Solutions 4

a) Let $\sim$ be the equivalence relation on I^2 for which $(K^2, \mathcal{O}_{K^2})$ is $(I^2/\sim, \mathcal{O}_{I^2/\sim})$. Let

$$I^2 \xrightarrow{\pi} K^2$$

be the quotient map. We have the following results from the course.

(1) We have that $(I, \mathcal{O}_I)$ is compact.

(2) A product of compact topological spaces is compact.

(3) A quotient of a compact topological space is compact.

Together, (1)–(3) imply that $(K^2, \mathcal{O}_{K^2})$ is compact. We also have the following results from the course.

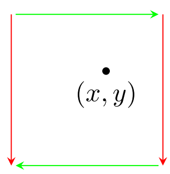
(1 bis) We have that $(I, \mathcal{O}_I)$ is connected.

(2 bis) A product of connected topological spaces is connected.

(3 bis) A quotient of a connected topological space is connected.

Together, (1 bis)–(3 bis) imply that $(K^2, \mathcal{O}_{K^2})$ is connected.

To prove that that $(K^2, \mathcal{O}_{K^2})$ is locally homeomorphic to an open rectangle, there are various cases to consider. Firstly, suppose that (x, y) does not belong to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})}(I^2)$.

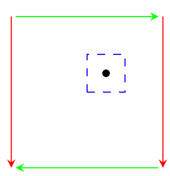


Let U be a subset of I^2 which has the following properties.

(1) We have that U is an open rectangle.

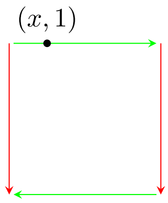
(2) We have that x belongs to U.

(3) We have that $U \cap \partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})}(I^2)$ is empty.



Then $\pi(U)$ is a neighbourhood of $\pi(x,y)$ in K^2 with respect to $\mathcal{O}_{K^2}$, and we have that $\left(\pi(U), \mathcal{O}_{\pi(U)}\right)$ is homeomorphic to an open rectangle, where $\mathcal{O}_{\pi(U)}$ is the subspace topology on $\pi(U)$ with respect to $\left(K^2, \mathcal{O}_{K^2}\right)$.

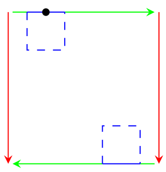
Secondly, let us consider a point $(x,1)$ such that x belongs to I.



Suppose that a and b are real numbers, and that $0 < a < x < b < 1$. Let U be, for example, the subset

$$\left(]a,b[\times \left]\tfrac{3}{4},1\right]\right) \cup \left(]1-b,1-a[\times \left[0,\tfrac{1}{4}\right[\right)$$

of I^2.

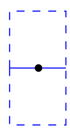


The following hold.

(1) We have that $\pi^{-1}\left(\pi(U)\right) = U$, and that U belongs to $\mathcal{O}_{I^2}$. Thus $\pi(U)$ belongs to $\mathcal{O}_{K^2}$.

(2) We have that x belongs to U, and thus that $\pi(x)$ belongs to $\pi(U)$.

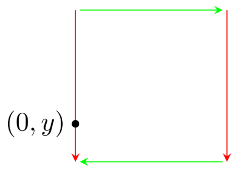
(3) We have that $\left(\pi(U), \mathcal{O}_{\pi(U)}\right)$ is homeomorphic to an open rectangle, where $\mathcal{O}_{\pi(U)}$ is the subspace topology on $\pi(U)$ with respect to $\left(K^2, \mathcal{O}_{K^2}\right)$. Intuitively, the two pieces of U are glued together as follows.



A detailed proof is not needed.

The case that we have a point $(x, 0)$ such that x belongs to I is similar.

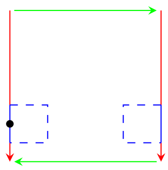
Thirdly, let us consider a point $(0, y)$ such that y belongs to I.



Suppose that $0 < a < x < b < 1$. Let U be, for example, the subset

$$\left(\left[0, \tfrac{1}{4}\right[\times \left]a, b\right[\right) \cup \left(\left]\tfrac{3}{4}, 1\right] \times \left]a, b\right[\right)$$

of I^2.

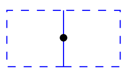


The following hold.

(1) We have that $\pi^{-1}(\pi(U)) = U$, and that U belongs to $\mathcal{O}_{I^2}$. Thus $\pi(U)$ belongs to $\mathcal{O}_{K^2}$.

(2) We have that x belongs to U, and thus that $\pi(x)$ belongs to $\pi(U)$.

(3) We have that $\left(\pi(U), \mathcal{O}_{\pi(U)}\right)$ is homeomorphic to an open rectangle, where $\mathcal{O}_{\pi(U)}$ is the subspace topology on $\pi(U)$ with respect to $\left(K^2, \mathcal{O}_{K^2}\right)$. Intuitively, the two pieces of U are glued together as follows.

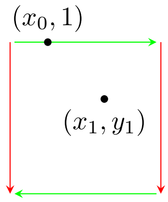


A detailed proof is not needed.

The case that we have a point $(x, 1)$ such that x belongs to I is similar.

To prove that $(K^2, \mathcal{O}_{K^2})$ is Hausdorff, there are various ways to proceed, but the most hands on is consider various cases, in a similar way as in the proof that $(K^2, \mathcal{O}_{K^2})$ is locally homeomorphic to an open rectangle.

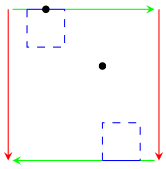
For example, suppose that we have a point $(x_0, 1)$, where x_0 belongs to I, and a point (x_1, y_1) which does not belong to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})}(I^2)$.



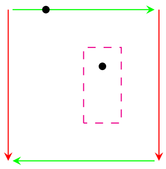
Suppose that $0 < a_0 < x_0 < b_0 < 1$, and that $0 < c < \min\{y_1, 1 - y_1\}$. Let U_0 be, for example, the subset

$$(]a, b[\times]1 - c, 1]) \cup (]1 - b, 1 - a[\times [0, c[)$$

of I^2.



Suppose that $0 < a_1 < x_1 < b_1 < 1$. Let U_1 be, for example, the subset $]a_1, b_1[\times]c, 1 - c[$ of I^2.



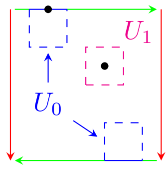
The following hold.

(1) We have that $\pi(U_0)$ is a neighbourhood of $\pi(x_0, 1)$ in K^2 with respect to $\mathcal{O}_{K^2}$.

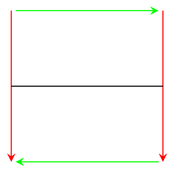
(2) We have that $\pi(U_1)$ is a neighbourhood of $\pi(x_1, y_1)$ in K^2 with respect to $\mathcal{O}_{K^2}$.

(3) We have that $\pi(U_0) \cap \pi(U_1)$ is empty.

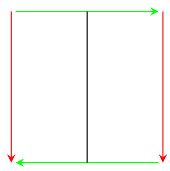
The other cases are similar. It is not necessary that you give full details, as I have done here. It is sufficient to draw a 'generic' picture for each case, such as the following picture for the case above, as long as you write that you are drawing subsets U_0 and U_1 of I^2, such that $\pi(U_0)$ and $\pi(U_1)$ give the required neighbourhoods in K^2 with respect to $\mathcal{O}_{K^2}$ of $\pi(x_0, 1)$ and $\pi(x_1, y_1)$ respectively.



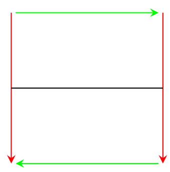
b) For example, we may take C to be a subset such as $I \times \left\{\frac{1}{2}\right\}$.



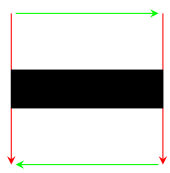
Or a subset such as $\left\{\frac{1}{2}\right\} \times I$.



c) Suppose that we take C to be a subset such as $I \times \left\{\frac{1}{2}\right\}$.

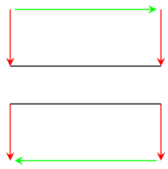


We thicken C.

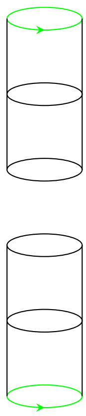


This thickening is a cylinder.

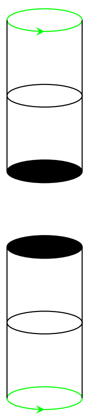
We cut out the interior of this cylinder.



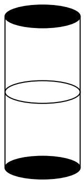
In other words, we have a pair of cylinders, as follows.



We now glue in discs to the circles left after we removed the interior of our cylinder.

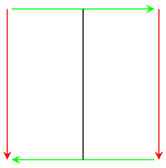


Making the remaining glueing, we see that we have obtained $(S^2, \mathcal{O}_{S^2})$.

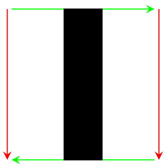


A quick way to see that we must obtain $(S^2, \mathcal{O}_{S^2})$ by performing surgery with respect to C is to calculate, using a Δ-complex structure as in parts d) and e), that $(K^2, \mathcal{O}_{K^2})$ has Euler characteristic 0, and to remember that performing a surgery in which we cut out the interior of a cylinder, and glue discs to the two circles we obtain, increases Euler characteristic by 2.

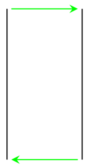
Suppose now that we take C to be a subset such as $\left\{\frac{1}{2}\right\} \times I$.



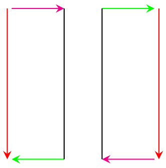
We thicken C.



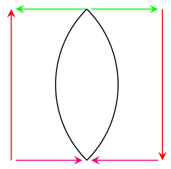
This thickening is a Möbius band.



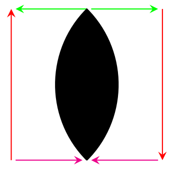
We cut out the interior of this Möbius band.



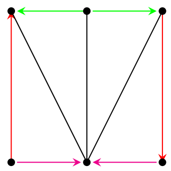
In other words, we have the following.



We glue a disc to the circle left after we removed the interior of the Möbius band.



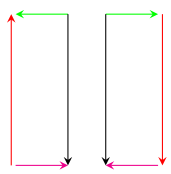
The following is a Δ-complex structure on this gadget.



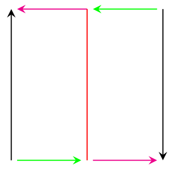
Its Euler characteristic is $3 - 6 + 4 = 1$. The only surface on the classification with this Euler characteristic is the real projective plane.

Alternatively, we could appeal to the fact that the Euler characteristic of $(K^2, \mathcal{O}_{K^2})$ is 0, and the fact that performing a surgery in which we cut out the interior of a Möbius band and glue a disc to the resulting circle increases Euler characateristic by 1.

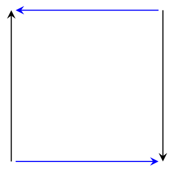
As yet another approach, we can observe that we can obtain the gadget we arrived at after glueing a disc onto the circle left after removing the interior of the Möbius band by glueing as follows.



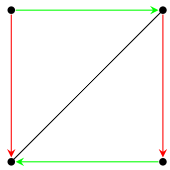
We obtain the same by glueing as follows.



This is exactly the real projective plane.



d) A Δ-complex structure on $(K^2, \mathcal{O}_{K^2})$ is depicted below.



e) Using the Δ-complex structure of part d), we calculate that the Euler characteristic is: $1 - 3 + 2 = 0$.

f) By the classification of surfaces, every surface is homeomorphic to either an n-handlebody, which has Euler characteristic $2 - 2n$, or an n-cross cap, which has Euler characteristic $2 - n$. Thus there are two surfaces, up to homeomorphism, with Euler characteristic 0: the 1-handlebody, or in other words the torus $(T^2, \mathcal{O}_{T^2})$, and the 2-cross cap, or in other words $(K^2, \mathcal{O}_{K^2})$.

Discussion

Surgery

It is very important that you feel confident that you can carry out a surgery argument both when given a picture of a surface in $\mathbb{R}^3$, as in Revision Question 10, and when given a 'glueing diagram' as in this question. In particular, it is important that you can carry out surgery in the case that the thickening of our circle is a Möbius band, as well as when the thickening is a cylinder (which it will always be when the surface can be drawn in $\mathbb{R}^3$).

The Euler characteristic is a very powerful tool, which you should keep close to hand! It can be used in different ways, as in part c), to determine which surface we obtain after a surgery, and it is precisely this interaction of the Euler characteristic with surgery that underlies the proof of the classification of surfaces.