Solutions and Discussion

The solutions here are not presented in as much detail as if they had appeared in the lecture notes. This is to help indicate the level of detail that I am looking for on the exam.

It is good to consider how you would in principle go about giving a proof with all the details, even if you do not actually write this proof down.

Just let me know if you would like me to elaborate upon or clarify anything.

Solutions 1

a) Yes, $(X, \mathcal{O}_X)$ is compact. By a result from the course, a subset of $\mathbb{R}^2$ is compact if and only if it is bounded and closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. The set X has both of these properties.

b) Yes, $(X, \mathcal{O}_X)$ is Hausdorff. This is a consequence of the following results from the course.

(1) $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff.

(2) Products of Hausdorff topological spaces are Hausdorff. By (1), we thus have that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff.

(3) Every subspace of a Hausdorff topological space is Hausdorff.

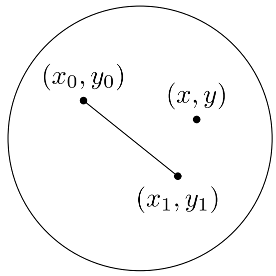
c) Yes, $(X, \mathcal{O}_X)$ is locally compact. By a result from the course, every compact Hausdorff topological space is locally compact. That $(X, \mathcal{O}_X)$ is locally compact thus follows from a) and b).

d) Suppose that (x_0, y_0) and (x_1, y_1) belong to $D^2 \setminus \{(x, y)\}$. Suppose that (x, y) does not lie on the straight line between these two points, or in other words in the image of the map

$$I \xrightarrow{f_{(x_0,y_0),(x_1,y_1)}} D^2$$

given by

$$t \mapsto (1-t)(x_0, y_0) + t(x_1, y_1).$$



This map is a product of polynomial maps, namely

$$t \mapsto (x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0)) \,.$$

Hence it is continuous. We also have that

$$f_{(x_0,y_0),(x_1,y_1)}(0) = (x_0, y_0),$$

and that

$$f_{(x_0,y_0),(x_1,y_1)}(1) = (x_1, y_1).$$

Thus the map

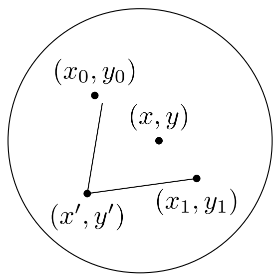
$$I \longrightarrow D^2 \setminus \{(x, y)\}$$

given by

$$t \mapsto f_{(x_0,y_0),(x_1,y_1)}(t)$$

defines a path from (x_0, y_0) to (x_1, y_1) in $(D^2 \setminus \{(x, y)\}, \mathcal{O}_{D^2 \setminus \{(x,y)\}})$.

Suppose now that (x, y) does belong to the image of $f_{(x_0,y_0),(x_1,y_1)}$. Let (x', y') be a point of D^2 which does not belong to the image of $f_{(x_0,y_0),(x_1,y_1)}$. Then $f_{(x_0,y_0),(x',y')}$ is a path from (x_0, y_0) to (x', y') in $(D^2 \setminus \{(x, y)\}, \mathcal{O}_{D^2 \setminus \{(x,y)\}})$, and $f_{(x',y'),(x_1,y_1)}$ is a path from (x', y') to (x_1, y_1) in $(D^2 \setminus \{(x, y)\}, \mathcal{O}_{D^2 \setminus \{(x,y)\}})$.



By as result from the course, we can ‘concatenate’ these two paths to obtain a path from (x_0, y_0) to (x_1, y_1) in $(D^2 \setminus (x, y), \mathcal{O}_{D^2 \setminus \{(x,y)\}})$.

e) No, $(X, \mathcal{O}_X)$ is not homeomorphic to $(Y, \mathcal{O}_Y)$. We have the following two facts from the course.

(1) If

$$X \xrightarrow{f} Y$$

is a homeomorphism, then the map

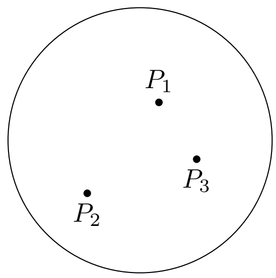
$$X \setminus A \longrightarrow Y \setminus f(A)$$

given by $x \mapsto f(x)$ is a homeomorphism for any subset A of X, where $X \setminus A$ has the subspace topology with respect to $(X, \mathcal{O}_X)$, and $Y \setminus f(A)$ has the subspace topology with respect to $(Y, \mathcal{O}_Y)$.

(2) Homeomorphisms preserve connectedness.

Suppose that P_0, P_1, and P_2 belong to

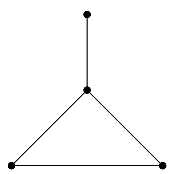
$$\{(2, y) \mid 1 < y < 3\}.$$



Then $X \setminus \{P_0, P_1, P_2\}$, equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$, is connected. However, by removing any three points from Y, and equipping this set with the subspace topology with respect to $(Y, \mathcal{O}_Y)$, we obtain a topological space which is not connected.

That $(X, \mathcal{O}_X)$ is not homeomorphic to $(Y, \mathcal{O}_Y)$ thus follows from (1) and (2).

f) A Δ-complex structure on $(X, \mathcal{O}_X)$ is depicted below. The triangle is to be regarded as a 2-simplex.



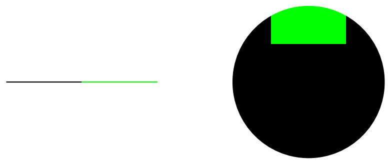
The Euler characteristic of $(X, \mathcal{O}_X)$ is $4 - 4 + 1 = 1$.

g) Yes, $\pi(A)$ belongs to $\mathcal{O}_{Z/\sim}$. We have that $\pi^{-1}(\pi(A))$ is the union of

$$\{(x,0) \mid -1 < x \leq 0\}$$

and

$$\{(x,y) \mid \|(x-2,y)\| \leq 1\} \cap \left(\left]\tfrac{3}{2}, \tfrac{5}{2}\right[\times \left]\tfrac{1}{2}, \tfrac{3}{2}\right[\right).$$

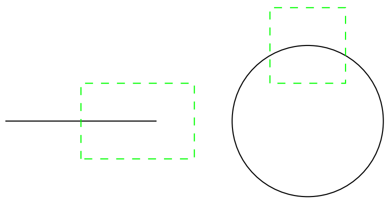


This union is the intersection with Z of, for example, the union of

$$\left]-1, \tfrac{1}{2}\right[\times \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[$$

and

$$\left]\tfrac{3}{2}, \tfrac{5}{2}\right[\times \left]\tfrac{1}{2}, \tfrac{3}{2}\right[.$$



Both of these two ‘open rectangles’ belong to $\mathcal{O}_{\mathbb{R}^2}$. Hence so does their union. By definition of $\mathcal{O}_Z$, we deduce that $\pi^{-1}(\pi(A))$ belongs to $\mathcal{O}_Z$. Hence $\pi(A)$ belongs to $\mathcal{O}_{Z/\sim}$.

h) Let

$$Z \xrightarrow{f} X$$

be the map given by

$$(x, y) \mapsto \begin{cases} (2, x+1) & \text{if } -2 \le x \le 0, \\ (x, y) & \text{otherwise.} \end{cases}$$

We have that $f(0,0) = f(2,1)$. Since this map is obtained by 'glueing' polynomial maps, it is continuous. By an exercise from the course, we deduce that the map

$$Z/{\sim} \xrightarrow{g} X$$

given by $[z] \mapsto f(z)$ is continuous. Moreover, g is evidently bijective.

The following hold.

(1) We have that $(Z, \mathcal{O}_Z)$ is compact, since it is a closed and bounded subset of $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Since a quotient of a compact topological space is compact, we deduce that $(Z/{\sim}, \mathcal{O}_{Z/\sim})$ is compact.

(2) By b), we have that $(X, \mathcal{O}_X)$ is Hausdorff.

By a result from the course, every continuous bijection with a compact source and a Hausdorff target is a homeomorphism. We conclude that g is a homeomorphism.

Discussion

Proving compactness

We have seen two principal ways to prove compactness.

(1) Begin with the fact that $(I, \mathcal{O}_I)$ is compact, and then appeal to the fact that compactness is preserved by the following 'canonical constructions': products, quotients, homeomorphisms, and closed subspaces.

(2) Use the characterisation of compact subsets of $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ as subsets which are bounded and closed in $\mathbb{R}^n$ with respect to $\mathcal{O}_{\mathbb{R}^n}$.

When asked to prove that a geometric example of a topological space is compact, it is a good idea to begin by considering which of these methods is most appropriate.

Proving that a topological space is Hausdorff

Similar remarks apply with regard to proving that a topological space is Hausdorff.

(1) Begin with the fact that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff, which you should make sure that you are able to prove, and then appeal the fact that being Hausdorff is preserved by the following 'canonical constructions: products, subspaces, and homeomorphisms'.

(2) Don't forget that being Hausdorff is not preserved by taking quotients in general. However, it is preserved under 'sufficiently nice' quotients, and it is important to recognise when the conditions for a quotient of Hausdorff topological space to be Hausdorff hold. This will be discussed in the solutions to a later revision question,

Demonstrating path connectedness

When proving that a geometric example of a topological space is path connected, a typical technique is the following.

(1) Use straight line paths 'as far as possible'. This is simply because straight line paths are easy to write down, and are evidently continuous (they are polynomial maps).

(2) Concatenate and reverse, as needed, straight line paths from 1) for any remaining paths.

In d), we use straight line paths as long as (x, y) does not lie on this straight line, and then use a concatenation of two such paths to treat the remaining cases.

Homeomorphisms in geometric examples

Whilst it will always be possible to use a different method, it is often very convenient when asked to prove that two geometric examples of topological spaces are homeomorphic, to use the fact that a continuous bijection in which the source is compact and the target is Hausdorff is a homeomorphism, as in h). This is particularly often useful when the source is a quotient.

You should be very careful, though, to appeal to this result only when the hypotheses are satisfied!

Repertoire of continuous maps

We typically rely only on the following to construct continuous maps. All of these facts are established in the exercises to Lecture 5.

1) Polynomial maps are continuous.

2) We can 'glue' continuous maps together: on disjoint pieces as in the definition of f in h); but also more generally when we have a union of open sets, and maps on each of these open sets, such that the maps agree wherever the open sets intersect; and similarly for a finite (or, more generally, locally finite, but don't worry too much about this) union of closed sets.

3) We can take products of continuous maps, as in d). In particular, maps to $\mathbb{R}^n$ which are polynomials 'in each component' are continuous.

4) Continuous maps which respect a given equivalence relation 'pass to the quotient', as in h).

We sometimes also make use of a continuous map

$$\mathbb{R} \xrightarrow{\phi} S^1$$

which 'travels around S^1' once for every interval $[n, n+1]$, where $n \in \mathbb{Z}$.