MA3002 Generell Topologi — Vår 2014 – Syllabus

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Contents

1 Overview

This document lists all the examinable material for the course this year. I have typically listed only definitions, proofs, and skills which I expect you to have acquired. This of course does not mean that you should ignore examples, exercises, and other supporting material!

References are to this year's lecture notes where available, and otherwise to last year's lecture notes. Thus 2014:8.2.1 refers to 8.2.1 in this year's lecture notes, and 2013:19.1 refers to 19.1 of the lecture notes for 2013. References to sections are of the form 2014:§1.4. References without the § symbol are always to individual definitions, remarks, examples, propositions, etc, and not to sections.

It is essential that you understand thoroughly all items marked with a star ⋆ in the 'Must know' column. These are the 'core' of the course, which you need to have a firm grasp of to obtain a satisfactory grade. In particular, you must be able to answer confidently the questions in the 'Exam questions' sections of the lecture notes on each of these topics. If something is listed, but not marked as 'Must know', it is still important, and may appear on the exam!

Although I shall not examine your ability to memorise proofs, it is a very good idea to work through the proofs in the lecture notes, as this will benefit your overall understanding of the material, and help you with being able to answer the exam questions. It is likely that one or two 'simple' proofs will asked for on the exam as parts of a question, which may not be marked as 'Must know'. These will be proofs that I believe that you should be able to work out for yourself, if you understand the material well.

To help prepare yourself for this, try to build up to being able to prove by yourself as many of the propositions in the lecture notes as you can, starting with the shortest. You will typically be allowed to quote 'big' results from the syllabus without proof, but it is very important that you are able to do so clearly and precisely.

I shall continue to work on the notes for this year up to the exam. However, it is likely that I will write up a little here and a little there, where I feel that it is most needed, rather than continue linearly, lecture by lecture, as I have done so far.

Thus you should not rely on, or wait for, the appearance of further notes. If you have not given the notes from a last year a try yet, I highly recommend that you do so: the students last year were universally happy with them.

This year's notes are a 'second draft' of last year's notes. Although they take a long time and lot of effort for me to prepare, and may look quite different in the end from last year, the differences are only quite minor, in the overall scheme of things. I should perhaps say that I have tried my hardest to keep up with the lecture notes for this year, but unfortunately just have not had enough time to do so.

If there is a topic which you feel is not sufficiently clear in last year's notes, particularly if this due to the fact that I presented the material in the lectures this year in a slightly different way, then you should, first and foremost, get in touch with me, either by email or in person. Please do not hesitate to do this: I will then prioritise writing up notes on these points. This, of course, also holds for this year's lecture notes.

As a second option, you can also consult other literature. There are lots of books available, some of which are listed in the Literature Guide. You may need my advice. Again, please do not hesitate to get in touch with me if so.

I am very happy to discuss any of the material of the course with each of you, and you can contact me at any time to do this, all the way through to the exam. Often we can make enormous progress when discussing things one-to-one, because I can present the material in a way that is appropriate to your individual needs. Just write an email to me, and we can arrange a time to meet.

Besides this, I would like to arrange one meeting with each of you individually before the exam, to see how you are feeling about your understanding of the course as a whole, and to help you with topics that you are struggling with. This meeting will be arranged via the course webpage.

I hope that you all feel that these arrangements will give you sufficient resources and opportunity to achieve the grade that you are aiming for on the exam. If you have concerns, please get in touch.

I have worked with many of you in the Exercise Classes, and feel certain that if you work hard over the next month or so, you should all be able to achieve, and heartily deserve, at least a B.

I have not listed material on advanced point set topology in this syllabus, as I have not heard from anyone that they plan to look into this. If you do, it is essential that you let me know as soon as possible; otherwise, I will not include a question on this material on the exam, as it would not serve any purpose.

2 Fundamentals

Topic	Reference	Must know
Definition of a topological space.	2014:1.1.1	$\star$
Deciding whether we have a topology in finite examples.	2014:§1.4	$\star$
Definition of the standard topology $\mathcal{O}_{\mathbb{R}}$ on $\mathbb{R}$.	2014:1.6.1	$\star$
Proof that $\mathcal{O}_{\mathbb{R}}$ is a topology.	2014:2.1.9, and the previous results on which the proof relies	
Definition of subspace topology.	2014:2.2.2	$\star$
Proof that subspace topology is a topology.	2014:2.2.4	
Deciding whether a set belongs to a subspace topology in both finite and geometric examples.	2014:§2.3, and 2014:§4.1	$\star$

Topic	Reference	Must know
Definition of product topology.	2014:3.1.2	⋆
Proof that product topology is a topology.	2014:3.1.3	
Deciding whether a set belongs to a product topology in both finite and geometric examples.	2014:§3.2, and 2014:§4.1	⋆
Definition of a continuous map.	2014:4.2.3	⋆
Deciding whether a map is continuous in finite examples.	2014:§4.3	⋆
Recognising when a map is continuous in geometric examples. Demonstrating that a map is not continuous in geometric examples. Awareness that one can 'build' continuous maps from polynomial maps in 'canonical ways'.	2014:§5.1	⋆
Giving simple 'abstract' proofs of continuity.	2014:5.2.2, 2014:5.3.1, and 2014:5.4.3, including proofs	⋆
Definition of quotient topology.	2014:6.1.4	⋆
Proof that quotient topology is a topology.	2014:6.1.5	
Awareness that quotient map is continuous.	2014:6.1.9	⋆
Deciding whether a set belongs to a quotient topology in both finite and geometric examples	2014:§6.2 and 2014:§6.3	⋆
Familiarity with quotient topologies on circle, cylinder, Möbius band, torus, Klein bottle, and sphere	2014:§6.4	⋆
Definition of a homeomorphism	2014:7.1.5	⋆
Deciding whether a map is a homeomorphism in finite examples. Awareness that a continuous bijection is not necessarily a homeomorphism.	2014:§7.2	⋆

Topic	Reference	Must know
Recognising homeomorphic topological spaces in geometric examples.	2014:§7.3 and 2014:§8.1	$\star$
Definition of a neighbourhood.	2014:8.2.1	$\star$
Definition of a limit point.	2014:8.3.1	$\star$
Deciding whether an element is a limit point in both finite and geometric examples.	2014:§8.4	$\star$
Definition of closure.	2014:8.5.1	$\star$
Calculating closure in both finite and geometric examples.	2014:§8.6	$\star$
Local characterisation of closed sets: a set is closed if and only if it is its own closure.	2014:9.1.1	Statement: $\star$
Definition of boundary.	2014:9.2.1	$\star$
Calculating boundary in both finite and geometric examples.	2014:§9.3 and 2014:§9.4	$\star$

3 Connectedness

Topic	Reference	Must know
Definition of a connected topological space.	2014:9.5.3	$\star$
Proving that a topological space is not connected in examples. Recognising when a finite topological space is connected.	2014:§9.6, 2014:§10.1, and 2014:§10.2	$\star$
A topological space is connected if and only if there does not exist a surjective, continuous map $$X \longrightarrow \{0,1\},$$ where $\{0,1\}$ has the discrete topology.	2014:10.3.1	Statement: $\star$

Topic	Reference	Must know
$(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected. Awareness that the proof is 'low-level'. The proof will not be examined.	2014:§10.4	$\star$.
The target of a continuous surjection with a connected source is connected.	2014:10.5.1	Statement: $\star$
Homeomorphisms preserve connectedness.	2014:10.5.2	$\star$
Quotient of a connected topological space is connected.	2014:10.5.3	$\star$ (including the proof)
Product of connected topological spaces is connected.	2014:10.7.1	Statement: $\star$
Proving that topological spaces are connected in geometric examples, building everything up in 'canonical ways' from the fact $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.	2014:§10.6 and 2014:§10.8	$\star$
Definition of a connected component of a topological space.	2014:11.3.4	$\star$
Calculating connected components in both finite and geometric examples.	2014:§11.4	$\star$
Proving that two topological spaces are not homeomorphic using connectedness, by removing points.	2014:§11.2, 2014:§11.5, and 2014:§12.1	$\star$
Definition of a locally connected topological space.	2014:12.3.1	$\star$
Proving that topological spaces are locally connected in examples.	2014:§12.3	$\star$
Awareness that a connected topological space is not necessarily locally connected.	2014:§12.4	
Definition of a path in a topological space.	2013:10.3	$\star$
Definition of a path connected topological space.	2013:10.14	$\star$
Awareness that we can 'concatenate' and 'reverse' paths.	2013:10.8, 2013:10.10, 2013:10.11, 2013:10.12	

Topic	Reference	Must know
Proving that topological spaces are path connected in both finite and geometric examples.	2013:10.15	$\star$
A path connected topological space is connected.	2013:10.17	Statement: $\star$

4 Hausdorffness

Topic	Reference	Must know
Definition of a Hausdorff topological space.	2014:13.1.1	$\star$
Proof that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff.	2014:13.2.1	$\star$
Subspace of a Hausdorff topological space is Hausdorff.	2014:13.3.1	$\star$
Product of Hausdorff topological spaces is Hausdorff.	2014:13.3.3	$\star$
Homeomorphisms preserve the property of being Hausdorff.	2014:13.3.8	$\star$
A topological space $(X, \mathcal{O}_X)$ is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$.	2014:14.1.3	
Awareness that a quotient of a Hausdorff topological space is not neessarily Hausdorff.	2014:§13.4	$\star$
Proving that topological spaces are Hausdorff in geometric examples, building everything up in 'canonical ways' from the fact $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff.	2014:§13.3	$\star$
Given a topological space $(X, \mathcal{O}_X)$ and an equivalence relation $\sim$ on X such that $(X/\sim, \mathcal{O}_{X/\sim})$ is Hausdorff, then $R_\sim$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$. Awareness that the converse does not hold.	2014:14.2.2	

5 Compactness

Topic	Reference	Must know
Definition of a compact topological space.	2014:14.3.1	$\star$
Awareness that every finite topological space is compact.	2014:14.3.4	
Proving that a topological space is not compact in examples. In particular, proving that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not compact.	2014:§14.4	$\star$
$(I, \mathcal{O}_I)$ is compact.	2013:13.2	Statement: $\star$
The target of a continuous surjection with a compact source is compact.	2013:12.10	Statement: $\star$
Homeomorphisms preserve compactness.	2013:12.11	$\star$
Quotient of a compact topological space is compact.	2013:13.14.	$\star$
Product of compact topological spaces is compact. Understanding of how the tube lemma can fail without the compactness assumption.	2013:14.1, 2013:14.2, 2013:14.4	$\star$ (excluding proofs of 2013:14.2 and 2013:14.4)
Proving that topological spaces are compact in geometric examples, building everything up in 'canonical ways' from the fact $(I, \mathcal{O}_I)$ is Hausdorff.	2013:13.15 and 2013:14.5	$\star$
Closed subspace of a compact topological space is compact.	2013:13.7	
Compact subset of a Hausdorff topological space is closed.	2013:13.11	

Topic	Reference	Must know
Given a compact topological space $(X, \mathcal{O}_X)$ and a Hausdorff topological space $(Y, \mathcal{O}_Y)$, a map $X \longrightarrow Y$ is a homeomorphism if and only if it is a continuous bijection.	2013:13.13	Statement: $\star$
A subset of $\mathbb{R}^n$ is compact with respect to $\mathcal{O}_{\mathbb{R}^n}$ if and only if it is closed and bounded.	2013:14.9.	Statement: $\star$
Proving that a topological space is compact using the previous result.	To come.	$\star$
Definition of a locally compact topological space.	2013:14.12	$\star$
Proving that a topological space is locally compact in geometric examples. In particular, proving that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally compact.	2013:14.13	$\star$
A topological space which is both compact and Hausdorff is locally compact.	2013:14.14	Statement: $\star$
Proving that a topological space is not locally compact in examples.	To come.	$\star$
Given a topological space $(X, \mathcal{O}_X)$, a locally compact topological space $(Y, \mathcal{O}_Y)$, and an equivalence relation $\sim$ on X, we have that $\big((X/\sim) \times Y, \mathcal{O}_{(X/\sim)\times Y}\big)$ is homeomorphic to $\big((X \times Y)/\approx, \mathcal{O}_{(X\times Y)/\approx}\big)$, where $\approx$ is generated by $(x_0, y) \approx (x_1, y)$ if $x_0 \sim x_1$, for all y which belong to Y.	To come.	
Using the previous result to prove that two topological spaces are homeomorphic in examples.	To come.	

Topic	Reference	Must know
Given a locally compact, Hausdorff topological space $(X, \mathcal{O}_X)$, and an equivalence relation $\sim$ on X such that $R_\sim$ is closed in $X \times X$ with respect to $\mathcal{O}_{X\times X}$, then $(X/\sim, \mathcal{O}_{X/\sim})$ is Hausdorff.	To come.	
Proving that topological spaces constructed via quotients are Hausdorff in examples, using the previous result.	To come.	$\star$
Definition of a one point compactification of a topological space.	To come.	$\star$
Finding one point compactifications in geometric examples.	To come.	$\star$
Every topological space has a one point compactification.	To come.	
A one point compactification of a locally compact topological space is Hausdorff.	To come.	

6 Classification of surfaces

Topic	Reference	Must know
Ability to equip a given topological space with a Δ-complex structure.	2013:23.7	$\star$
Definition of a surface.	2013:23.9	$\star$
Ability to decide whether a given topological space is a surface.	2013:23.13	$\star$
Definition of the Euler characteristic of a Δ-complex.	2013:23.14	$\star$
Awareness that Euler characteristic is independent of a choice of Δ-complex structure.	2013:23.15	
Calculating Euler characteristic in examples.	2013:23.17	$\star$

Topic	Reference	Must know
Statement of the classification of surfaces, including being able to define an n-handlebody and an n-crosscap.	2013:23.25	$\star$
Definition of a surgery on a surface.	To come.	$\star$
Awareness of how the proof of the classification of surfaces is carried out. In particular, ability to state precisely (without proof) the main facts which are appealed to, and an understanding of how they fit together.	To come.	
Ability to carry out a surgery argument to decide which of the surfaces in the statement of the classification, namely an n-handlebody or an n-crosscap for some $n \geq 0$, a given surface is homeomorphic to. Ability to determine in this way the Euler characteristic of a given surface.	To come.	$\star$
Awareness that the classification of surfaces does not say that we can deform any orientable surface to an n-handlebody *inside* $\mathbb{R}^3$.	To come.	

7 Knot theory

Topic	Reference	Must know
Definition of a knot.	2013:17.2	
Definition of a link.	2013:17.4	
Definition of isotopy of knots.	To come.	
Definition of the Reidemeister moves.	To come.	$\star$
Ability to demonstrate that two knots are isotopic via a sequence of Reidemeister moves.	2013:18.2	$\star$

Topic	Reference	Must know
Definition of the linking number of an oriented link.	2013:18.10	$\star$
Linking number is a link invariant.	2013:18.14	$\star$ (including the proof)
Ability to calculate linking numbers in examples, and thereby to prove that two given links are not isotopic.	2013:18.12 and 2013:18.16.	$\star$
Definition of m-colourability of a link.	2013:19.1	$\star$
Colourability is a link invariant.	2013:19.5	$\star$ (including proof)
Ability to find an m-colouring of a given link for some m, where m may be given or not given in advance.	2013:19.3	$\star$
Ability to prove that a given link is only m-colourable for certain m, and thereby to prove that two given links are not isotopic.	2013:19.7, 2013:19.9, and 2013:20.1	$\star$
Ability to calculate the bracket polynomial (both the version with two variables A and B, and the Laurent polynomial version) of a link from first principles.	2013:20.14	$\star$
Laurent polynomial version of the bracket polynomial is invariant under the R2-move and the R3-move.	2013:20.20	$\star$ (including proof)
Definition of the writhe of an oriented link.	2013:21.1	$\star$
Ability to calculate the writhe of a given oriented link.	2013:21.3	$\star$
Definition of the Jones polynomial.	2013:21.5	$\star$
Ability to calculate the Jones polynomial from first principles.	2013:21.6	$\star$
Jones polynomial is a link invariant.	2013:21.7	$\star$ (including proof).
Awareness that the Jones polynomial in t satisfies the skein relations.	2013:21.12 and 2013:22.2	$\star$ (excluding proof)

Topic	Reference	Must know
Ability to calculate the Jones polynomial in t using the skein relations, and thereby (or by calculating a Jones polynomial from first principles) to conclude that two given links are not isotopic.	2013:21.14 and 2013:22.1	$\star$
Jones polynomial of mirror image is obtained by replacing t by t^{-1}.	2013:22.3	$\star$ (including proof)
Ability to deduce from the latter that a given link is not isotopic to its mirror image.	2013:22.6	
Jones polynomial of a knot does not depend on the choice of orientation. Awareness that this is not the case for links.	2013:22.7 and 2013:22.9	$\star$ (including proof).
Ability to carry out Seifert's algorithm for a given knot to obtain a surface with boundary.	To come.	$\star$
Calculation of the Euler characteristic of a surface with boundary obtained via Seifert's algorithm.	To come.	$\star$
Ability to use the latter and the classification of surfaces to determine for which value of n a surface with boundary obtained via Seifert's algorithm is homeomorphic to an n-handlebody with a disc removed.	To come.	$\star$