MA3002 Generell Topologi — Outline Solutions to Spring 2014 Exam

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June 11, 2014

Contents

1 Problem 1

a)

Examines

Principally the definition of a topology. To a small degree, also the product topology.

Solution

No, not in general. A counterexample is obtained as follows.

Take both $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ to be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let U be $]0,1[$. We have that $]0,1[$ belongs to $\mathcal{O}_{\mathbb{R}}$. We have that $p_0^{-1}(U) = \,]0,1[\, \times \mathbb{R}$, and $p_1^{-1}(U) = \mathbb{R} \times \,]0,1[$. But

$$(]0,1[\, \times \mathbb{R}) \cup (\mathbb{R} \times \,]0,1[)$$

is neither of the form $p_0^{-1}(U_0)$ for any subset U_0 of $\mathbb{R}$, nor of the form $p_1^{-1}(U_1)$ for any subset U_1 of $\mathbb{R}$. Thus $p_0^{-1}(U) \cup p_1^{-1}(U)$ does not belong to $\mathcal{O}$. Since $p_0^{-1}(U)$ and $p_1^{-1}(U)$ both belong to $\mathcal{O}$, we conclude that $\mathcal{O}$ is not a topology.

Alternatively, we could observe that $]0,1[\, \times \,]0,1[\, = p_0^{-1}(U) \cap p_1^{-1}(U)$ does not belong to $\mathcal{O}$.

b)

Examines

Hausdorffness.

Solution

Every set containing b which belongs to $\mathcal{O}_X$ also contains a. Alternatively, every set containing d which belongs to $\mathcal{O}_X$ also contains a.

c)

Examines

Compactness, product topology, subspace topology.

Solution

For a given n which belongs to $\mathbb{N}$, let U_n be $Z \cap \left]-\frac{1}{2} + \frac{1}{n}, 1\right]$. Then $\{U_n\}_{n\in\mathbb{N}}$ is an open covering of Z with respect to $\mathcal{O}_Z$ which does not admit a finite subcovering.

d)

Examines

Homeomorphisms. Also, to a small degree, the definition of a quotient topology.

Solution

The map f is a homeomorphism. It is continuous by construction.

Moreover, since f is a surjection, we have that $f\left(f^{-1}(U)\right) = U$ for every U which belongs to $\mathcal{O}_Z$. We deduce that the inverse to f is continuous.

e)

Examines

Products and homeomorphisms preserve compactness.

Solution

No. Since $(I, \mathcal{O}_I)$ is compact, and a product of compact topological spaces is compact, we have that $(I^2, \mathcal{O}_{I^2})$ is compact. Moreover, homeomorphisms preserve compactness. Thus if $(I^2, \mathcal{O}_2)$ were homeomorphic to $(I^2, \mathcal{O})$, we would, since $(I^2, \mathcal{O})$ is homeomorphic to $(Z, \mathcal{O}_Z)$, have that $(Z, \mathcal{O}_Z)$ is compact. Part c) establishes that this is not the case.

2 Problem 2

a)

Examines

Connectedness, continuity.

Solution

The proof was given in the course (Proposition 10.3.1).

b)

Examines

Connectedness, product topology, subspace topology.

Solution

Let $U_0 = X \cap \left(\mathbb{R} \times \left]-5, -\frac{1}{4}\right[\right)$. Let $U_1 = X \cap \left(\mathbb{R} \times \left]-\frac{1}{4}, 1\right[\right)$. We have that $\mathbb{R}$, $\left]-5, -\frac{1}{4}\right[$, and $\left]-\frac{1}{4}, 1\right[$ belong to $\mathcal{O}_{\mathbb{R}}$. It follows immediately from the definition of a product topology that $\mathbb{R} \times \left]-5, -\frac{1}{4}\right[$ and $\mathbb{R} \times \left]-\frac{1}{4}, 1\right[$ belong to $\mathcal{O}_{\mathbb{R}^2}$. It follows immediately from the definition of a subspace topology that U_0 and U_1 belong to $\mathcal{O}_X$.

We also have that $X = U_0 \sqcup U_1$, and that both U_0 and U_1 are non-empty. Thus $(X, \mathcal{O}_X)$ is not connected.

There are other ways to proceed.

c)

Examines

Quotient topology, product topology, subspace topology.

Solution

No. We have that

$$\pi^{-1}\left(\pi(U)\right) = U \cup \left\{ \left(-\frac{2}{3}, 0\right), \left(-\frac{1}{3}, 0\right), \left(\frac{1}{3}, 0\right), \left(\frac{1}{3}, 0\right) \right\}.$$

This set does not belong to $\mathcal{O}_X$, since every open rectangle in $\mathbb{R}^2$ around $\left(-\frac{1}{3}, 0\right)$, for instance, which contains this point, also contains points of X which do not belong to U.

d)

Examines

Connectedness. Other topics, depending on the candidate's approach, such as path connectedness.

Solution

One approach is to argue that $(Y, \mathcal{O}_Y)$ is path connected: we saw in the course that travelling around a circle, and travelling along a straight line, define paths; and that we can compose and reverse paths. Then appeal to the fact that path connectedness implies connectedness.

An alternative approach is to observe firstly that the circle is homeomorphic to a quotient of $(I, \mathcal{O}_I)$, so is connected, since $(I, \mathcal{O}_I)$ is connected, since a quotient of a connected topological space is connected, and since homeomorphisms preserve connectedness. Secondly, each of the straight lines is homeomorphic to $(I, \mathcal{O}_I)$, so is connected. An inductive argument, appealing to the fact that a union of connected subsets of a topological space whose intersection is non-empty is connected, which was an exercise in the course, then demonstrates that $(Y, \mathcal{O}_Y)$ is connected.

Full details, in either approach, are not necessary.

e)

Examines

Connected components, homeomorphisms.

Solution

No. Let A be the set of eight points of Y given by $\{L_i \cap S^1\}_{1 \leq i \leq 8}$. Let $\mathcal{O}_{Y \setminus A}$ be the subspace topology on $Y \setminus A$ with respect to $(Y, \mathcal{O}_Y)$. Then $(Y \setminus A, \mathcal{O}_{Y \setminus A})$ has sixteen connected components. Suppose that we had a homeomorphism

$$Y \xrightarrow{f} X.$$

Let $\mathcal{O}_{(X/\sim) \setminus f(A)}$ be the subspace topology on $(X/\sim) \setminus f(A)$ with respect to $(X/\sim, \mathcal{O}_{X/\sim})$. Then f restricts to a homeomorphism

$$Y \setminus A \longrightarrow (X/\sim) \setminus f(A).$$

However, $\left((X/\sim) \setminus f(A), \mathcal{O}_{(X/\sim) \setminus f(A)}\right)$ has at most twelve connected components. Since homeomorphisms preserve numbers of connected components, we have a contradiction.

The solution given here is to the question that I intended to ask, namely whether $(X/\sim, \mathcal{O}_{X/\sim})$ is homeomorphic to $(Y, \mathcal{O}_Y)$. Unfortunately (from my point of view at least!) the actually question asked whether $(X, \mathcal{O}_X)$ is homeomorphic to $(Y, \mathcal{O}_Y)$, which is much easier: $(X, \mathcal{O}_X)$ is not connected by part b), but $(Y, \mathcal{O}_Y)$ is connected by part d), and homeomorphisms preserve connectedness, so $(X, \mathcal{O}_X)$ cannot be homeomorphic to $(Y, \mathcal{O}_Y)$.

If you did answer the problem of whether $(X/\sim, \mathcal{O}_{X/\sim})$ is homeomorphic to $(Y, \mathcal{O}_Y)$ instead, that is fine. In this case, I will mark your answer generously, so that you are not penalised for answering a harder problem.

3 Problem 3

a)

Examines

Closure, subspace topology.

Solution

If A is not closed in X with respect to $\mathcal{O}_X$, then there is an c which belongs to C but does not belong to A. By definition of a limit point, every neighbourhood of c in X with respect to $\mathcal{O}_X$ contains an element of A. Since A is a subset of C, we thus have that $C \cap U$ is not $\{c\}$ for any U which belongs to $\mathcal{O}_X$. We conclude, by definition of the subspace topology on C with respect to $\mathcal{O}_X$, that $\{c\}$ does not belong to $\mathcal{O}_C$.

b)

Examines

Closure, product topology, subspace topology.

Solution

The union of A, the set

$$\left\{ \left(\frac{1}{m}, 0 \right) \mid m \in \mathbb{N} \right\},$$

the set

$$\left\{ \left(0, \frac{1}{n} \right) \mid n \in \mathbb{N} \right\},$$

and $\{(0,0)\}$.

c)

Examines

Local compactness, Heine-Borel theorem (subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded). Possibly other topics, depending on the candidate's approach, such as Hausdorffness or closure.

Solution

Since X is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$, and is bounded, we have that $(X, \mathcal{O}_X)$ is compact. Since $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff, and since a subspace of a Hausdorff topological space is Hausdorff, we have that $(X, \mathcal{O}_X)$ is Hausdorff. The fact that $(X, \mathcal{O}_X)$ is both compact and Hausdorff implies that $(X, \mathcal{O}_X)$ is locally compact.

Alternatively, one can show it directly. Suppose that x belongs to X. Let $X \cap U$ be a neighbourhood of x in X with respect to $\mathcal{O}_X$, where U belongs to $\mathcal{O}_{\mathbb{R}^2}$. By definition of $\mathcal{O}_{\mathbb{R}^2}$, there is an open rectangle U' in $\mathbb{R}^2$, given by $]a_0, b_0[\times]a_1, b_1[$, which is a subset of U, and to which x belongs. Taking a slightly smaller open rectangle if necessary, we can, for convenience, assume that none of a_0, a_1, b_0, or b_1 are equal to $\frac{1}{n}$ for any n which belongs to $\mathbb{N}$, or to 0.

We have that $X \cap U'$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$, and is a subset of $X \cap U$. Moreover, we have that $X \cap U'$ is closed in X with respect to $\mathcal{O}_X$. In addition, $X \cap U'$ is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$, and is bounded. Thus $X \cap U'$ is a compact subset of $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. Since the subspace topology on $X \cap U'$ with respect to $(X, \mathcal{O}_X)$ is equal to the subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$, we deduce that $X \cap U'$ is a compact subset of X with respect to $\mathcal{O}_X$. Thus $X \cap U'$ satisfies the condition for local compactness at x with respect to $X \cap U$.

d)

Examines

Boundary, subspace topology.

Solution

The union of the set

$$\left\{ \left(\tfrac{1}{2}, y\right) \mid 0 \leq y \leq \tfrac{1}{2} \right\},$$

the set

$$\left\{ \left(x, \tfrac{1}{2}\right) \mid \tfrac{1}{2} \leq x \leq 1 \right\},$$

and the set

$$\left\{ \left(\tfrac{1}{8}, 0\right), \left(\tfrac{3}{8}, 0\right) \right\}.$$

e)

Examines

Compactness, Hausdorffness, Heine-Borel theorem, homeomorphisms.

Solution

No. Since X is closed and bounded in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$, we have that $(X, \mathcal{O}_X)$ is compact. Since $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff, and since a subspace of a Hausdorff topological space is Hausdorff, we have that $(Z, \mathcal{O}_Z)$ is Hausdorff. Every continuous bijection with a compact source and a Hausdorff target is a homeomorphism.

However, $(Z, \mathcal{O}_Z)$ is not compact. For instance, it is not closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$, since every element of

$$\left(B \cup \partial_{(Y,\mathcal{O}_Y)}(B)\right) \cap \left((\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q}) \right)$$

is a limit point, and does not belong to Z. Since homeomorphisms preserve compactness, we thus have that $(X, \mathcal{O}_X)$ is not homeomorphic to $(Z, \mathcal{O}_Z)$. Alternatively, one can appeal to local compactness to see this.

4 Problem 4

a)

Examines

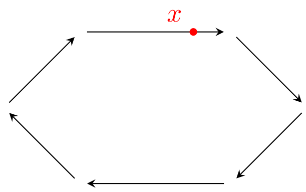
Surfaces, compactness, Hausdorffness, connectedness, homeomorphisms, quotient topologies.

Solution

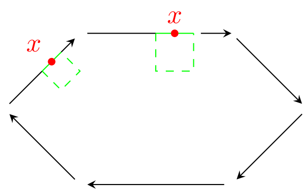
We have that X is closed and bounded in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. Hence $(X, \mathcal{O}_X)$ is compact. Since a quotient of a compact topological space is compact, we deduce that $(X/\sim, \mathcal{O}_{X/\sim})$ is compact.

We have that $(X, \mathcal{O}_X)$ is connected. To see this, we can demonstrate that it is path connected, and then appeal to the fact that path connectedness implies connectedness. Alternatively, we can observe that $(X, \mathcal{O}_X)$ is homeomorphic to $(I^2, \mathcal{O}_{I^2})$, and then appeal to the fact that homeomorphisms preserve connectedness, and that $(I^2, \mathcal{O}_{I^2})$ is connected, since $(I, \mathcal{O}_I)$ is connected and products of connected topological spaces are connected. Since a quotient of a connected topological space is connected, the fact that $(X, \mathcal{O}_X)$ is connected implies that $(X/\sim, \mathcal{O}_{X/\sim})$ is connected.

To demonstrate that $(X/\sim, \mathcal{O}_{X/\sim})$ is locally homeomorphic to an open rectangle, there are several cases to consider. For instance, suppose that x is a point of X as follows.



Then a neighbourhood of $\pi(x)$ which is homeomorphic to an open rectangle is $\pi(A)$ for a subset A of X as depicted below.



Full marks are to be awarded if the candidate considers the representative cases: where x is the middle of an edge, as in the case just considered; where x is a point on the boundary where two edges meet; and where x is in the interior of X.

To demonstrate that $(X/\sim, \mathcal{O}_{X/\sim})$ is Hausdorff, there are again several cases to consider. The two neighbourhoods that are needed in each case can be constructed in a similar way as in the proof that $(X/\sim, \mathcal{O}_{X/\sim})$ is locally homeomorphic to an open rectangle, making sure in addition that they do not intersect.

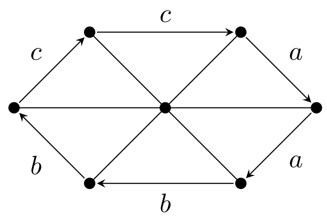
b)

Examines

Δ-complexes, Euler characteristic.

Solution

A Δ-complex structure is depicted below.



All the vertices on the boundary are glued together. Thus the Euler characteristic is: $2 - 9 + 6 = -1$.

c)

Examines

Classification of surfaces, Euler characteristic.

Solution

The 3-cross cap: three disjoint Möbius bands glued onto a sphere. This is the only surface in the statement of the classification of surfaces which has Euler characteristic -1.

As an alternative, one could give a geometric 'cut and paste' argument to prove that we have the 3-cross cap. This is much harder: if the candidate carries this out, they can be awarded bonus marks for part d), but no more than the allocated 5 marks are to be awarded for part d).

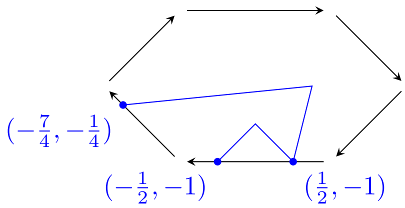
d)

Examines

Surgery, Euler characteristic, quotient topology, homeomorphisms.

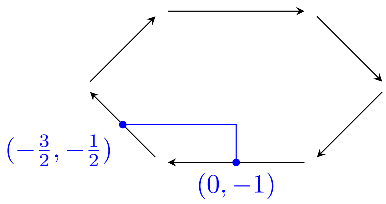
Solution

One possibility is to take C to be $\pi(A)$, where A is a subset of X of the kind pictured below.

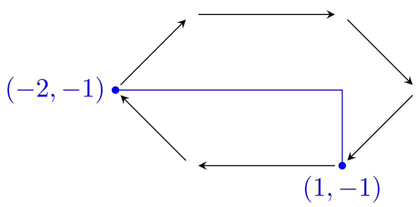


We have that $\left((X/\sim) \setminus C, \mathcal{O}_{(X/\sim)\setminus C}\right)$ is connected, where $\mathcal{O}_{(X/\sim)\setminus C}$ is the subspace topology on $(X/\sim) \setminus C$ with respect to $\left(X/\sim, \mathcal{O}_{X/\sim}\right)$. Thickening C gives a cylinder, and hence surgery with respect to C increases the Euler characteristic by 2.

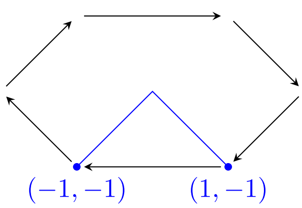
If we were to take C to be $\pi(A)$, where A is the following subset of X, then C would not have the second required property: thickening C gives a Möbius band rather than a cylinder, and thus surgery with respect to it increases the Euler characteristic by 1 rather than 2.



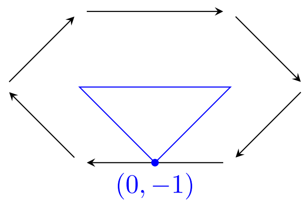
If we were to take C to be $\pi(A)$, where A is the following subset of X, then C also would not have the second required property: $\left((X/\sim) \setminus C, \mathcal{O}_{(X/\sim)\setminus C}\right)$ would not be connected.



For the same reason, if we were to take C to be $\pi(A)$, where A is a variation on the above curve in which we have any of the six points labelled in the figure in part a) instead of $(-2, 0)$, then C also would not have the second required property.



For the same reason once more, if we were to take C to be $\pi(A)$, where A is a subset of X of the following kind, then C also would not have the second required property.



This part of the question is harder than I had intended. My original solution was erroneuous, and I therefore do not wish to penalise you for making the same mistake!

After marking a few scripts, I have decided that the fairest way to mark this question is to mark it out of 5 as planned, as many have earned high marks on this question in any case, but to not take into account lost marks on this question when making my overall decision on your grade.

5 Problem 5

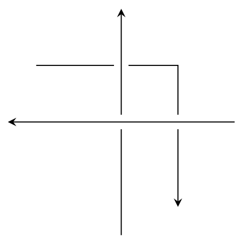
a)

Examines

Reidemeister moves.

Solution

We can replace the three arcs pictured in the problem by the following ones.



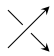
b)

Examines

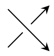
Linking number. Proving that something is a link invariant by checking that it is invariant under the Reidemeister moves.

Solution

We adopt the convention that

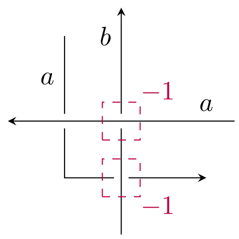


has sign -1, and that

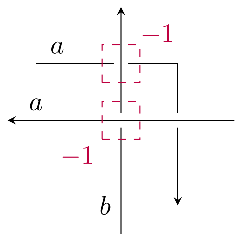


has sign $+1$.

When calculating the linking numbeer, we look only at the signs of crossings between distinct components. Before applying the R3 move, we have the following signs.



After applying the R3 move, we have the following signs.



Thus the crossings contribute the same sum of signs to the linking number in each case. It follows that the linking number is unchanged by applying the R3 move.

c)

Examines

Knot colouring. Using knot colouring to prove that two links are not isotopic.

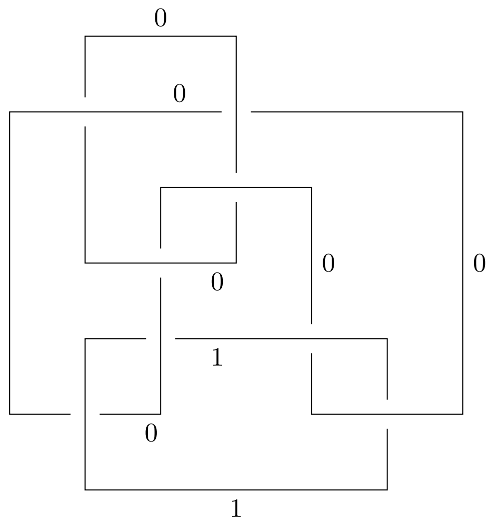
Solution

There is a serious mistake in the question: both links are m-colourable if and only if $2 \mid m$ (in fact all links with more than one component are 2-colourable!). A correct question could have asked: can we use knot colouring to prove that 6^2_3 is not isotopic to 8^3_4?

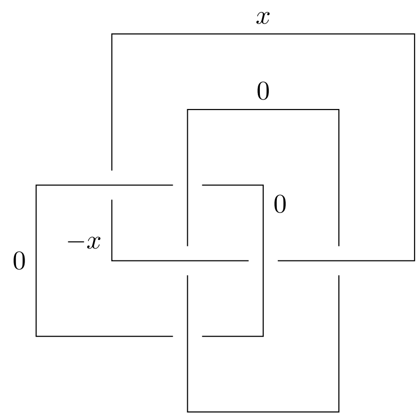
The marking will take this into account as follows. If you have attempted Problem 5, I will consider four different marks when making my qualitative decision on the grade: your actual mark (quite a few students still obtained good marks on this question, some even full marks); your mark on the best three other problems that you attempted, adjusted to be out of 100; your mark obtained by taking into account only parts a), b), and d) of Problem 5, adjusted to be out of 25; and your mark if awarded full marks for 5 (c). Not all of these marks will be appropriate for consideration in all cases, the last especially.

Below is the correct part of the original solution.

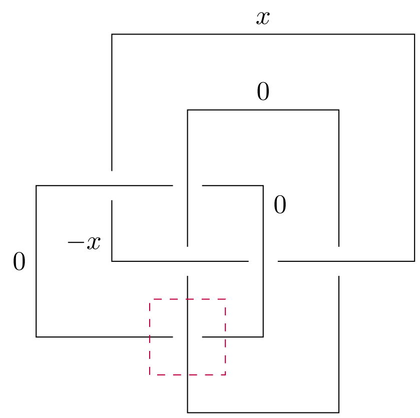
The link 8^3_4 is 2-colourable. An example of a 2-colouring is given below.



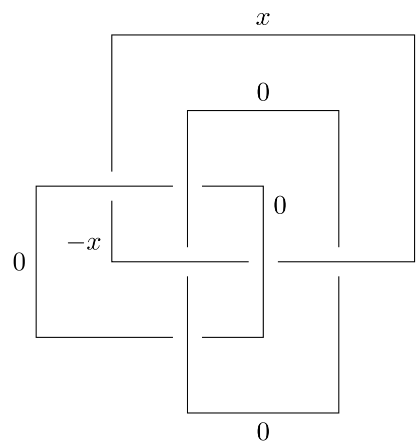
Here is a proof that if link 6^3_2 is m-colourable for any m, then $2 \mid m$. By a result from the course, we can choose the integer assigned to one of the arcs to be 0. Denoting the integer assigned to another of the arcs by x, we must have the following, mod m.



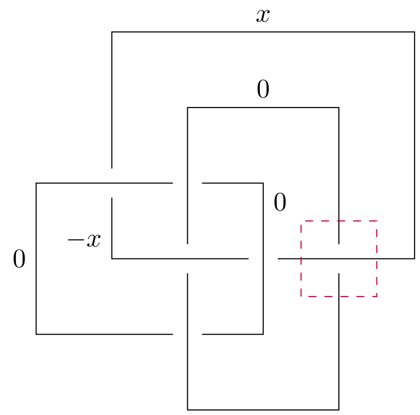
Consider the following crossing.



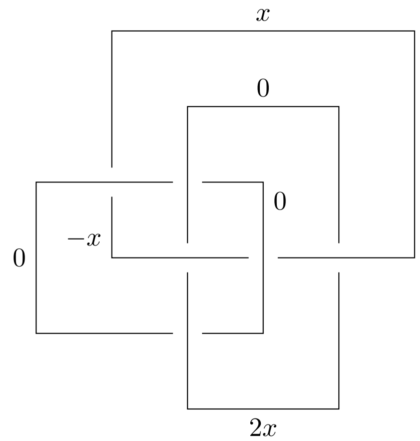
If we have a colouring, then we must have the following, mod m.



On the other hand, consider the following crossing.

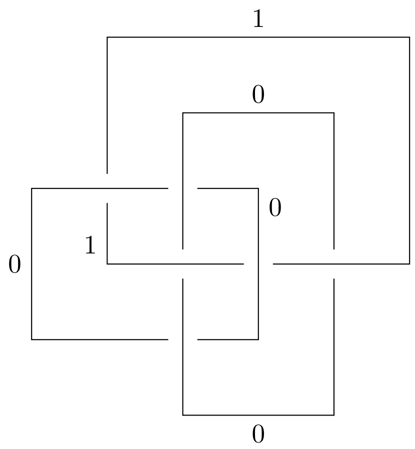


If we have a colouring, then we must have the following, mod m.



We deduce that $2x \equiv 0 \pmod{m}$. We then have that $2x = km$ for an integer k. Since 2 is a prime, we have, by the fundamental theorem of arithmetic, that either $2 \mid k$ or $2 \mid m$. If $2 \mid k$, then $x \equiv 0 \pmod{m}$, and we have that 0 is assigned to all the arcs, contradicting the fact that we have a colouring. Thus we have that $2 \mid m$.

An actual 2-colouring of 6^3_2 is given as follows.



One can prove, in a similar way as for 6^3_2, that if 8^3_4 is m-colourable, then $2 \mid m$. Thus we cannot use knot colouring to prove that 6^3_2 is not isotopic to 8^3_4.

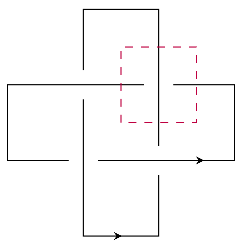
d)

Examines

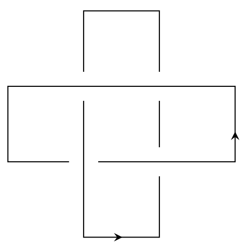
Jones polynomial, skein relations, using the Jones polynomial to prove that two links are not isotopic.

Solution

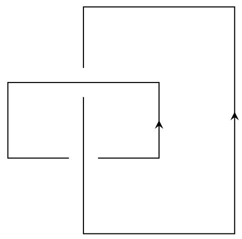
We outline one way to proceed. We begin with the following crossing.



We have that $V_{\times}(t)$ is the Jones polynomial of 4^2_1. We have that $V_{\asymp}(t)$ is the Jones of the following link,

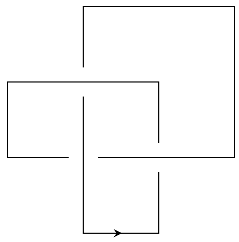


This link is isotopic to the following Hopf link.



We calculate, using the skein relations, that the Jones polynomial of this Hopf link is $-t^{\frac{1}{2}} - t^{\frac{5}{2}}$.

We have that $V_{\asymp}(t)$ is the Jones polynomial of the following trefoil.



We calculate, using the skein relations, that the Jones polynomial of this trefoil is $t + t^3 - t^4$. By the second skein relation, we have that

$$t^{-1}V_{\times}(t) - t\left(-t^{\frac{1}{2}} - t^{\frac{5}{2}}\right) = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\left(t + t^3 - t^4\right).$$

Thus we have that

$$t^{-1}V_{\times}(t) + t^{\frac{3}{2}} + t^{\frac{7}{2}} = t^{\frac{3}{2}} + t^{\frac{7}{2}} - t^{\frac{9}{2}} - t^{\frac{1}{2}} - t^{\frac{5}{2}} + t^{\frac{7}{2}}.$$

Hence we have that

$$t^{-1}V_{\times}(t) = -t^{\frac{1}{2}} - t^{\frac{5}{2}} + t^{\frac{7}{2}} - t^{\frac{9}{2}}.$$

We conclude that

$$V_{\times}(t) = -t^{\frac{3}{2}} - t^{\frac{7}{2}} + t^{\frac{9}{2}} - t^{\frac{11}{2}}.$$

If Solomon's knot were isotopic to the unlink with the two components, they would have the same Jones polynomial. This is not the case.

It is also possible to calculate the Jones polynomial of Solomon's knot directly, by calculating its bracket polynomial and its writhe.