Solutions 2

a) Let

$$\mathbb{R} \xrightarrow{f} X$$

be the map given by $x \mapsto (x, x^2)$. Since f is a polynomial map in each component, it is continuous. Let

$$X \xrightarrow{g} \mathbb{R}$$

be the map given by $(x, y) \mapsto x$. Since g is the restriction to X of the projection map

$$\mathbb{R}^2 \longrightarrow \mathbb{R}$$

given by $(x, y) \mapsto x$, it is continuous. We have that

$$\begin{aligned} g\left(f(x)\right) &= g\left(x, x^2\right) \\ &= x. \end{aligned}$$

Thus $g \circ f = id_{\mathbb{R}}$. We also have that

$$\begin{aligned} f\left(g(x, y)\right) &= f(x) \\ &= (x, x^2). \end{aligned}$$

Thus $f \circ g = id_X$.

b) One possibility is to take Y to be

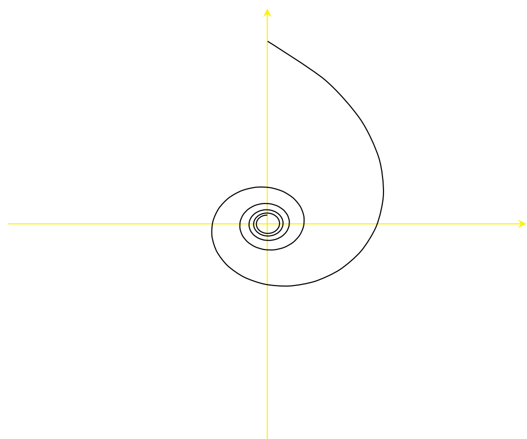
$$\left\{(x, 0) \mid 0 < x < 1\right\}.$$

Then $(0, 0)$ and $(0, 1)$ are limit points of Y in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. It is straightforward, by a similar argument to that of part a), to prove that $(Y, \mathcal{O}_Y)$ is homeomorphic to the open interval $]0, 1[$, equipped with its subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. We proved in the lectures that an open interval with its subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

For a more exotic example, one could take Y to be the spiral given by image of the map

$$]0, \infty[\xrightarrow{f} \mathbb{R}^2$$

defined by $t \mapsto \frac{\phi(t)}{t}$, where ϕ is the 'travelling around the circle' map constructed in the exercises to Lecture 5. Then $(0,0)$ and $(0,1)$ are limit points of Y in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ which does not belong to Y. Thus Y is not closed in $\mathbb{R}^2$. It is not important to prove this carefully, nor to prove that $(Y, \mathcal{O}_Y)$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, but if you would like a challenge and have attempted the other revision questions, you might like to have a go at this!



c) For the first example given in b), the closure of Y in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ is the union of Y and $\{(0,0),(1,0)\}$, or in other words

$$\{(x,0) \mid 0 \leq x \leq 1\}.$$

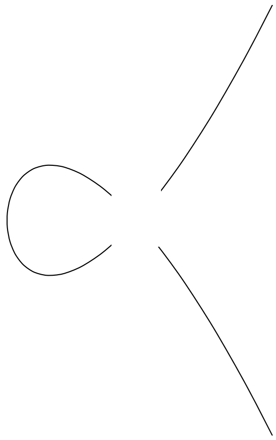
For the second examplei given in b), the closure of Y in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ is the union of Y and $\{(0,0),(0,1)\}$ in the second example.

d) No, $(Z, \mathcal{O}_Z)$ is not homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, by the following argument.

(1) If there were a homeomorphism between these topological spaces, then there would be a homeomorphism between $Z \setminus \{(0,0)\}$, equipped with the subspace topology $\mathcal{O}_{Z \setminus \{0,0\}}$ with respect to $(Z, \mathcal{O}_Z)$, and $\mathbb{R} \setminus \{x\}$, equipped with the subspace topology $\mathcal{O}_{\mathbb{R} \setminus \{x\}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, for some x which belongs to $\mathbb{R}$.

(2) Homeomorphic topological spaces have the same number of connected components.

(3) The topological space $(Z \setminus \{0,0\}, \mathcal{O}_{Z \setminus \{0,0\}})$ has three connected components.



(4) For every x which belongs to $\mathbb{R}$, the topological space $(\mathbb{R} \setminus \{x\}, \mathcal{O}_{\mathbb{R}\setminus\{x\}})$ has two connected components.

e) We can argue as follows.

(1) The map

$$\mathbb{R} \xrightarrow{f} Z$$

given by $x \mapsto (x^2 - 1, x^3 - x)$ is continuous, since it is a polynomial map in each component.

(2) Moreover, f is surjective.

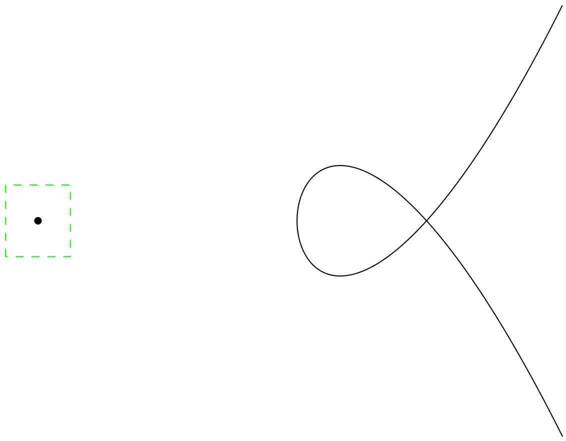
(3) By a result from the course, we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.

(4) By another result from the course, the target of a surjective, continuous map with a connected source is connected.

f) We have that $x^2 - 1 \geq -1$ for all $x \in \mathbb{R}$. Thus, for instance, the subset U of $\mathbb{R}^2$ given by

$$\left]-\tfrac{13}{4}, -\tfrac{11}{4}\right[\times \left]-\tfrac{1}{4}, \tfrac{1}{4}\right[$$

is a neighbourhood of $(-3, 0)$ in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ such that $U \cap Z$ is empty, since $-\frac{11}{4} < -1$.



Discussion

Closedness

Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. We have two equivalent definitions of what it means for A to be closed in X with respect to $\mathcal{O}_X$.

(1) That $X \setminus A$ belongs to $\mathcal{O}_X$.

(2) That every limit point of A in X with respect to $\mathcal{O}_X$ belongs to A.

When trying to prove that a given subset is closed or not closed, keep both points of view in mind, and decide which definition would be most convenient to use.

Closedness interacts in important ways with compactness and Hausdorffness. Thus it is very important that you have a firm grasp of deciding whether or not a given set is closed, and that you can calculate the closure of a given set.

Proving that two topological spaces are not homeomorphic

Remember that to prove that two topological spaces are not homeomorphic, it is not enough to demonstrate that any particular map is not a homeomorphism. Thus, in part d), it is not enough to demonstrate that the map

$$\mathbb{R} \longrightarrow Z$$

given by $x \mapsto (x^2 - 1, x^3 - x)$ is not a homeomorphism. Even though we then cannot carry out an analogue of the argument of part a), there might, a priori, be a different map between $\mathbb{R}$ and Z which we could prove to be a homeomorphism. Thus we have to demonstrate that there is no map at all between $\mathbb{R}$ and Z which is a homeomorphism.

For this, we need to make use of an 'invariant' of our topological spaces. This can be a property (connectedness, compactness, Hausdorffness, etc) of topological spaces that is preserved by homeomorphisms; or it can be a number, or some other gadget, which we associate to topological spaces, and which is preserved by homeomorphisms. In part d), the number of connected components is the invariant that we make use of (after removing a point).

When asked whether two topological spaces are homeomorphic, run through the properties of topological spaces that we have covered in the course, and try to figure out whether one of the topological spaces has one of these properties, but the other does not. See whether you can apply the removing points technique, or whether, in appropriate situations (when we have surfaces, for instance), their Euler characteristics differ.