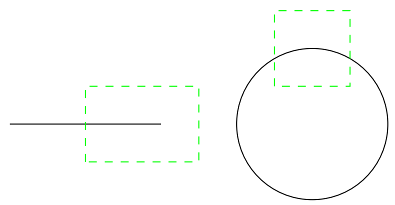


This union is the intersection with Z of, for example, the union of

$$\left]-1, \tfrac{1}{2}\right[\times \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[$$

and

$$\left]\tfrac{3}{2}, \tfrac{5}{2}\right[\times \left]\tfrac{1}{2}, \tfrac{3}{2}\right[.$$



Both of these two 'open rectangles' belong to $\mathcal{O}_{\mathbb{R}^2}$. Hence so does their union. By definition of $\mathcal{O}_Z$, we deduce that $\pi^{-1}(\pi(A))$ belongs to $\mathcal{O}_Z$. Hence $\pi(A)$ belongs to $\mathcal{O}_{Z/\sim}$.

h) Let

$$Z \xrightarrow{f} X$$

be the map given by

$$(x, y) \mapsto \begin{cases} (2, x+1) & \text{if } -2 \leq x \leq 0, \\ (x, y) & \text{otherwise.} \end{cases}$$

We have that $f(0,0) = f(2,1)$. Since this map is obtained by 'glueing' polynomial maps, it is continuous. By an exercise from the course, we deduce that the map

$$Z/\sim \xrightarrow{g} X$$

given by $[z] \mapsto f(z)$ is continuous. Moreover, g is evidently bijective.

The following hold.

(1) We have that $(Z, \mathcal{O}_Z)$ is compact, since it is a closed and bounded subset of $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Since a quotient of a compact topological space is compact, we deduce that $(Z/\sim, \mathcal{O}_{Z/\sim})$ is compact.

(2) By b), we have that $(X, \mathcal{O}_X)$ is Hausdorff.

By a result from the course, every continuous bijection with a compact source and a Hausdorff target is a homeomorphism. We conclude that g is a homeomorphism.

Discussion

Proving compactness

We have seen two principal ways to prove compactness.

(1) Begin with the fact that $(I, \mathcal{O}_I)$ is compact, and then appeal to the fact that compactness is preserved by the following 'canonical constructions': products, quotients, homeomorphisms, and closed subspaces.

(2) Use the characterisation of compact subsets of $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ as subsets which are bounded and closed in $\mathbb{R}^n$ with respect to $\mathcal{O}_{\mathbb{R}^n}$.

When asked to prove that a geometric example of a topological space is compact, it is a good idea to begin by considering which of these methods is most appropriate.

Proving that a topological space is Hausdorff

Similar remarks apply with regard to proving that a topological space is Hausdorff.

(1) Begin with the fact that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff, which you should make sure that you are able to prove, and then appeal the fact that being Hausdorff is preserved by the following 'canonical constructions: products, subspaces, and homeomorphisms'.

(2) Don't forget that being Hausdorff is not preserved by taking quotients in general. However, it is preserved under 'sufficiently nice' quotients, and it is important to recognise when the conditions for a quotient of Hausdorff topological space to be Hausdorff hold. This will be discussed in the solutions to a later revision question,

Demonstrating path connectedness

When proving that a geometric example of a topological space is path connected, a typical technique is the following.

(1) Use straight line paths 'as far as possible'. This is simply because straight line paths are easy to write down, and are evidently continuous (they are polynomial maps).

(2) Concatenate and reverse, as needed, straight line paths from 1) for any remaining paths.

In d), we use straight line paths as long as (x, y) does not lie on this straight line, and then use a concatenation of two such paths to treat the remaining cases.

Homeomorphisms in geometric examples

Whilst it will always be possible to use a different method, it is often very convenient when asked to prove that two geometric examples of topological spaces are homeomorphic, to use the fact that a continuous bijection in which the source is compact and the target is Hausdorff is a homeomorphism, as in h). This is particularly often useful when the source is a quotient.

You should be very careful, though, to appeal to this result only when the hypotheses are satisfied!

Repertoire of continuous maps

We typically rely only on the following to construct continuous maps. All of these facts are established in the exercises to Lecture 5.

1) Polynomial maps are continuous.

2) We can 'glue' continuous maps together: on disjoint pieces as in the definition of f in h); but also more generally when we have a union of open sets, and maps on each of these open sets, such that the maps agree wherever the open sets intersect; and similarly for a finite (or, more generally, locally finite, but don't worry too much about this) union of closed sets.

3) We can take products of continuous maps, as in d). In particular, maps to $\mathbb{R}^n$ which are polynomials 'in each component' are continuous.

4) Continuous maps which respect a given equivalence relation 'pass to the quotient', as in h).

We sometimes also make use of a continuous map

$$\mathbb{R} \xrightarrow{\phi} S^1$$

which 'travels around S^1' once for every interval $[n, n+1]$, where $n \in \mathbb{Z}$.

Solutions 2

a) Let

$$\mathbb{R} \xrightarrow{f} X$$

be the map given by $x \mapsto (x, x^2)$. Since f is a polynomial map in each component, it is continuous. Let

$$X \xrightarrow{g} \mathbb{R}$$

be the map given by $(x, y) \mapsto x$. Since g is the restriction to X of the projection map

$$\mathbb{R}^2 \longrightarrow \mathbb{R}$$

given by $(x, y) \mapsto x$, it is continuous. We have that

$$\begin{aligned} g(f(x)) &= g\left(x, x^2\right) \\ &= x. \end{aligned}$$

Thus $g \circ f = id_{\mathbb{R}}$. We also have that

$$\begin{aligned} f(g(x,y)) &= f(x) \\ &= (x, x^2). \end{aligned}$$

Thus $f \circ g = id_X$.

b) One possibility is to take Y to be

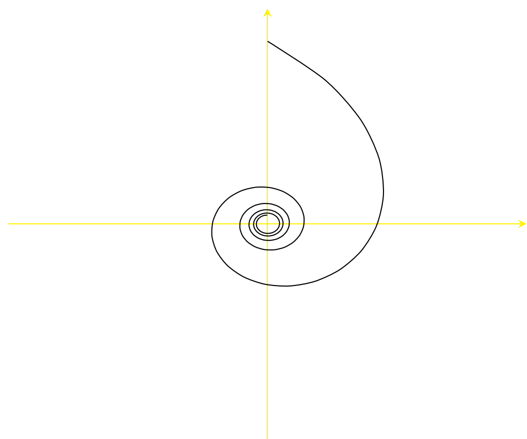
$$\{(x, 0) \mid 0 < x < 1\}.$$

Then $(0,0)$ and $(0,1)$ are limit points of Y in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. It is straightforward, by a similar argument to that of part a), to prove that $(Y, \mathcal{O}_Y)$ is homeomorphic to the open interval $]0,1[$, equipped with its subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. We proved in the lectures that an open interval with its subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

For a more exotic example, one could take Y to be the spiral given by image of the map

$$]0, \infty[\xrightarrow{f} \mathbb{R}^2$$

defined by $t \mapsto \frac{\phi(t)}{t}$, where ϕ is the 'travelling around the circle' map constructed in the exercises to Lecture 5. Then $(0,0)$ and $(0,1)$ are limit points of Y in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ which does not belong to Y. Thus Y is not closed in $\mathbb{R}^2$. It is not important to prove this carefully, nor to prove that $(Y, \mathcal{O}_Y)$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, but if you would like a challenge and have attempted the other revision questions, you might like to have a go at this!



c) For the first example given in b), the closure of Y in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ is the union of Y and $\{(0,0),(1,0)\}$, or in other words

$$\{(x,0) \mid 0 \leq x \leq 1\}.$$

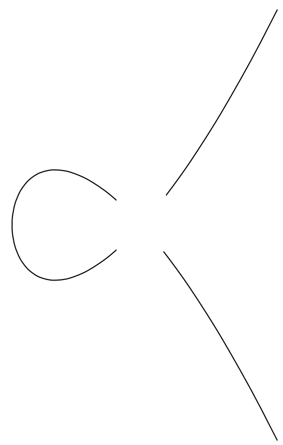
For the second examplei given in b), the closure of Y in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ is the union of Y and $\{(0,0),(0,1)\}$ in the second example.

d) No, $(Z, \mathcal{O}_Z)$ is not homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, by the following argument.

   (1) If there were a homeomorphism between these topological spaces, then there would be a homeomorphism between $Z \setminus \{(0,0)\}$, equipped with the subspace topology $\mathcal{O}_{Z\setminus\{0,0\}}$ with respect to $(Z, \mathcal{O}_Z)$, and $\mathbb{R} \setminus \{x\}$, equipped with the subspace topology $\mathcal{O}_{\mathbb{R}\setminus\{x\}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, for some x which belongs to $\mathbb{R}$.

   (2) Homeomorphic topological spaces have the same number of connected components.

   (3) The topological space $(Z \setminus \{0,0\}, \mathcal{O}_{Z\setminus\{0,0\}})$ has three connected components.



(4) For every x which belongs to $\mathbb{R}$, the topological space $(\mathbb{R} \setminus \{x\}, \mathcal{O}_{\mathbb{R}\setminus\{x\}})$ has two connected components.

e) We can argue as follows.

(1) The map

$$\mathbb{R} \xrightarrow{f} Z$$

given by $x \mapsto (x^2 - 1, x^3 - x)$ is continuous, since it is a polynomial map in each component.

(2) Moreover, f is surjective.

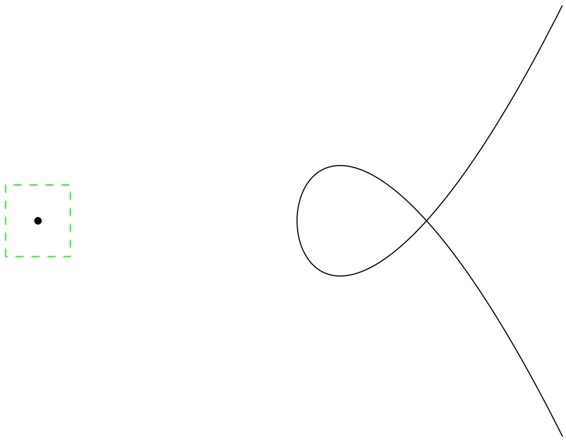
(3) By a result from the course, we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.

(4) By another result from the course, the target of a surjective, continuous map with a connected source is connected.

f) We have that $x^2 - 1 \geq -1$ for all $x \in \mathbb{R}$. Thus, for instance, the subset U of $\mathbb{R}^2$ given by

$$\left]-\tfrac{13}{4}, -\tfrac{11}{4}\right[\times \left]-\tfrac{1}{4}, \tfrac{1}{4}\right[$$

is a neighbourhood of $(-3, 0)$ in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ such that $U \cap Z$ is empty, since $-\frac{11}{4} < -1$.



Discussion

Closedness

Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. We have two equivalent definitions of what it means for A to be closed in X with respect to $\mathcal{O}_X$.

(1) That $X \setminus A$ belongs to $\mathcal{O}_X$.

(2) That every limit point of A in X with respect to $\mathcal{O}_X$ belongs to A.

When trying to prove that a given subset is closed or not closed, keep both points of view in mind, and decide which definition would be most convenient to use.

Closedness interacts in important ways with compactness and Hausdorffness. Thus it is very important that you have a firm grasp of deciding whether or not a given set is closed, and that you can calculate the closure of a given set.

Proving that two topological spaces are not homeomorphic

Remember that to prove that two topological spaces are not homeomorphic, it is not enough to demonstrate that any particular map is not a homeomorphism. Thus, in part d), it is not enough to demonstrate that the map

$$\mathbb{R} \longrightarrow Z$$

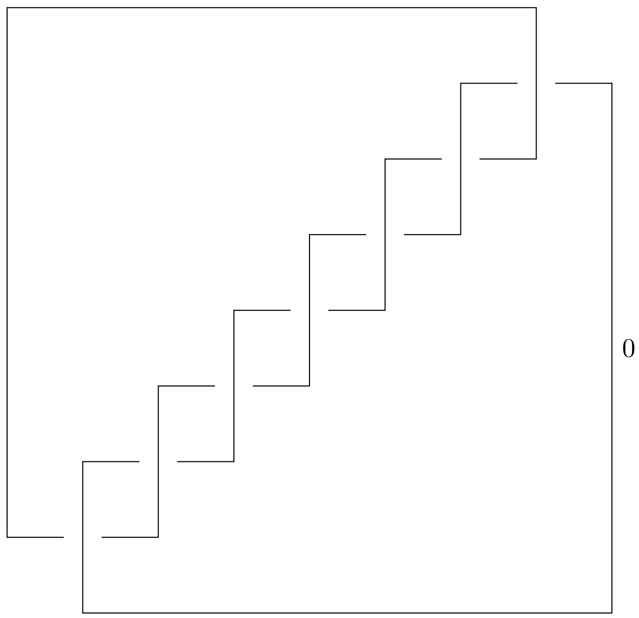
given by $x \mapsto (x^2 - 1, x^3 - x)$ is not a homeomorphism. Even though we then cannot carry out an analogue of the argument of part a), there might, a priori, be a different map between $\mathbb{R}$ and Z which we could prove to be a homeomorphism. Thus we have to demonstrate that there is no map at all between $\mathbb{R}$ and Z which is a homeomorphism.

For this, we need to make use of an 'invariant' of our topological spaces. This can be a property (connectedness, compactness, Hausdorffness, etc) of topological spaces that is preserved by homeomorphisms; or it can be a number, or some other gadget, which we associate to topological spaces, and which is preserved by homeomorphisms. In part d), the number of connected components is the invariant that we make use of (after removing a point).

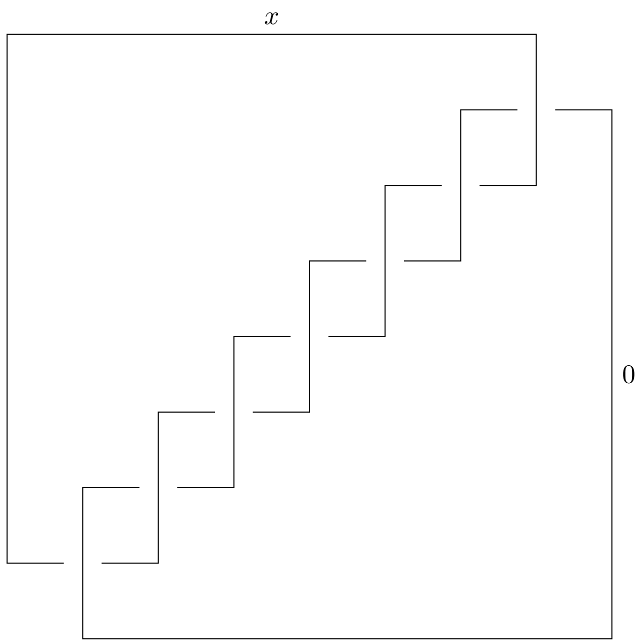
When asked whether two topological spaces are homeomorphic, run through the properties of topological spaces that we have covered in the course, and try to figure out whether one of the topological spaces has one of these properties, but the other does not. See whether you can apply the removing points technique, or whether, in appropriate situations (when we have surfaces, for instance), their Euler characteristics differ.

Solutions 3

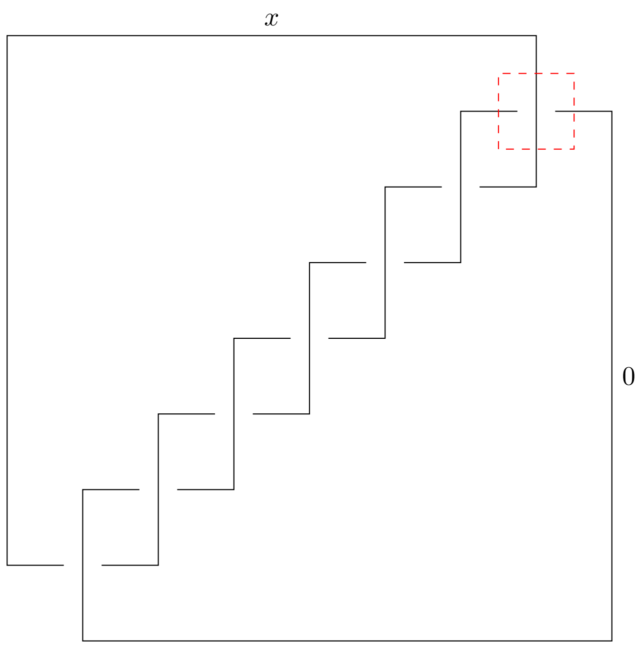
a) Suppose that 7_1 is m-colourable. By a result from the course, we can assume that the integer assigned to one of the arcs is 0, for instance the arc indicated below.



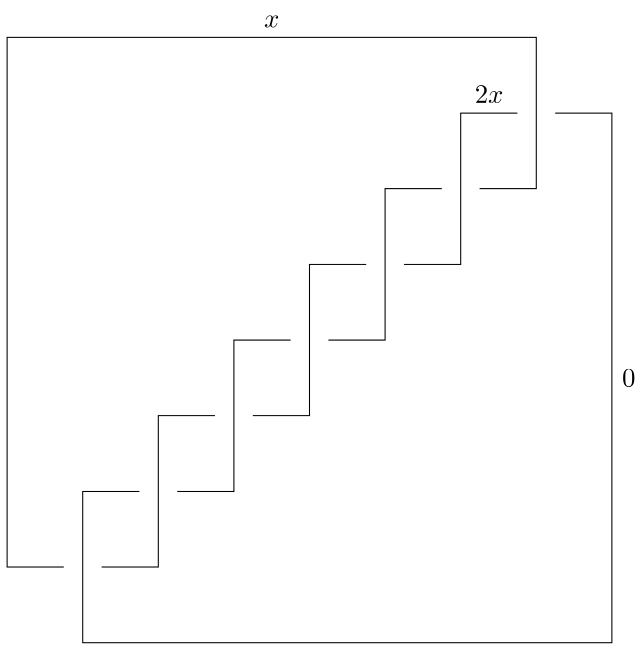
Suppose that the integer, mod m, assigned to the arc indicated below is x.



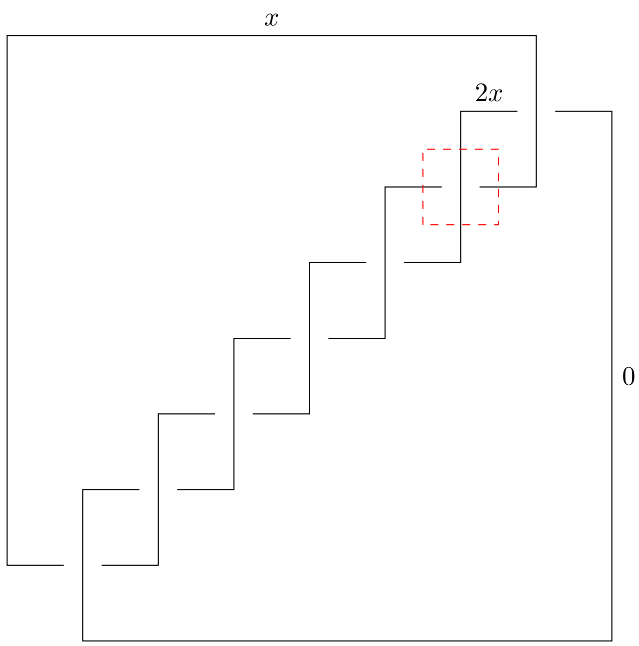
Consider the following crossing.



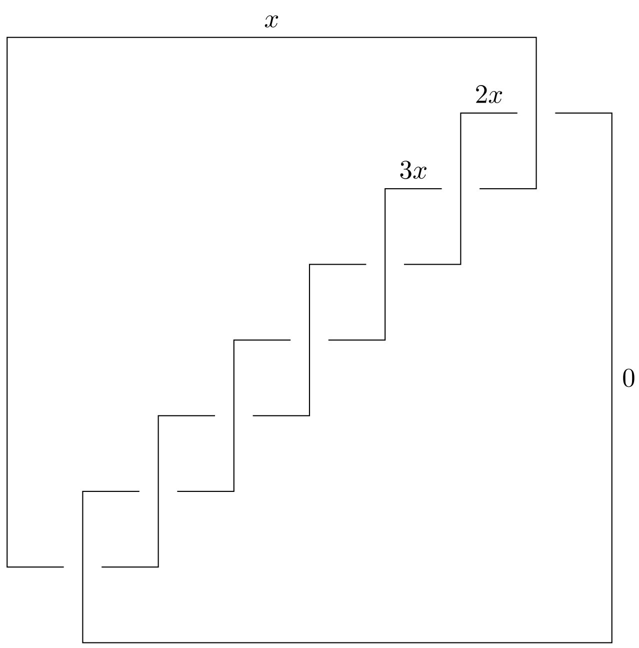
By the condition for a knot colouring which must hold at this crossing, we have that the integer assigned to the indicated arc is $2x \pmod{m}$.



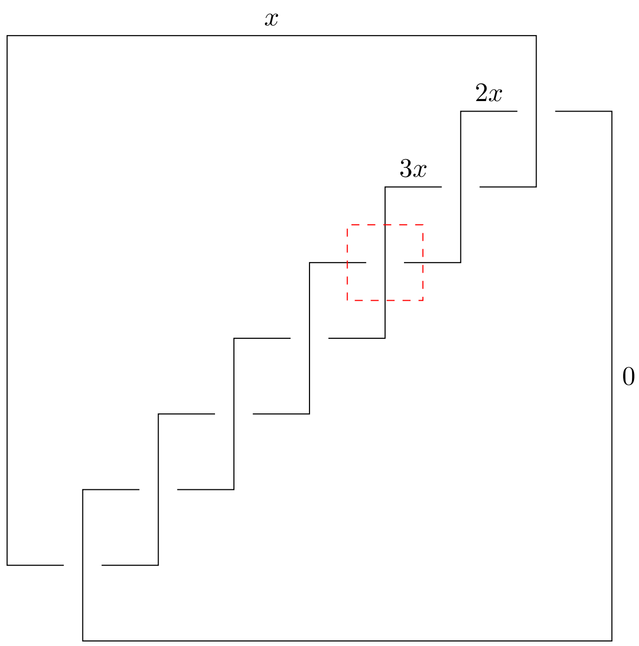
Consider the following crossing.



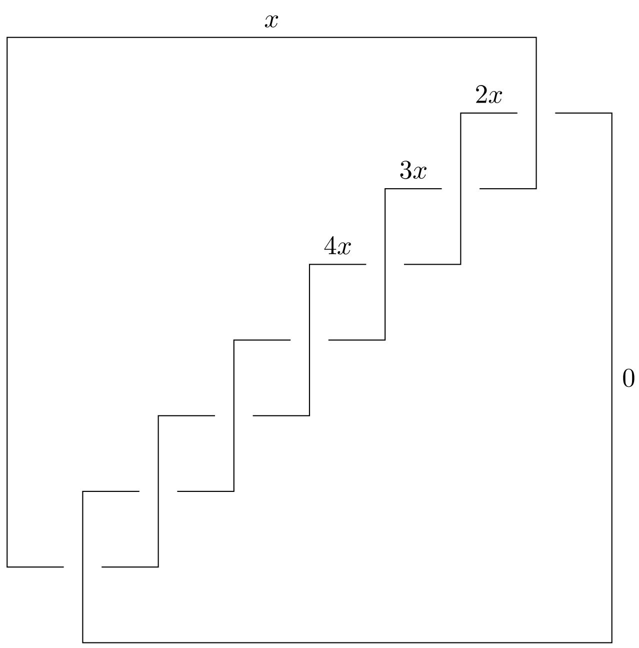
By the condition for a knot colouring which must hold at this crossing, we have that the integer assigned to the indicated arc is $4x - x \pmod{m}$, namely $3x \pmod{m}$.



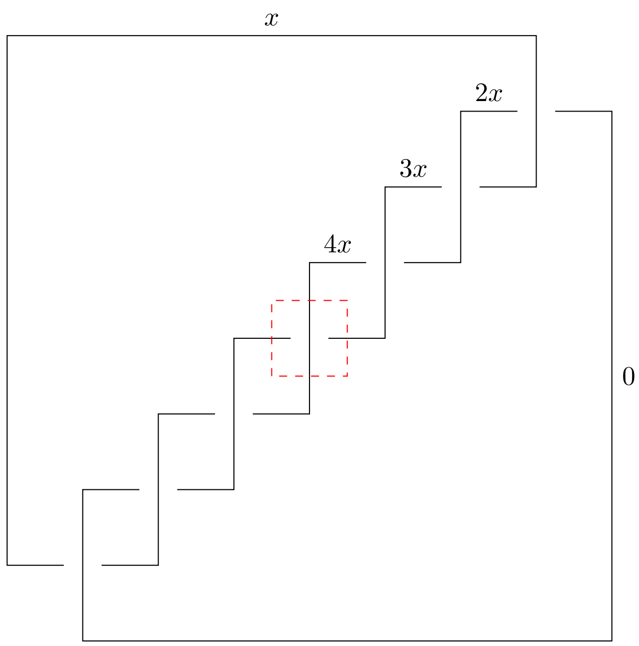
Consider the following crossing.



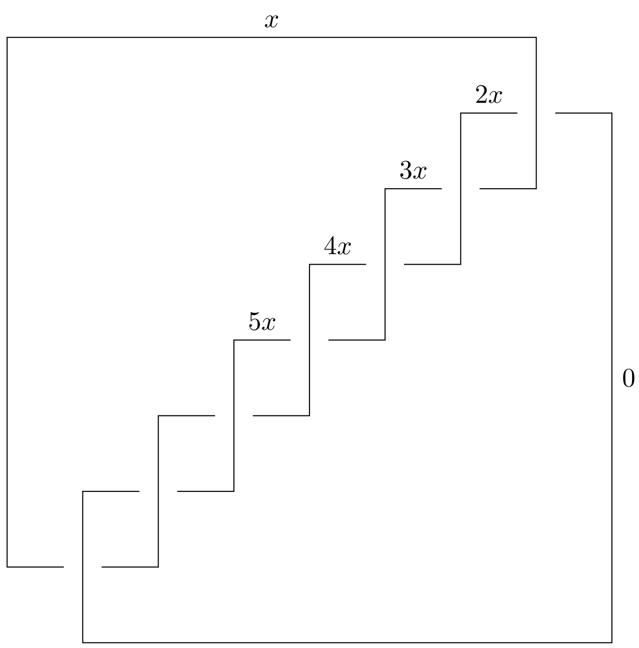
By the condition for a knot colouring which must hold at this crossing, we have that the integer assigned to the indicated arc is $6x - 2x \pmod{m}$, namely $4x \pmod{m}$.



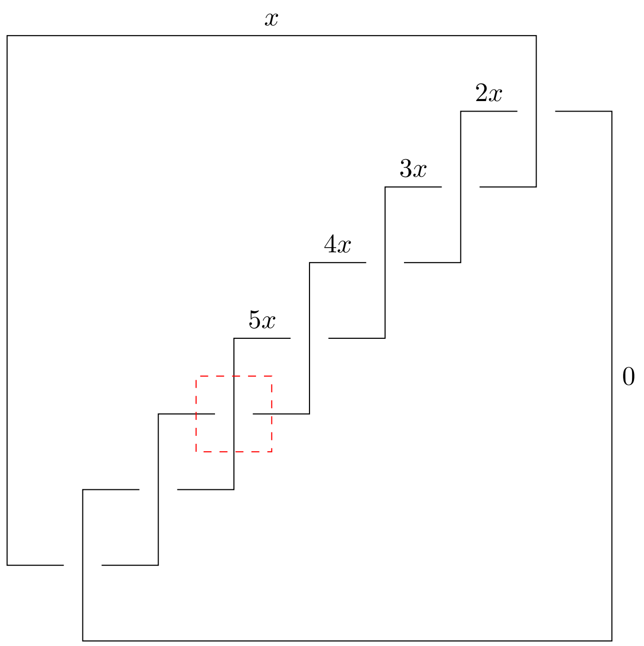
Consider the following crossing.



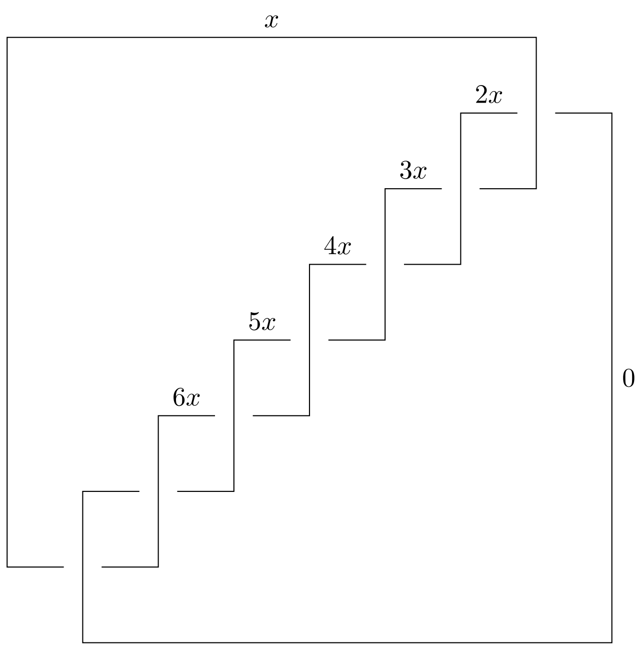
By the condition for a knot colouring which must hold at this crossing, we have that the integer assigned to the indicated arc is $8x - 3x \pmod{m}$, namely $5x \pmod{m}$.



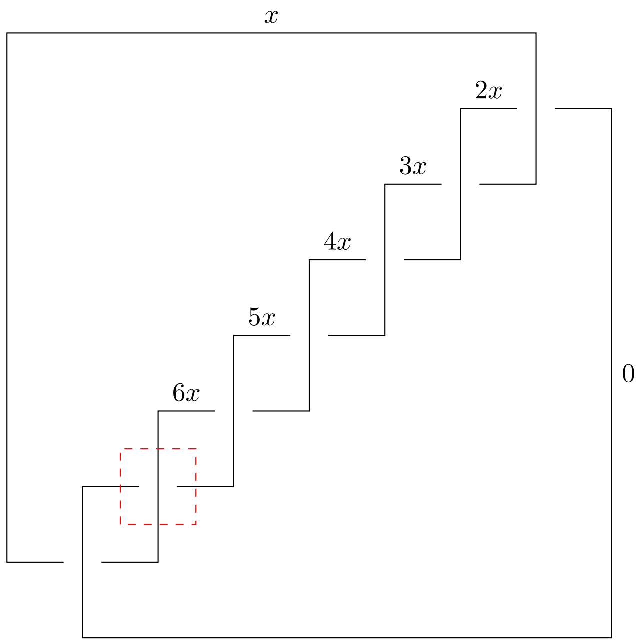
Consider the following crossing.



By the condition for a knot colouring which must hold at this crossing, we have that the integer assigned to the indicated arc is $10x - 4x \pmod{m}$, namely $6x \pmod{m}$.



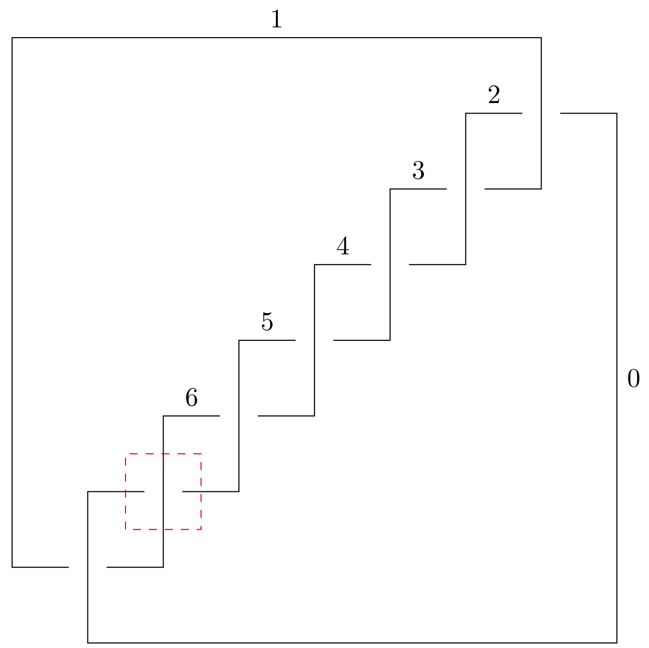
Consider the following crossing.



We have already assigned integers to all the arcs involved in this crossing. By the condition for a knot colouring which must hold at this crossing, we have that $12x \equiv 5x + 0 \pmod{m}$. We deduce that $7x \equiv 0 \pmod{m}$.

By definition of equivalence mod m, we thus have that there is an integer k such that $7x = km$. Since 7 is a prime, we have, by the the fundamental theorem of arithmetic, that either $7 \mid k$ or $7 \mid m$. Suppose that $7 \mid k$. Then we have that $x = \frac{k}{7} \cdot m$, and $\frac{k}{7}$ is an integer. Thus we have that $x \equiv 0 \pmod{m}$. Then 0 is assigned to every arc mod m, which contradicts one of the conditions for a colouring.

We conclude that $7 \mid m$. Conversely, a 7-colouring of 7_1 is obtained by, for instance, taking x to be 1 in the previous figure.



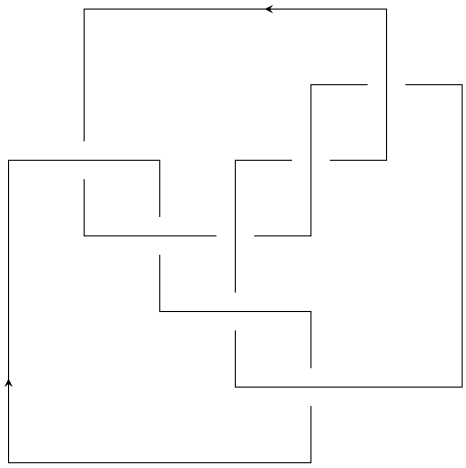
A $7k$-colouring for any integer k is obtained by multiplying the integer assigned to each arc in the previous figure by k.

b) Omitted.

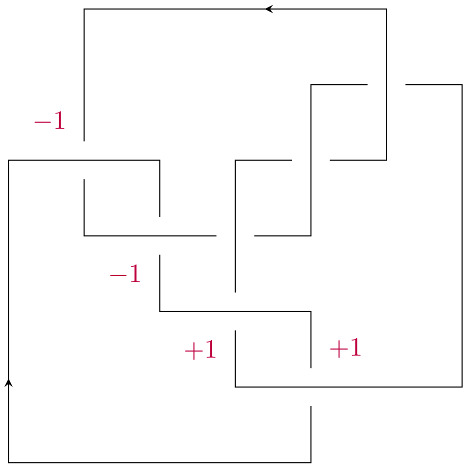
c) No. By a result from the course, if 6_1 is isotopic to 7_1, then 6_1 is m-colourable for an integer m if and only if 7_1 is m-colourable. By part b), we have that 6_1 is 15-colourable. Hence, if 6_1 is isotopic to 7_1, then 7_1 is 15-colourable. Since 15 is not divisible by 7, this contradicts part a).

d) An argument using the Jones polynomial of each knot is needed. Omitted.

e) The linking number of 7^2_5 is 0. Suppose, for instance, that we choose the following orientation.



The signs of the crossings between distinct components are as indicated.



We then calculate: $\frac{1}{2} \cdot |-1-1+1+1| = 0$.

f) We calculated in the lectures that the linking number of 2^2_1 is 1. By a result from the course, if 7^2_5 were isotopic to 2^2_1, then their linking numbers would be the equal. Since the linking number of 7^2_5 is 0 by part e), we conclude that 7^2_5 is not isotopic to 2^2_1.

g) No. The unlink with two components also has linking number 0.

Discussion

Knot colouring

The solution to part a) is in the style that I used in the lectures this year. It is much more efficient than the approach taken in last year's lecture notes.

It is very important that you have a clear understanding of the conclusion of the proof that $7 \mid m$ in the solution to part a), using the fundamental theorem of arithmetic and so on, and that you feel confident that you can carry out this kind of argument.

Linking number

Don't forget: only count the signs between *distinct* crossings! Remember also that one needs an orientation to calculate the linking number of a link, but any orientation can be chosen. This is by contrast with the Jones polynomial of a link, in which the answer depends upon the orientation, unless the link is a knot.

Solutions 4

a) Let $\sim$ be the equivalence relation on I^2 for which $(K^2, \mathcal{O}_{K^2})$ is $(I^2/\sim, \mathcal{O}_{I^2/\sim})$. Let

$$I^2 \xrightarrow{\pi} K^2$$

be the quotient map. We have the following results from the course.

(1) We have that $(I, \mathcal{O}_I)$ is compact.

(2) A product of compact topological spaces is compact.

(3) A quotient of a compact topological space is compact.

Together, (1)–(3) imply that $(K^2, \mathcal{O}_{K^2})$ is compact. We also have the following results from the course.

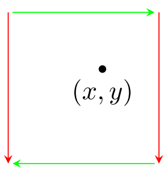
(1 bis) We have that $(I, \mathcal{O}_I)$ is connected.

(2 bis) A product of connected topological spaces is connected.

(3 bis) A quotient of a connected topological space is connected.

Together, (1 bis)–(3 bis) imply that $(K^2, \mathcal{O}_{K^2})$ is connected.

To prove that that $(K^2, \mathcal{O}_{K^2})$ is locally homeomorphic to an open rectangle, there are various cases to consider. Firstly, suppose that (x, y) does not belong to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})}(I^2)$.

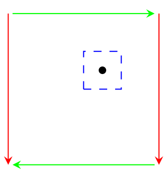


Let U be a subset of I^2 which has the following properties.

(1) We have that U is an open rectangle.

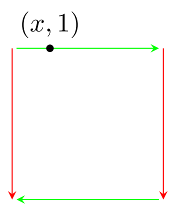
(2) We have that x belongs to U.

(3) We have that $U \cap \partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})}(I^2)$ is empty.



Then $\pi(U)$ is a neighbourhood of $\pi(x, y)$ in K^2 with respect to $\mathcal{O}_{K^2}$, and we have that $\left(\pi(U), \mathcal{O}_{\pi(U)}\right)$ is homeomorphic to an open rectangle, where $\mathcal{O}_{\pi(U)}$ is the subspace topology on $\pi(U)$ with respect to $\left(K^2, \mathcal{O}_{K^2}\right)$.

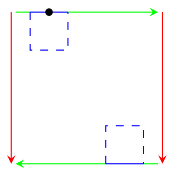
Secondly, let us consider a point $(x, 1)$ such that x belongs to I.



Suppose that a and b are real numbers, and that $0 < a < x < b < 1$. Let U be, for example, the subset

$$\left(]a, b[\times \left]\tfrac{3}{4}, 1\right]\right) \cup \left(]1-b, 1-a[\times \left[0, \tfrac{1}{4}\right[\right)$$

of I^2.

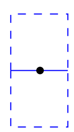


The following hold.

(1) We have that $\pi^{-1}(\pi(U)) = U$, and that U belongs to $\mathcal{O}_{I^2}$. Thus $\pi(U)$ belongs to $\mathcal{O}_{K^2}$.

(2) We have that x belongs to U, and thus that $\pi(x)$ belongs to $\pi(U)$.

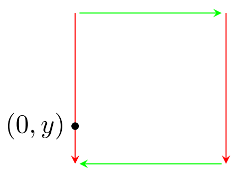
(3) We have that $\left(\pi(U), \mathcal{O}_{\pi(U)}\right)$ is homeomorphic to an open rectangle, where $\mathcal{O}_{\pi(U)}$ is the subspace topology on $\pi(U)$ with respect to $\left(K^2, \mathcal{O}_{K^2}\right)$. Intuitively, the two pieces of U are glued together as follows.



A detailed proof is not needed.

The case that we have a point $(x, 0)$ such that x belongs to I is similar.

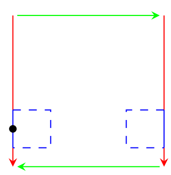
Thirdly, let us consider a point $(0, y)$ such that y belongs to I.



Suppose that $0 < a < x < b < 1$. Let U be, for example, the subset

$$\left(\left[0, \tfrac{1}{4}\right[\times \left]a, b\right[\right) \cup \left(\left]\tfrac{3}{4}, 1\right] \times \left]a, b\right[\right)$$

of I^2.

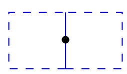


The following hold.

(1) We have that $\pi^{-1}(\pi(U)) = U$, and that U belongs to $\mathcal{O}_{I^2}$. Thus $\pi(U)$ belongs to $\mathcal{O}_{K^2}$.

(2) We have that x belongs to U, and thus that $\pi(x)$ belongs to $\pi(U)$.

(3) We have that $\left(\pi(U), \mathcal{O}_{\pi(U)}\right)$ is homeomorphic to an open rectangle, where $\mathcal{O}_{\pi(U)}$ is the subspace topology on $\pi(U)$ with respect to $\left(K^2, \mathcal{O}_{K^2}\right)$. Intuitively, the two pieces of U are glued together as follows.

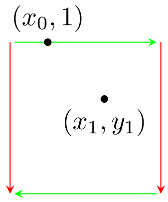


A detailed proof is not needed.

The case that we have a point $(x, 1)$ such that x belongs to I is similar.

To prove that $(K^2, \mathcal{O}_{K^2})$ is Hausdorff, there are various ways to proceed, but the most hands on is consider various cases, in a similar way as in the proof that $(K^2, \mathcal{O}_{K^2})$ is locally homeomorphic to an open rectangle.

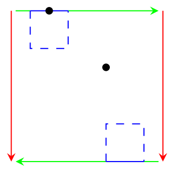
For example, suppose that we have a point $(x_0, 1)$, where x_0 belongs to I, and a point (x_1, y_1) which does not belong to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})}(I^2)$.



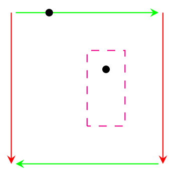
Suppose that $0 < a_0 < x_0 < b_0 < 1$, and that $0 < c < \min\{y_1, 1 - y_1\}$. Let U_0 be, for example, the subset

$$(]a, b[\times]1 - c, 1]) \cup (]1 - b, 1 - a[\times [0, c[)$$

of I^2.



Suppose that $0 < a_1 < x_1 < b_1 < 1$. Let U_1 be, for example, the subset $]a_1, b_1[\times]c, 1 - c[$ of I^2.



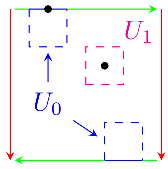
The following hold.

(1) We have that $\pi(U_0)$ is a neighbourhood of $\pi(x_0, 1)$ in K^2 with respect to $\mathcal{O}_{K^2}$.

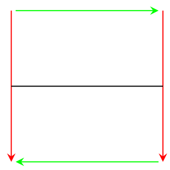
(2) We have that $\pi(U_1)$ is a neighbourhood of $\pi(x_1, y_1)$ in K^2 with respect to $\mathcal{O}_{K^2}$.

(3) We have that $\pi(U_0) \cap \pi(U_1)$ is empty.

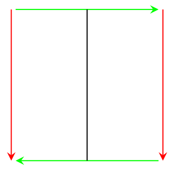
The other cases are similar. It is not necessary that you give full details, as I have done here. It is sufficient to draw a 'generic' picture for each case, such as the following picture for the case above, as long as you write that you are drawing subsets U_0 and U_1 of I^2, such that $\pi(U_0)$ and $\pi(U_1)$ give the required neighbourhoods in K^2 with respect to $\mathcal{O}_{K^2}$ of $\pi(x_0, 1)$ and $\pi(x_1, y_1)$ respectively.



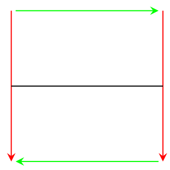
b) For example, we may take C to be a subset such as $I \times \left\{\frac{1}{2}\right\}$.



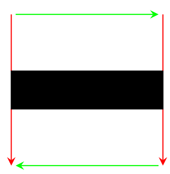
Or a subset such as $\left\{\frac{1}{2}\right\} \times I$.



c) Suppose that we take C to be a subset such as $I \times \left\{\frac{1}{2}\right\}$.

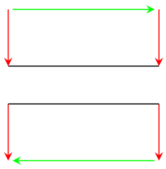


We thicken C.

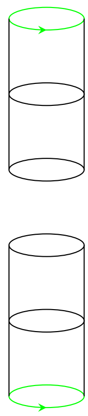


This thickening is a cylinder.

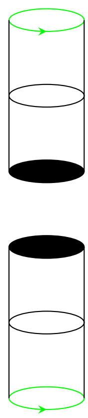
We cut out the interior of this cylinder.



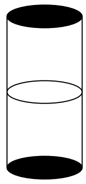
In other words, we have a pair of cylinders, as follows.



We now glue in discs to the circles left after we removed the interior of our cylinder.

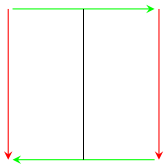


Making the remaining glueing, we see that we have obtained $(S^2, \mathcal{O}_{S^2})$.

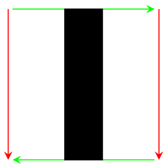


A quick way to see that we must obtain $(S^2, \mathcal{O}_{S^2})$ by performing surgery with respect to C is to calculate, using a Δ-complex structure as in parts d) and e), that $(K^2, \mathcal{O}_{K^2})$ has Euler characteristic 0, and to remember that performing a surgery in which we cut out the interior of a cylinder, and glue discs to the two circles we obtain, increases Euler characteristic by 2.

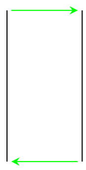
Suppose now that we take C to be a subset such as $\left\{\frac{1}{2}\right\} \times I$.



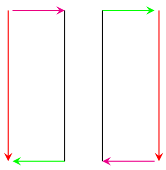
We thicken C.



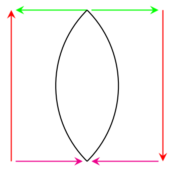
This thickening is a Möbius band.



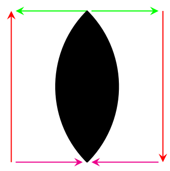
We cut out the interior of this Möbius band.



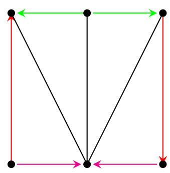
In other words, we have the following.



We glue a disc to the circle left after we removed the interior of the Möbius band.



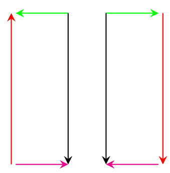
The following is a Δ-complex structure on this gadget.



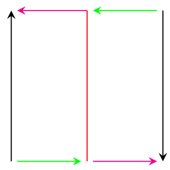
Its Euler characteristic is $3 - 6 + 4 = 1$. The only surface on the classification with this Euler characteristic is the real projective plane.

Alternatively, we could appeal to the fact that the Euler characteristic of $(K^2, \mathcal{O}_{K^2})$ is 0, and the fact that performing a surgery in which we cut out the interior of a Möbius band and glue a disc to the resulting circle increases Euler characateristic by 1.

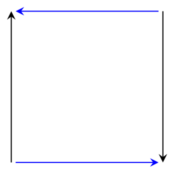
As yet another approach, we can observe that we can obtain the gadget we arrived at after glueing a disc onto the circle left after removing the interior of the Möbius band by glueing as follows.



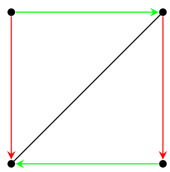
We obtain the same by glueing as follows.



This is exactly the real projective plane.



d) A Δ-complex structure on $(K^2, \mathcal{O}_{K^2})$ is depicted below.



e) Using the Δ-complex structure of part d), we calculate that the Euler characteristic is: $1 - 3 + 2 = 0$.

f) By the classification of surfaces, every surface is homeomorphic to either an n-handlebody, which has Euler characteristic $2 - 2n$, or an n-cross cap, which has Euler characteristic $2 - n$. Thus there are two surfaces, up to homeomorphism, with Euler characteristic 0: the 1-handlebody, or in other words the torus $(T^2, \mathcal{O}_{T^2})$, and the 2-cross cap, or in other words $(K^2, \mathcal{O}_{K^2})$.

Discussion

Surgery

It is very important that you feel confident that you can carry out a surgery argument both when given a picture of a surface in $\mathbb{R}^3$, as in Revision Question 10, and when given a 'glueing diagram' as in this question. In particular, it is important that you can carry out surgery in the case that the thickening of our circle is a Möbius band, as well as when the thickening is a cylinder (which it will always be when the surface can be drawn in $\mathbb{R}^3$).

The Euler characteristic is a very powerful tool, which you should keep close to hand! It can be used in different ways, as in part c), to determine which surface we obtain after a surgery, and it is precisely this interaction of the Euler characteristic with surgery that underlies the proof of the classification of surfaces.